Common fixed point result for weakly compatible mappings

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Abstract. In this note a common fixed point theorem for weakly compatible mappings in a metric space is proved which, improves the results of Ciric [3], Fisher [4], Pant [8],[9] and Popa et al.[10].

Resumen. En esta nota se prueba un teorema de punto fijo común para las funciones débilmente compatibles en un espacio métrico que mejora los resultados de Ciric [3], Fisher [4], Pant [8],[9] y Popa et al.[10].

1.Introduction

The concept of commuting mappings has proven useful for generalizing fixed point theorems in the context of metric spaces. Sessa [11] defined two self-maps S and T of a metric space (X, d) to be weakly commuting if and only if $d(STx, TSx) \leq d(Sx, Tx)$ for each x in X. Jungck [5] further weakened this property by introducing the concept of compatibility of mappings, which asserts that S and T are said to be compatible if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$. Jungk et al. [7] have defined S and T to be compatible of type (A) if $\lim_{n\to\infty} d(TSx_n, SSx_n) = 0$ and $\lim_{n\to\infty} d(STx_n, TTx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$. Clearly weakly commuting mappings are compatible of type (A). By [7], Ex.2.2, it follows that this implication is not reversible and it also follows by Ex.2.1 and Ex.2.2 that the notion of compatible mappings and compatible mappings of type (A) are independent. Recently, Jungck and Rhoades [6] introduced the concept of weakly compatible mappings as, two selfmaps S and T of a metric space (X, d) are said to be weakly compatible if they commute at their coincidence points; i.e. if $Su = Tu$ for some $u \in X$, then $STu = TSu$. Notice that every commuting pair of maps are weakly commuting, weakly commuting pair of maps are compatible and compatible pair of maps are weakly compatible, but the converse of each need not be true. [see for instance, [1] and [10]].

The main object of the paper is to prove a common fixed point theorem for weakly compatible mappings in a complete metric space which improves several known results.

2. Preliminaries

Recently, Popa et al. [10] proved the following theorem :

Let $\mathcal{F}$ be the set of all functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that
(i) f is isotone, i.e. if $t_1 \leq t_2$, then $f(t_1) \leq f(t_2)$ for all $t_1, t_2 \in \mathbb{R}_+$;
(ii) f is upper semi-continuous;
(iii) $f(t) < t$ for all $t > 0$.

Theorem 2.1 Let A, B, S, and T be self mappings of a complete metric space (X, d) such that

$$(2.1) \qquad A(X) \subset T(X) \quad \textit{and} \quad B(X) \subset S(X),$$

(2.2) the inequality

$$[1+p\, d(Sx,Ty)]d(Ax,By) \leq p\, max\{d(By,Ty)d(Ax,Sx), d(Ax,Ty)d(Sx,By)\}$$

$$+ f\big(\max\{d(Sx,Ty), d(By,Ty), d(Ax,Sx), \frac{1}{2}[d(Ax,Ty)+d(Sx,By)]\}\big)$$

holds for all $x, y \in X$, where $p \geq 0$ and $f \in \mathcal{F}$.
(2.3) one of A, B, S, and T is continuous;
(2.4) A and B weakly commutes with S and T respectively;
then A, B, S and T have a unique common fixed point q in X. Further q is the common fixed point of A and S and of B and T.

Now we can define a sequence $\{y_n\}$ with the help of (2.1), by choosing an arbitrary $x_0 \in X$ as follows :

$$(2.5) \qquad y_{2n} = Tx_{2n+1} = Ax_{2n} \quad \textit{and} \quad y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}$$

for $n = 1, 2, 3, \ldots$ (see, for instance, [12]).

The following lemmas help us to establish our results :

Lemma 2.2[2] Let $f \in \mathcal{F}$ and $\{\xi_n\}$ be a sequence of non-negative real numbers. If $\xi_{n+1} \leq f(\xi_n\}$ for $n \in N$, then the sequence $\{\xi_n\}$ converges to 0.

Lemma 2.3 If we denote $d_n = d(y_n, y_{n+1})$ then $lim_{n\to\infty} d_n = 0$.
Proof. The inequality (2.2) with $x = x_{2n}$ and $y = x_{2n+1}$ implies

$$[1+pd(y_{2n-1}, y_{2n})]d(y_{2n}, y_{2n+1}) \leq p\, d(y_{2n+1}, y_{2n})d(y_{2n}, y_{2n-1})$$

$$+f\big(\max\{d(y_{2n-1}, y_{2n}), d(y_{2n+1}, y_{2n}), d(y_{2n-1}, y_{2n}), \frac{1}{2}[d(y_{2n}, y_{2n})+d(y_{2n-1}, y_{2n+1})]\}\big),$$

which implies

$$d_{2n} \leq f\big(\max\{d_{2n-1}, d_{2n}, \frac{1}{2}d_{2n-1}, \frac{1}{2}d_{2n}\}\big),$$

If $d_{2n} > d_{2n-1}$, then we have

$$d_{2n} \leq f\big(d_{2n})\big) < d_{2n},$$

a contradiction. Thus, we must have $d_{2n} \leq d_{2n-1}$.
Then using this inequality the condition (2.2) yields

$$d_{2n} \leq f(d_{2n-1}). \tag{2.6}$$

for $n = 0, 1, 2 \ldots$ Similarly taking $x = x_{2n+2}$ and $y = x_{2n+1}$ in (2.2), we get

$$[1+pd(y_{2n+1}, y_{2n})]d(y_{2n+2}, y_{2n+1}) \leq p\, d(y_{2n+1}, y_{2n})d(y_{2n+2}, y_{2n+1})$$

$$+ f\big(\max\{d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n})), d(y_{2n+1}, y_{2n+2}),$$

$$\frac{1}{2}[d(y_{2n+2}, y_{2n}) + d(y_{2n+1}, y_{2n+1})]\}\big),$$

which implies

$$d_{2n+1} \leq f\big(\max\{d_{2n}, d_{2n+1}, \frac{1}{2}[d_{2n} + d_{2n+1}]\}\big).$$

If $d_{2n+1} > d_{2n}$, then we have

$$d_{2n+1} \leq f\big(d_{2n+1})\big) < d_{2n+1},$$

a contradiction. Thus, we must have $d_{2n+1} \leq d_{2n}$.
Then using this inequality the condition (2.2) yields

$$d_{2n+1} \leq f(d_{2n}). \tag{2.7}$$

for $n = 1, 2, 3, \ldots$ From equations (2.6) and (2.7), we obtain

$$d_{n+1} \leq f(d_n),$$

for $n = 1, 2, 3, \ldots$. And so, by Lemma 2.2, we get $lim_{n\to\infty} \mathrm{d}_n = 0$.

Lemma 2.4 The sequence $\{y_n\}$ defined by (2.5) is a Cauchy sequence.
Proof. Suppose subsequence $\{y_{2n}\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ such that for each even integer 2k, there exist even integers 2m(k) and 2n(k) with $2m(k) > 2n(k)$ such that

$$d(y_{2m(k)}, y_{2n(k)}) > \epsilon \tag{2.8}$$

For each even integer 2k, let 2m(k) be the least even integer exceeding 2n(k) satisfying (2.8), that is

$$d(y_{2n(k)}, y_{2m(k)-2}) \leq \epsilon \ \ and \ \ d(y_{2n(k)}, y_{2m(k)}) > \epsilon. \tag{2.9}$$

Then for each even integer 2k, we have

$$\begin{aligned} \epsilon <& d(y_{2n(k)}, y_{2m(k)}) \\ \leq& d(y_{2n(k)}, y_{2m(k)-2}) + d(y_{2m(k)-2}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2m(k)}) \\ \leq& \epsilon + d_{2m(k)-2} + d_{2m(k)-1}. \end{aligned}$$

Hence from Lemma 2.3 and (2.9), it follows that

$$lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)}) = \epsilon. \tag{2.10}$$

By the triangular inequality, we have

$$|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \leq d(y_{2m(k)-1}, y_{2m(k)}),$$

and

$$|d(y_{2n(k)+1}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \leq d(y_{2m(k)}, y_{2m(k)-1}) + d(y_{2n(k)}, y_{2n(k)+1}).$$

By Lemma 2.2 and equation (2.9), we obtain

$$lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)-1}) = \epsilon \ and \ lim_{k\to\infty} d(y_{2m(k)+1}, y_{2m(k)-1}) = \epsilon. \tag{2.11}$$

Now using (2.2) with $x = x_{2n(k)}$ and $y = x_{2m(k)-1}$, we have

$$\begin{aligned} [1+p\, d(Sx_{2n(k)}, Tx_{2m(k)-1})] d(Ax_{2n(k)}, Bx_{2m(k)-1}) \leq& \, p\, max\{d(Bx_{2m(k)-1}, Tx_{2m(k)-1}) \\ & d(Ax_{2n(k)}, Sx_{2n(k)}), d(Ax_{2n(k)}, Tx_{2m(k)-1}) d(Sx_{2n(k)}, Bx_{2m(k)-1})\} \\ +f\big(\max\{d(Sx_{2n(k)}, Tx_{2m(k)-1}),& d(Bx_{2m(k)-1}, Tx_{2m(k)-1})), d(Ax_{2n(k)}, Sx_{2n(k)}), \\ & \frac{1}{2}[d(Ax_{2n(k)}, Tx_{2m(k)-1}) + d(Sx_{2n(k)}, Bx_{2m(k)-1})]\}\big), \end{aligned}$$

or

$$\begin{aligned} [1 + p\ d(y_{2n(k)-1}, y_{2m(k)-1})] d(y_{2n(k)}, Bx_{2m(k)-1}) \leq& \, p\ max\{d(y_{2n(k)-1}, Tx_{2n(k)}) \\ & d(y_{2m(k)-2}, y_{2m(k)-1}), d(y_{2n(k)-1}, y_{2m(k)-1}) d(y_{2m(k)-1}, y_{2n(k)})\} \end{aligned}$$

$$+ f\Big(\max\{d(y_{2n(k)-1}, y_{2m(k)-1}), d(y_{2n(k)-1}, y_{2n(k)})),$$

$$d(y_{2n(k)}, y_{2n(k-1)}), \frac{1}{2}[d(y_{2n(k)-1}, y_{2m(k)-1}) + d(y_{2m(k)-2}, y_{2n(k)})]\}\Big).$$

Letting $k \to \infty$ and using Lemma 2.2, equations (2.10) and (2.11), we have

$$[1 + p\ \epsilon]\epsilon \leq p\ \epsilon^2 + f\big(\max\{\epsilon, 0, 0, \epsilon\}\big),$$

or

$$\epsilon \leq f\big(\epsilon\big) < \epsilon,$$

a contradiction. Hence $\{y_{2n}\}$ is a Cauchy sequence.

3. Main Results

Now we present our main results. Throughout this section, suppose (X, d) denotes a complete metric space.

Theorem 3.1 Let A, B, S and T be self maps of a complete metric space (X, d) satisfying (2.1) and (2.2) and suppose one of the mappings A, B, S and T is continuous. If the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point q in X.

Proof. Since X is complete, it follows from Lemma 2.4 that the sequence $\{y_n\}$ converges to a point q in X. On the other hand, the sub sequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converges to the point q. Now suppose that A is continuous. Then the sequences $\{ASx_{2n}\}$ and $\{A^2x_{2n}\}$ converges to Aq. Since the pair $\{A, S\}$ is weakly compatible, it follows from proposition 2.8[9] that $lim_{n\to\infty} SSx_{2n} = Aq$.
Now using (2.2) we have,

$$[1{+}p\, d(SSx_{2n}, Tx_{2n+1})]d(ASx_{2n}, Bx_{2n+1}) \leq p\, max\{d(Bx_{2n+1}, Tx_{2n+1})$$

$$d(ASx_{2n}, SSx_{2n}), d(ASx_{2n}, Tx_{2n+1})d(SSx_{2n}, Bx_{2n+1})\}$$

$$+ f\Big(\max\{d(SSx_{2n}, Tx_{2n+1}), d(Bx_{2n+1}, Tx_{2n+1}),$$

$$d(ASx_{2n}, SSx_{2n}), \frac{1}{2}[d(ASx_{2n}, Tx_{2n+1}) + d(SSx_{2n}, Bx_{2n+1})]\}\Big).$$

Letting $n \to \infty$, we have

$$[1{+}p\, d(Aq, q)]d(Aq, q) \leq p\, max\{d(q, q)d(Aq, Aq), d(Aq, q)d(Aq, q)\}$$

$$+f\Big(\max\{d(Aq, q), d(q, q)), d(Aq, Aq), \frac{1}{2}[d(Aq, q) + d(Aq, q)]\}\Big),$$

or

$$d(Aq,q) \le f\big(d(Aq,q)\big) < d(Aq,q),$$

which implies $Aq = q$.
Since $A(X) \subset T(X)$, there exists a point $u \in X$ such that $Tu = q$. Using the inequality (2.2), we have

$$[1+p\, d(SSx_{2n},Tu)]d(ASx_{2n},Bu) \le p\, max\{d(Bu,Tu)d(ASx_{2n},SSx_{2n}),$$
$$d(ASx_{2n},Tu)d(SSx_{2n},Bu)\} + f\big(\max\{d(SSx_{2n},Tu), d(Bu,Tu),$$
$$d(ASx_{2n},SSx_{2n}), \frac{1}{2}[d(ASx_{2n},Tu) + d(SSx_{2n},Bu)]\}\big).$$

Letting $n \to \infty$, we have

$$[1+p\, d(q,q)]d(q,Bu) \le p\, max\{d(Bu,q)d(q,q), d(q,q)d(q,Bu)\}$$
$$+ f\big(\max\{d(q,q), d(Bu,q)), d(q,q), \frac{1}{2}[d(q,q) + d(q,Bu)]\}\big),$$

or

$$d(Bq,q) \le f\big(d(Bq,q)\big) < d(Bq,q),$$

which implies $Bq = q$.
Since the pair (B,T) is weakly compatible and $Bu = Tu = q$, we have $d(BTu,TBu) = 0$ whenever $Tu = Bu$, and so $Bq = BTu = TBu = Tq$.
From inequality (2.2), we have

$$[1+p\, d(Sx_{2n},Tq)]d(Ax_{2n},Bq) \le p\, max\{d(Bq,Tq)d(Ax_{2n},Sx_{2n}),$$
$$d(Ax_{2n},Tq)d(Sx_{2n},Bq)\} + f\big(\max\{d(Sx_{2n},Tq), d(Bq,Tq)),$$
$$d(Ax_{2n},Sx_{2n}), \frac{1}{2}[d(Ax_{2n},Tq) + d(Sx_{2n},Bq)]\}\big).$$

Letting $n \to \infty$, we have

$$[1+p\, d(q,Tq)]d(q,Tq) \le p\, max\{d(Bq,Tq)d(q,q), d(q,Tq)d(q,Tq)\}$$
$$+f\big(\max\{d(q,Tq), d(Tq,Tq)), d(q,q), \frac{1}{2}[d(q,Tq) + d(q,Tq)]\}\big),$$

which implies,

$$d(q,Tq) \le f\big(d(q,Tq)\big) < d(q,Tq),$$

and so, $Tq = q = Bq$.
Similarly, since $B(X) \subset S(X)$, there exists a point $u' \in X$ such that $Su' = q$, and from (2.2), we have

$$d(Au',q) \le f\big(d(q,Au')\big) < d(q,Au')$$

which implies that, $Su' = Au' = q$.
Since the pair (A, S) is weakly compatible, we have $d(ASu, SAu) = 0$ whenever $Au = Su$. So $q = Aq = ASu' = SAu' = Sq$. Thus q is a common fixed point of A, B, S and T. The same conclusion holds if we suppose the mapping B is continuous.

For uniqueness of common fixed point. Let A and S have another fixed point z. Then from (2.2), we have

$$[1+p\, d(Sz,Tq)]d(Az,Bq) \leq p\, max\{d(Bq,Tq)d(Az,Sz), d(Az,Tq)d(Sz,Bq)\}$$
$$+f\big(\max\{d(Sz,Tq), d(Bq,Tq)), d(Az,Sz), \frac{1}{2}[d(Az,Tq)+d(Sz,Bq)]\}\big),$$

or

$$[1+p\, d(z,q)]d(z,q) \leq p\, d(z,q)d(z,q)\}+f\big(\max\{d(z,q), 0, 0, \frac{1}{2}[d(z,q)+d(z,q)]\}\big),$$

or

$$d(z,q) \leq f\big(d(z,q)\big) < d(z,q)$$

It follows from the above that, $z = q$. Similarly q is the unique common fixed point of B and T.

By setting $p = 0$ in Theorem 3.1, we obtain the following corollary:

Corollary 3.2 Let A, B, S and T be self maps of a complete metric space (X, d) satisfying (2.1) and

$$d(Ax,By) \leq f\big(\max\{d(Sx,Ty), d(By,Ty)), \frac{1}{2}[d(Ax,Ty)+d(Sx,By)]\}\big) \tag{3.1}$$

holds for all $x, y \in X$, where $f \in \mathcal{F}$. Suppose one of the mappings A, B, S and T is continuous. If the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.

Remark 3.3 If we take $A = B$ and $S = T = I_X$ (the identity mapping on X), then the above Corollary improves the result of Pant [8] and [9].

Again by setting $p = 0$ and $f(t) = \lambda t$; $0 < \lambda < 1$ in Theorem 3.1, we obtain the following corollary:

Corollary 3.4 Let A, B, S and T be self maps of a complete metric space (X, d) satisfying (2.1) and

$$d(Ax,By) \leq \lambda \max\{d(Sx,Ty), d(By,Ty)), \frac{1}{2}[d(Ax,Ty) + d(Sx,By)]\} \tag{3.2}$$

holds for all $x, y \in X$. Suppose one of the mappings A, B, S and T is continuous. If the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.

Remark 3.5 If we take $A = B$ and $S = T = I_X$ (the identity mapping on X), the above Corollary improves the result of Ciric [3] and Fisher [4].

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