Analytic functions associated with Caputos fractional differentiation defined by Hilbert space operator

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Abstract. In this paper, we introduce a new class of functions which are analytic and univalent with negative coefficients defined by using certain fractional operators described in the Caputo sense. Characterization property, the results on modified Hadamard product and integral transforms are discussed. Further, distortion theorem and radii of starlikeness and convexity are also determined here.

Resumen. En este trabajo, presentamos una nueva clase de funciones que son analíticas y univalente con coeficientes negativos, definidos usando ciertos operadores fraccionarios en el sentido de Caputo. Discutimos la propiedad de caracterización, los resultados sobre el producto de Hadamard modificado y transformaciones integrales. Además, determinamos el teorema de distorsión y los radios de "starlikeness" y convexidad.

1 Introduction

Fractional calculus operators have recently found interesting application in the theory of analytic functions. The classical definition of fractional calculus and their other generalizations have fruitfully been applied in obtaining, the characterization properties, coefficient estimates and distortion inequalities for various subclasses of analytic functions. Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

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which are analytic in the open disc $U = \{z : z \in \mathbb{C};\ |z| < 1\}$. Also denote by $\mathcal{T}$, a subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0; z \in U \tag{1.2}$$

introduced and studied by Silverman [9]. For functions $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum\limits_{k=2}^{\infty} b_k z^k$, we define the convolution product (or Hadamard) of f and g by

$$(f * g)(z) = (g * f)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \quad z \in U. \tag{1.3}$$

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on$\mathcal{H}$. For a complex-valued function f analytic in a domain $\mathbb{E}$ of the complex z-plane containing the spectrum $\sigma(\mathbb{P})$ of the bounded linear operator $\mathbb{P}$, let $f(\mathbb{P})$ denote the operator on $\mathcal{H}$ defined by Dunford [3],

$$f(\mathbb{P}) = \frac{1}{2\pi i} \int\limits_{\mathcal{C}} (z\mathbb{I} - \mathbb{P})^{-1} f(z) dz, \tag{1.4}$$

where $\mathbb{I}$ is the identity operator on $\mathcal{H}$ and $\mathcal{C}$ is a positively-oriented simple rectifiable closed contour containing the spectrum $\sigma(\mathbb{P})$ in the interior domain. The operator $f(\mathbb{P})$ can also be defined by the following series:

$$f(\mathbb{P}) = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} \mathbb{P}^n$$

which converges in the norm topology (cf. [3]).

Now we look at the Caputos [2]definition which shall be used throughout the paper. Caputos definition of the fractional-order derivative is defined as

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int\limits_{a}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} \tag{1.5}$$

where $n - 1 < Re(\alpha) \leq n, n \in N$, and the parameter α is allowed to be real or even complex, a is the initial value of the function f .

We recall the following definitions [6] .

Definition 1. *[6] Let the function $f(z)$ be analytic in a simply - connected region of the $z-$ plane containing the origin. The fractional integral of f of* *order μ is defined by*

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\mu}} d\xi, \quad \mu > 0, \tag{1.6}$$

where the multiplicity of $(z-\xi)^{1-\mu}$ is removed by requiring $log(z-\xi)$ to be real when $z-\xi > 0$.

Definition 2. *[6] The fractional derivatives of order μ, is defined for a function $f(z)$, by*

$$D_z^{\mu} f(z) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\mu}} d\xi, \quad 0 \le \mu < 1, \tag{1.7}$$

where the function $f(z)$ is constrained, and the multiplicity of the function $(z-\xi)^{-\mu}$ is removed as in Definition 1.

Definition 3. *Under the hypothesis of Definition 2, the fractional derivative of order $n+\mu$ is defined by*

$$D_z^{n+\mu} f(z) = \frac{d^n}{dz^n} D_z^{\mu} f(z), \quad (0 \le \mu < 1 \ ; \ n \in N_0). \tag{1.8}$$

With the aid of the above definitions, and their known extensions involving fractional derivative and fractional integrals, Srivastava and Owa [13] introduced the operator Ω^δ $(\delta \in \mathbb{R}; \delta \neq 2, 3, 4, \dots) : \mathcal{A} \to \mathcal{A}$ defined by

$$\Omega^\delta f(z) = \Gamma(2-\delta) z^\delta D_z^\delta f(z) = z + \sum_{n=2}^{\infty} \Phi(n,\delta) a_n z^n \tag{1.9}$$

where

$$\Phi(n,\delta) = \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}. \tag{1.10}$$

For $f \in \mathcal{A}$ and various choices of δ, , we get different operators

$$\Omega^0 f(z) := f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.11}$$

$$\Omega^1 f(z) := z f'(z) = z + \sum_{k=2}^{\infty} k a_k z^k \tag{1.12}$$

$$\Omega^j f(z) := \Omega(\Omega^{j-1} f(z)) = z + \sum_{k=2}^{\infty} k^j a_k z^k, (j = 1, 2, 3, ...) \tag{1.13}$$

which is known as Salagean operator[7] .Also note that

$$\Omega^{-1}f(z) = \frac{2}{z}\int_0^z f(t)dt := z + \sum_{k=2}^{\infty}\left(\frac{2}{k+1}\right)a_k z^k$$

and

$$\Omega^{-j}f(z) := \Omega^{-1}(\Omega^{-j+1}f(z)) := z + \sum_{k=2}^{\infty}\left(\frac{2}{k+1}\right)^j a_k z^k, (j = 1, 2, 3, ...) \quad (1.14)$$

called Libera integral operator.We note that the Libera integral operator is generalized as Bernardi integral operator given by Bernardi[1],

$$\frac{1+\nu}{z^\nu}\int_0^z t^{\nu-1}f(t)dt := z + \sum_{k=2}^{\infty}\left(\frac{1+\nu}{k+1}\right)a_k z^k, (\nu = 1, 2, 3, ...).$$

Making use of these results Recently Salah and Darusin [8], introduced the following operator

$$\mathcal{J}_\mu^\eta = \frac{\Gamma(2+\eta-\mu)}{\Gamma(\eta-\mu)}z^{\mu-\eta}\int_0^z \frac{\Omega^\eta f(t)}{(z-t)^{\mu+1-\eta}}dt \quad (1.15)$$

where η(real number) and $(\eta - 1 < \mu < \eta < 2)$. By simple calculations for functions $f(z) \in \mathcal{A}$, we get

$$\mathcal{J}_\mu^\eta f(z) = z + \sum_{k=2}^{\infty}\frac{(\Gamma(k+1))^2\Gamma(2+\eta-\mu)\Gamma(2-\eta)}{\Gamma(k+\eta-\mu+1)\Gamma(k-\eta+1)}a_k z^k \quad (z \in U), \quad (1.16)$$

and for the sake of brevity we let

$$C_k(\eta,\mu) = \frac{(\Gamma(k+1))^2\Gamma(2+\eta-\mu)\Gamma(2-\eta)}{\Gamma(k+\eta-\mu+1)\Gamma(k-\eta+1)} \quad (1.17)$$

and

$$C_2(\eta,\mu) = \frac{4\Gamma(2+\eta-\mu)\Gamma(2-\eta)}{\Gamma(3+\eta-\mu)\Gamma(1-\eta)}$$

unless otherwise stated.

Further, note that $\mathcal{J}_0^0 f(z) = f(z)$ and $\mathcal{J}_1^1 f(z) = zf'(z)$. In this paper, by making use of the operator $\mathcal{J}_\mu^\eta$ we introduced a new subclass of analytic functions with negative coefficients and discuss some interesting properties of this generalized function class.

For $0 \le \alpha < \frac{1}{2}$, we let $\mathcal{J}_\mu^\eta(\lambda,\alpha)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) and satisfying the inequality

$$\left\|\frac{\mathcal{J}_{\mu,\eta}^\lambda(\mathbb{P}) - 1}{\mathcal{J}_{\mu,\eta}^\lambda(\mathbb{P}) - (2\alpha-1)}\right\| < 1 \quad (1.18)$$

where

$$\mathcal{J}^{\lambda}_{\mu,\eta}f(\mathbb{P}) = \frac{\mathbb{P}(\mathcal{J}^{\eta}_{\mu}f(\mathbb{P}))'}{\mathcal{J}^{\eta}_{\mu}f(\mathbb{P})} + \frac{\lambda\mathbb{P}^2(\mathcal{J}^{\eta}_{\mu}f(\mathbb{P}))''}{\mathcal{J}^{\eta}_{\mu}f(\mathbb{P})}, \tag{1.19}$$

$0 \le \lambda \le 1$, $\mathcal{J}^{\eta}_{\mu}f(z)$ is given by (1.16) . We further let $\mathcal{T}\mathcal{J}^{\eta}_{\mu}(\lambda,\alpha) = \mathcal{J}^{\eta}_{\mu}(\lambda,\alpha) \cap \mathcal{T}$.

In the following section we obtain coefficient estimates for $f \in \mathcal{T}\mathcal{J}^{\eta}_{\mu}(\lambda,\alpha)$.

2 Coefficient Bounds

Theorem 1. *Let the function f be defined by (1.2). Then $f \in \mathcal{T}\mathcal{J}^{\eta}_{\mu}(\lambda,\alpha)$ if and only if*

$$\sum_{k=2}^{\infty}(k[1+\lambda(k-1)]-\alpha)C_k(\eta,\mu)a_k \le (1-\alpha). \tag{2.1}$$

The result is sharp for the function

$$f(z) = z - \frac{(1-\alpha)}{(k[1+\lambda(k-1)]-\alpha)C_k(\eta,\mu)}z^k, \quad k \ge 2. \tag{2.2}$$

Proof. Suppose f satisfies (2.1). Then for $\|z\|$,

$$\begin{aligned}
&\left\|\mathcal{J}^{\lambda}_{\mu,\eta}(\mathbb{P}) - 1\right\| < \left\|\mathcal{J}^{\lambda}_{\mu,\eta}(\mathbb{P}) + 1 - 2\alpha\right\| \\
&= \left\|\frac{z - \sum_{k=2}^{\infty} kC_k(\eta,\mu)a_k z^k - \lambda\sum_{k=2}^{\infty} k(k-1)C_k(\eta,\mu)a_k z^k}{z - \sum_{k=2}^{\infty} C_k(\eta,\mu)a_k z^k} - 1\right\| \\
&< \left\|2(1-\alpha) - \frac{z - \sum_{k=2}^{\infty} kC_k(\eta,\mu)a_k z^k - \lambda\sum_{k=2}^{\infty} k(k-1)C_k(\eta,\mu)a_k z^k}{z - \sum_{k=2}^{\infty} C_k(\eta,\mu)a_k z^k}\right\| \\
&\le \sum_{k=2}^{\infty}(k[1+\lambda(k-1)]-1)\ C_k(\eta,\mu)a_k \le 2(1-\alpha) - \sum_{k=2}^{\infty}(k[1+\lambda(k-1)]+(1-2\alpha))C_k(\eta,\mu)a_k \\
&= \sum_{k=2}^{\infty}(k[1+\lambda(k-1)]-\alpha)C_k(\eta,\mu)a_k - (1-\alpha) \\
&\le 0, \quad \text{by (2.1)}.
\end{aligned}$$

Hence, by maximum modulus theorem and (1.18), $f \in \mathcal{T}\mathcal{J}^{\eta}_{\mu}(\lambda,\alpha)$. To prove the converse, assume that

$$\begin{aligned}
&\left\|\frac{\mathcal{J}^{\lambda}_{\mu,\eta}(\mathbb{P}) - 1}{\mathcal{J}^{\lambda}_{\mu,\eta}(\mathbb{P}) + 1 - 2\alpha}\right\| = \left\|\frac{-\sum\limits_{k=2}^{\infty}(k[1+\lambda(k-1)]-1)C_k(\eta,\mu)a_k z^{k-1}}{2(1-\alpha) - \sum\limits_{k=2}^{\infty}(k[1+\lambda(k-1)]+(1-2\alpha))C_k(\eta,\mu)a_k z^{k-1}}\right\| \\
&\le 1, \quad z \in U.
\end{aligned}$$

Equivalently,

$$\operatorname{Re}\left\{\frac{-\sum\limits_{k=2}^{\infty}(k[1+\lambda(k-1)]-1)C_k(\eta,\mu)a_k z^{k-1}}{2(1-\alpha)-\sum\limits_{k=2}^{\infty}(k[1+\lambda(k-1)]+(1-2\alpha))C_k(\eta,\mu)a_k z^{k-1}}\right\}<1. \quad (2.3)$$

Since $\operatorname{Re}(z)\leq \|z\|$ for all z. Choose values of z on the real axis so that $\mathcal{J}^{\lambda}_{\mu,\eta}(\mathbb{P})$ is real. Upon clearing the denominator in (2.3) and letting $\|z\|=\mathbb{P}=r\mathbb{I}(0<r<1)$ and letting $r\to 1^-$, we obtain the desired assertion (2.1). □

Corollary 1. *If $f(z)$ of the form (1.2) is in $\mathcal{TJ}^{\eta}_{\mu}(\lambda,\alpha)$, then*

$$a_k \leq \frac{(1-\alpha)}{(k[1+\lambda(k-1)]-\alpha)C_k(\eta,\mu)}, \quad k\geq 2, \quad (2.4)$$

with equality only for functions of the form (2.2).

In the following theorem we state the distortion bounds and extreme point results for functions $f\in\mathcal{TJ}^{\eta}_{\mu}(\lambda,\alpha)$ without proof.

Theorem 2. *If $f\in\mathcal{TJ}^{\eta}_{\mu}(\lambda,\alpha)$, then*

$$r-\frac{(1-\alpha)}{[2(1+\lambda)-\alpha]C_2(\eta,\mu)}r^2 \leq \|f(\mathbb{P})\| \leq r+\frac{(1-\alpha)}{[2(1+\lambda)-\alpha]C_2(\eta,\mu)}r^2 \quad (2.5)$$

$$1-\frac{2(1-\alpha)}{[2(1+\lambda)-\alpha]C_2(\eta,\mu)}r \leq \|f'(\mathbb{P})\| \leq 1+\frac{2(1-\alpha)}{[2(1+\lambda)-\alpha]C_2(\eta,\mu)}r \quad (2.6)$$

The bounds in (2.5) and (2.6) are sharp, since the equalities are attained by the function

$$f(z)=z-\frac{(1-\alpha)}{[2(1+\lambda)-\alpha]C_2(\eta,\mu)}z^2 \quad z=\pm r. \quad (2.7)$$

Theorem 3. (Extreme Points) *Let $f_1(z)=z$ and $f_k(z)=z-\frac{(1-\alpha)}{(k[1+\lambda(k-1)]-\alpha)C_k(\eta,\mu)}z^k$, $k\geq 2$, for $0\leq\alpha<\frac{1}{2}$, and $0\leq\lambda\leq 1$. Then $f(z)$ is in the class $\mathcal{TJ}^{\eta}_{\mu}(\lambda,\alpha)$ if and only if it can be expressed in the form $f(z)=\sum\limits_{k=1}^{\infty}\omega_k f_k(z)$, where $\omega_k\geq 0$ and $\sum\limits_{k=1}^{\infty}\omega_k=1$.*

3 Radius of Starlikeness and Convexity

The radii of close-to-convexity, starlikeness and convexity for the class $\mathcal{TJ}_{\mu}^{\eta}(\lambda,\alpha)$ are given in this section.

Theorem 4. *Let the function $f(z)$ defined by (1.2) belong to the class $\mathcal{TJ}_{\mu}^{\eta}(\lambda,\alpha)$. Then $f(z)$ is close-to-convex of order δ $(0 \le \delta < 1)$ in the disc $|z| < r_1$, where*

$$r_1 := \left[\frac{(1-\delta)(k[1+\lambda(k-1)]-\alpha)\ C_k(\eta,\mu)}{k(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \ge 2). \tag{3.1}$$

The result is sharp, with extremal function $f(z)$ given by (2.2).

Proof. Given $f \in T$ and f is close-to-convex of order δ, we have

$$\|f'(\mathbb{P})-1\| < 1-\delta. \tag{3.2}$$

For the left hand side of (3.2) we have

$$\|f'(\mathbb{P})-1\| \le \sum_{k=2}^{\infty} k a_k \|\mathbb{P}\|^{k-1}.$$

The last expression is less than $1-\delta$ if

$$\sum_{k=2}^{\infty} \frac{k}{1-\delta} a_k \|\mathbb{P}\|^{k-1} < 1.$$

Using the fact, that $f \in \mathcal{TJ}_{\mu}^{\eta}(\lambda,\alpha)$ if and only if

$$\sum_{k=2}^{\infty} \frac{(k[1+\lambda(k-1)]-\alpha)a_k\ C_k(\eta,\mu)}{(1-\alpha)} \le 1.$$

We can say (3.2) is true if

$$\frac{k}{1-\delta}\|\mathbb{P}\|^{k-1} \le \frac{(k[1+\lambda(k-1)]-\alpha)C_k(\eta,\mu)}{(1-\alpha)}.$$

Or, equivalently,

$$\|\mathbb{P}\|^{k-1} = r^{k-1} = \left[\frac{(1-\delta)(k[1+\lambda(k-1)]-\alpha)\ C_k(\eta,\mu)}{k(1-\alpha)}\right]$$

which completes the proof. □

Theorem 5. *Let $f \in \mathcal{TJ}_{\mu}^{\eta}(\lambda,\alpha)$. Then*

1. *f is starlike of order $\delta (0 \leq \delta < 1)$ in the disc $\|z\| < r_2$; that is, Re $\left\{\frac{zf'(z)}{f(z)}\right\} > \delta$, where*

$$r_2 = \inf_{k \geq 2} \left\{ \frac{(1-\delta)(k[1+\lambda(k-1)] - \alpha)\ C_k(\eta,\mu)}{(1-\alpha)(k-\delta)} \right\}^{\frac{1}{k-1}}.$$

2. *f is convex of order δ $(0 \leq \delta < 1)$ in the disc $|z| < r_3$, that is Re $\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \delta$, where*

$$r_3 = \inf_{k \geq 2} \left\{ \frac{(1-\delta)(k[1+\lambda(k-1)] - \alpha)\ C_k(\eta,\mu)}{(1-\alpha)k(k-\delta)} \right\}^{\frac{1}{k-1}}.$$

Each of these results are sharp for the extremal function $f(z)$ given by (2.2).

Proof. Given $f \in \mathcal{T}$ and f is starlike of order δ, we have

$$\left\| \frac{\mathbb{P}f'(\mathbb{P})}{f(\mathbb{P})} - 1 \right\| < 1 - \delta, \quad (\mathbb{P} = r_2 \mathbb{I}\ (0 < r_2 < 1)) \tag{3.3}$$

For the left hand side of (3.3) we have

$$\left\| \frac{\mathbb{P}f'(\mathbb{P})}{f(\mathbb{P})} - 1 \right\| \leq \frac{\sum\limits_{k=2}^{\infty}(k-1)a_k\ \|\mathbb{P}\|^{k-1}}{1 - \sum\limits_{k=2}^{\infty} a_k\ \|\mathbb{P}\|^{k-1}}.$$

The last expression is less than $1 - \delta$, if

$$\sum_{k=2}^{\infty} \frac{k-\delta}{1-\delta} a_k\ \|\mathbb{P}\|^{k-1} < 1.$$

Using the fact, that $f \in \mathcal{T}\mathcal{J}_{\mu}^{\eta}(\lambda, \alpha)$, if and only if

$$\sum_{k=2}^{\infty} \frac{(k[1+\lambda(k-1)] - \alpha)a_k\ C_k(\eta,\mu)}{(1-\alpha)} < 1.$$

We can say (3.3) is true if

$$\frac{k-\delta}{1-\delta}\|\mathbb{P}\|^{k-1} < \frac{(k[1+\lambda(k-1)] - \alpha)\ C_k(\eta,\mu)}{(1-\alpha)}.$$

Or, equivalently,

$$\|\mathbb{P}\|^{k-1} < \frac{(1-\delta)(k[1+\lambda(k-1)] - \alpha)\ C_k(\eta,\mu)}{(1-\alpha)(k-\delta)}$$

which yields the starlikeness of the family.
(2) Using the fact that f is convex if and only if zf' is starlike, we can prove (2), on lines similar the proof of (1). $\square$

4 Integral transform of the class $\mathcal{TJ}_{\mu}^{\eta}(\lambda,\alpha)$

For $f \in \mathcal{TJ}_{\mu}^{\eta}(\lambda,\alpha)$ we define the integral transform

$$V_{\mu}(f)(z) = \int_0^1 \mu(t)\frac{f(tz)}{t}dt,$$

where μ is real valued, non-negative weight function normalized so that $\int_0^1 \mu(t)dt = 1$. Since special cases of $\mu(t)$ are particularly interesting such as $\mu(t) = (1+c)t^c$, $c > -1$, for which V_μ is known as the Bernardi operator, and

$$\mu(t) = \frac{(c+1)^{\delta}}{\mu(\delta)}t^c\left(log\frac{1}{t}\right)^{\delta-1},\ c>-1,\ \delta \geq 0$$

which gives the Komatu operator. For more details see [4].

First we show that the class $\mathcal{TJ}_{\mu}^{\eta}(\lambda,\alpha)$ is closed under $V_\mu(f)$.

Theorem 6. *Let $f \in \mathcal{TJ}_{\mu}^{\eta}(\lambda,\alpha)$. Then $V_\mu(f) \in \mathcal{TJ}_{\mu}^{\eta}(\lambda,\alpha)$.*

Proof. By definition, we have

$$\begin{aligned} V_\mu(f) &= \frac{(c+1)^{\delta}}{\mu(\delta)}\int_0^1(-1)^{\delta-1}t^c(logt)^{\delta-1}\left(z-\sum_{k=2}^{\infty}a_kz^kt^{k-1}\right)dt \\ &= \frac{(-1)^{\delta-1}(c+1)^{\delta}}{\mu(\delta)}\lim_{r\to 0^+}\left[\int_r^1 t^c(logt)^{\delta-1}\left(z-\sum_{k=2}^{\infty}a_kz^kt^{k-1}\right)dt\right], \end{aligned}$$

and a simple calculation gives

$$V_\mu(f)(z) = z - \sum_{k=2}^{\infty}\left(\frac{c+1}{c+k}\right)^{\delta}a_kz^k.$$

We need to prove that

$$\sum_{k=2}^{\infty}\frac{(k[1+\lambda(k-1)]-\alpha)\ C_k(\eta,\mu)}{(1-\alpha)}\left(\frac{c+1}{c+k}\right)^{\delta}a_k < 1. \tag{4.1}$$

On the other hand by Theorem 1, $f \in \mathcal{TJ}_{\mu}^{\eta}(\lambda,\alpha)$ if and only if

$$\sum_{k=2}^{\infty}\frac{(k[1+\lambda(k-1)]-\alpha)a_k\ C_k(\eta,\mu)}{(1-\alpha)} < 1.$$

Hence $\frac{c+1}{c+n} < 1$. Therefore (4.1) holds and the proof is complete. $\square$

Next we provide a starlike condition for functions in $\mathcal{TJ}_{\mu}^{\eta}(\lambda,\alpha)$ and $V_{\mu}(f)$ on lines similar to Theorem 5 .

Theorem 7. *Let* $f \in \mathcal{TJ}_{\mu}^{\eta}(\lambda,\alpha)$. *Then*
(i) $V_{\mu}(f)$ *is starlike of order* $0 \leq \gamma < 1$ *in* $|z| < R_1$ *where*

$$R_1 = \inf_k \left[\left(\frac{c+k}{c+1}\right)^{\delta} \frac{(1-\gamma)(k[1+\lambda(k-1)]-\alpha)\ C_k(\eta,\mu)}{(1-\alpha)(k-\gamma)} \right]^{\frac{1}{k-1}}$$

(ii). $V_{\mu}(f)$ *is convex of order* $0 \leq \gamma < 1$ *in* $|z| < R_2$ *where*

$$R_2 = \inf_k \left[\left(\frac{c+k}{c+1}\right)^{\delta} \frac{(1-\gamma)(k[1+\lambda(k-1)]-\alpha)\ C_k(\eta,\mu)}{(1-\alpha)(k-\gamma)} \right]^{\frac{1}{k-1}}.$$

5 Integral Means Inequalities

In this section, we obtain integral means inequalities for the functions in the family $\mathcal{TJ}_{\mu}^{\eta}(\lambda,\alpha)$.

Lemma 1. *[5] If the functions* f *and* g *are analytic in* U *with* $g \prec f$, *then for* $\kappa > 0$, *and* $0 < r < 1$,

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^{\kappa} d\theta \leq \int_0^{2\pi} \left|f(re^{i\theta})\right|^{\kappa} d\theta. \tag{5.1}$$

In [9], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family $\mathcal{T}$. He applied this function to resolve his integral means inequality, conjectured in [10] and settled in [11], that

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\kappa} d\theta \leq \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^{\kappa} d\theta,$$

for all $f \in \mathcal{T}$, $\kappa > 0$ and $0 < r < 1$. In [11], he also proved his conjecture for the subclasses of starlike functions of order α and convex functions of order α.

Applying Lemma 1, Theorem 1 and Theorem 3, we prove the following result.

Theorem 8. *Suppose* $f(z) \in \mathcal{TJ}_{\mu}^{\eta}(\lambda,\alpha)$ *and* $f_2(z)$ *is defined by* $f_2(z) = z - \frac{(1-\alpha)}{[2(1+\lambda)-\alpha]C_2(b,\mu)} z^2$, *Then for* $z = re^{i\theta}$, $0 < r < 1$, *we have*

$$\int_0^{2\pi} \|f(z)\|^{\kappa} d\theta \leq \int_0^{2\pi} \|f_2(z)\|^{\kappa} d\theta. \tag{5.2}$$

Proof. For $f(z) = z - \sum\limits_{k=2}^{\infty} a_k z^k$, (5.2) is equivalent to proving that

$$\int\limits_0^{2\pi} \left\| 1 - \sum_{k=2}^{\infty} a_k z^{k-1} \right\|^{\kappa} d\theta \leq \int\limits_0^{2\pi} \left\| 1 - \frac{(1-\alpha)}{[2(1+\lambda)-\alpha]C_2(b,\mu)} z \right\|^{\kappa} d\theta.$$

By Lemma 1, it suffices to show that

$$1 - \sum_{k=2}^{\infty} a_k \|\mathbb{P}\|^{k-1} \prec 1 - \frac{(1-\alpha)}{[2(1+\lambda)-\alpha]\|C_2(b,\mu)\|} \|\mathbb{P}\|.$$

Setting

$$1 - \sum_{k=2}^{\infty} a_k \|\mathbb{P}\|^{k-1} = 1 - \frac{(1-\alpha)}{[2(1+\lambda)-\alpha]\|C_2(b,\mu)\|} w(z), \tag{5.3}$$

and using (2.1), we obtain

$$\begin{aligned} \|w(z)\| &= \left\| \sum_{k=2}^{\infty} \frac{(1-\alpha)}{(k[1+k\lambda-\lambda]-\alpha)C_k(b,\mu)} a_k z^{k-1} \right\| \\ &\leq \|\mathbb{P}\| \sum_{k=2}^{\infty} \frac{(1-\alpha)}{(k[1+k\lambda-\lambda]-\alpha)\|C_k(b,\mu)\|} |a_k| \\ &\leq \|\mathbb{P}\|. \end{aligned}$$

This completes the proof Theorem 8. □

6 Modified Hadamard Products

Let the functions $f_j(z)(j = 1, 2)$ be defined by (1.2). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) \ = \ z - \sum_{k=2}^{\infty} a_{k,1} a_{k,2} \ z^k.$$

Using the techniques of Schild and Silverman [12], we prove the following results.

Theorem 9. *For functions $f_j(z)(j = 1, 2)$ defined by (1.2), let $f_1 \in \mathcal{TJ}_{\mu}^{\eta}(\lambda, \alpha)$ and $f_2 \in \mathcal{TJ}_{\mu}^{\eta}(\lambda, \gamma)$. Then $(f_1 * f_2) \in \mathcal{TJ}_{\mu}^{\eta}(\lambda, \xi)$ where*

$$\xi = 1 - \frac{(3+2\lambda)(1-\alpha)(1-\gamma)}{(2+2\lambda-\gamma)(2+2\lambda-\alpha)C_2(\eta,\mu) - (1-\alpha)(1-\gamma)}.$$

Proof. In view of Theorem 1, it suffice to prove that

$$\sum_{k=2}^{\infty}\frac{(k[1+\lambda(k-1)]-\xi)C_k(\eta,\mu)}{(1-\xi)}a_{k,1}a_{k,2}\leq 1,\ \ (0\leq\xi<1)$$

where ξ is defined by (6.1). On the other hand, under the hypothesis, it follows from (2.1) and the Cauchy's-Schwarz inequality that

$$\sum_{k=2}^{\infty}\frac{(k[1+\lambda(k-1)]-\gamma)^{\frac{1}{2}}(k[1+\lambda(k-1)]-\alpha)^{\frac{1}{2}}C_k(\eta,\mu)}{\sqrt{(1-\alpha)(1-\gamma)}}\sqrt{a_{k,1}a_{k,2}}\ \leq 1. \tag{6.1}$$

We need to find the largest ξ such that

$$\begin{aligned}&\sum_{k=2}^{\infty}\frac{(k[1+\lambda(k-1)]-\xi)C_k(\eta,\mu)}{(1-\xi)}a_{k,1}a_{k,2}\\ \leq\ &\sum_{k=2}^{\infty}\frac{(k[1+\lambda(k-1)]-\gamma)^{\frac{1}{2}}(k[1+\lambda(k-1)]-\alpha)^{\frac{1}{2}}C_k(\eta,\mu)}{\sqrt{(1-\alpha)(1-\gamma)}}\sqrt{a_{k,1}a_{k,2}}\end{aligned}$$

or, equivalently that

$$\sqrt{a_{k,1}a_{k,2}}\quad\leq\quad\frac{(k[1+\lambda(k-1)]-\gamma)^{\frac{1}{2}}(k[1+\lambda(k-1)]-\alpha)^{\frac{1}{2}}}{\sqrt{(1-\alpha)(1-\gamma)}}\frac{1-\xi}{(k[1+\lambda(k-1)]-\xi)},\ \ (k\geq 2).$$

By view of (6.1) it is sufficient to find largest ξ such that

$$\begin{aligned}&\frac{\sqrt{(1-\alpha)(1-\gamma)}}{C_k(\eta,\mu)(k[1+\lambda(k-1)]-\gamma)^{\frac{1}{2}}(k[1+\lambda(k-1)]-\alpha)^{\frac{1}{2}}}\\ \leq\quad&\frac{(k[1+\lambda(k-1)]-\gamma)^{\frac{1}{2}}(k[1+\lambda(k-1)]-\alpha)^{\frac{1}{2}}}{\sqrt{(1-\alpha)((1-\gamma))}}\times\frac{1-\xi}{(k[1+\lambda(k-1)]-\xi)}\end{aligned}$$

which yields

$$\xi\leq 1-\frac{(k[1+\lambda(k-1)]+1)(1-\alpha)(1-\gamma)}{(k[1+\lambda(k-1)]-\gamma)(k[1+\lambda(k-1)]-\alpha)C_k(\eta,\mu)-(1-\alpha)(1-\gamma)} \tag{6.2}$$

for $k\geq 2$ it is an increasing function of $k\ \ (k\geq 2)$ for $0\leq\alpha<1; 0<\beta\leq 1; 0\leq\lambda\leq 1$ and letting $k=2$ in (6.2), we have

$$\xi=1-\frac{(3+2\lambda)(1-\alpha)(1-\gamma)}{(2+2\lambda-\gamma)(2+2\lambda-\alpha)C_2(\eta,\mu)-(1-\alpha)(1-\gamma)}.$$

□

Theorem 10. *Let the function $f(z)$ defined by (1.2) be in the class $\mathcal{T}\mathcal{J}_{\mu}^{\eta}(\lambda,\alpha)$. Also let $g(z) = z - \sum_{k=2}^{\infty} |b_k| z^k$ for $|b_k| \leq 1$. Then $(f * g) \in \mathcal{T}\mathcal{J}_{\mu}^{\eta}(\lambda,\alpha)$.*

Proof. Since

$$\begin{aligned} & \sum_{k=2}^{\infty}(k[1+\lambda(k-1)]-\alpha)C_k(\eta,\mu)|a_k b_k| \\ \leq\ & \sum_{k=2}^{\infty}(k[1+\lambda(k-1)]-\alpha)C_k(\eta,\mu)a_k \\ \leq\ & (1-\alpha) \end{aligned}$$

it follows that $(f * g) \in \mathcal{T}\mathcal{J}_{\mu}^{\eta}(\lambda,\alpha)$, by the view of Theorem 1. □

Theorem 11. *Let the functions $f_j(z)(j = 1,2)$ defined by (1.2) be in the class $\in \mathcal{T}\mathcal{J}_{\mu}^{\eta}(\lambda,\alpha)$. Then the function $h(z)$ defined by $h(z) = z - \sum_{k=2}^{\infty}(a_{k,1}^2 + a_{k,2}^2)z^k$ is in the class $\in \mathcal{T}\mathcal{J}_{\mu}^{\eta}(\lambda,\xi)$, where*

$$\xi = 1 - \frac{2(1-\alpha)^2[2(1+\lambda)-1]}{[2(1+\lambda)-\alpha]^2 C_2(\eta,\mu) - 2(1-\alpha)^2}$$

.

Proof. By virtue of Theorem 1, it is sufficient to prove that

$$\sum_{k=2}^{\infty}\frac{(k[1+\lambda(k-1)]-\xi)C_k(\eta,\mu)}{(1-\xi)}(a_{k,1}^2 + a_{k,2}^2) \leq 1 \tag{6.3}$$

where $f_j \in \mathcal{T}\mathcal{J}_{\mu}^{b}(\lambda,\xi,)$ we find from (2.1) and Theorem 1, that

$$\sum_{k=2}^{\infty}\left[\frac{(k[1+\lambda(k-1)]-\alpha)C_k(\eta,\mu)}{(1-\alpha)}\right]^2 a_{k,j}^2 \leq \left[\sum_{k=2}^{\infty}\frac{(k[1+\lambda(k-1)]-\alpha)C_k(\eta,\mu)}{(1-\alpha)}a_{k,j}\right]^2 \tag{6.4}$$

which yields

$$\sum_{k=2}^{\infty}\frac{1}{2}\left[\frac{(k[1+\lambda(k-1)]-\alpha)C_k(\eta,\mu)}{(1-\alpha)}\right]^2 (a_{k,1}^2 + a_{k,2}^2) \leq 1. \tag{6.5}$$

On comparing (6.4) and (6.5), it is easily seen that the inequality (6.3) will be satisfied if

$$\frac{(k[1+\lambda(k-1)]-\xi)C_k(\eta,\mu)}{(1-\xi)} \leq \frac{1}{2}\left[\frac{(k[1+\lambda(k-1)]-\alpha)C_k(\eta,\mu)}{(1-\alpha)}\right]^2, \quad \text{for } k \geq 2.$$

That is an increasing function of k $(k \geq 2)$. Taking $k = 2$ in (6.6), we have,

$$\xi = 1 - \frac{2(1-\alpha)^2[2(1+\lambda)-1]}{[2(1+\lambda)-\alpha]^2 C_2(\eta,\mu) - 2(1-\alpha)^2}, \tag{6.6}$$

which completes the proof. □

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