

ARTÍCULOS

(Dirichlet-Neumann)-Schwarz problem for the nonhomogeneous tri-analytic equation

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Abstract. The goal of this paper is to solve a combined boundary value problem for the nonhomogeneous tri-analytic equation namely the (Dirichlet-Neumann)-Schwarz problem. In order to obtain the solution and solvability conditions we use an iteration's process involving those corresponding to equations of lower order.

Resumen. El objetivo de este artículo es el de resolver un problema de valores de frontera combinado para la ecuación no-homogénea tri-analítica, es decir, el problema de (Dirichlet-Neumann)-Schwarz. Con el fin de obtener la solución y las condiciones de solubilidad, utilizamos un proceso iterativo que involucra las ecuaciones correspondientes de orden inferior.

Keywords: Combined boundary value problems, tri-analytic equations, solvability conditions.

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1 Introduction

Basic boundary value problems in complex analysis are the Schwarz, the Dirichlet and the Neumann boundary value problems. The Schwarz problem is the simplest form of the Riemann-Hilbert problem. The Dirichlet problem is related to the Riemann jump problem and the Neumann problem is connected to the Dirichlet problem. In the last ten years these boundary value problems have been investigated in different domains. Particularly in [2] Heinrich Begehr started a systematic investigation on the unit disc \mathbb{D} of the complex plane for complex model equations using integral representation formulas. He also introduced there the study of combined boundary value problems which are boundary value problems involving different boundary conditions. In those papers he used an iteration's process to get a solution for the problems involving operators of higher order. Using this method integral representations for solutions to higher order partial differential equations can be obtained from the representation integral formulas for those corresponding to first order equations. This method

has been used many times, see [1, 2, 6] and references therein. In [1, 3], the authors studied Dirichlet, Neumann and Schwarz boundary value problems for equations of higher order and some combined problems taking into account only two different boundary conditions for those equations.

Following their idea we solve in this paper a combined boundary value problem for a higher order equation involving a particular combination of three different boundary conditions: Dirichlet, Neumann and Schwarz conditions. For do it we apply the Begehr's method. Because of the laborious calculations to be done, we have restricted our study to the inhomogeneous tri-analytic equation.

The combined problem studied in this paper is (Dirichlet-Neumann)-Schwarz problem and in order to give an explicit integral representation for the solution, the unit disc of the complex plane is taken as the domain. For the inhomogeneous Cauchy-Riemann equation in a regular domain D belonging to the complex plane, the Schwarz problem is well-posed while the Dirichlet-Neumann problem is overdetermined. Therefore we have to look for solvability conditions. This last also happens in the combined (Dirichlet-Neumann)-Schwarz problem in the unit disc and hence we have to determine solvability conditions.

The integral representation of the solution and the solvability conditions (found using the Begehr's method) for our combined problem are original results and represent our main contribution in this research.

The problem solved in this paper may be used as a starting point to study others combined problems, combined problems in other bounded or unbounded domains and also for higher order equations. Combined problems can model physical and engineering problems involving different geometries, see [4, 5].

In order to solve our combined problem we will need the following problems whose proofs can be found in [2].

THEOREM 1.1. *The Schwarz problem for the inhomogeneous analytic equation in the unit disc*

$$\partial_{\bar{z}}\omega = f \text{ in } \mathbb{D}, \quad \operatorname{Re}(\omega) = \gamma \text{ on } \partial\mathbb{D}, \quad \operatorname{Im}(\omega)(0) = c$$

for $f \in L_1(\mathbb{D}, \mathbb{C})$, $\gamma \in C(\partial\mathbb{D}, \mathbb{R})$, $c \in \mathbb{R}$, has a unique solution. The solution is given by the Cauchy-Schwarz-Pompeiu formula

$$\omega(z) = ic + \frac{1}{2\pi i} \int_{|\zeta|=1} \gamma(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} - \frac{1}{2\pi} \int_{|\zeta|<1} \frac{\zeta + z}{\zeta - z} \frac{f(\zeta)}{\zeta} + \frac{1 + z\bar{\zeta}}{1 - z\bar{\zeta}} \frac{\overline{f(\zeta)}}{\bar{\zeta}} d\xi d\eta, \quad (1)$$

for $\zeta = \xi + i\eta$.

THEOREM 1.2. *The Dirichlet-Neumann problem for the inhomogeneous Bitsadze equation in the unit disc*

$$\partial_{\bar{z}}^2 \omega = f \text{ on } \mathbb{D}, \quad \omega = \gamma_0 \text{ in } \partial\mathbb{D}, \quad \partial_\nu \partial_{\bar{z}} \omega = \gamma_1 \text{ on } \partial\mathbb{D}, \quad \partial_{\bar{z}} \omega(0) = c,$$

where ∂_ν is the Neumann operator defined by $\partial_\nu = (z\partial_z + \bar{z}\partial_{\bar{z}})$, is uniquely solvable for $f \in L_1(\mathbb{D}, \mathbb{C}) \cap C(\partial\mathbb{D}, \mathbb{C})$, $\gamma_0, \gamma_1 \in C(\partial\mathbb{D}, \mathbb{C})$, and $c \in \mathbb{C}$ if and only if

$$c - \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\gamma_0(\zeta)}{1 - \bar{z}\zeta} d\zeta + \frac{1}{\pi} \int_{|\zeta|<1} \frac{1 - |\zeta|^2}{1 - \bar{z}\zeta} \frac{f(\zeta)}{\zeta} d\xi d\eta = 0. \quad (2)$$

and

$$\frac{1}{2\pi i} \int_{|\zeta|=1} (\gamma_1(\zeta) - \bar{\zeta}f(\zeta)) \frac{1}{(1 - \bar{z}\zeta)} \frac{d\zeta}{\zeta} + \frac{1}{\pi} \int_{|\zeta|<1} \frac{\bar{z}f(\zeta)}{(1 - \bar{z}\zeta)^2} d\xi d\eta = 0. \quad (3)$$

The solution then is given by

$$\begin{aligned} \omega(z) = c\bar{z} + \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\gamma_0(\zeta)}{\zeta - z} d\zeta + \\ \frac{1}{2\pi i} \int_{|\zeta|=1} [\gamma_1(\zeta) - \bar{\zeta}f(\zeta)] \log(1 - z\bar{\zeta}) \frac{1 - |z|^2}{z} \frac{d\zeta}{\zeta} \\ + \frac{1}{\pi} \int_{|\zeta|<1} \frac{|\zeta|^2 - |z|^2}{(\zeta - z)} \frac{f(\zeta)}{\zeta} d\xi d\eta. \quad (4) \end{aligned}$$

The Schwarz problem for the inhomogeneous polyanalytic equation is proved in [2]. For the same equation the Dirichlet-Neumann problem is proved in [6]. In [1] using a non-iterative process, the Schwarz-Dirichlet-half-Neumann- $(n-2)$ problem and the Dirichlet-half-Neumann- $(n-2)$ -Schwarz problem for the inhomogeneous polyanalytic equation have been studied.

The mathematical tools which we use in this paper are classical results of the complex analysis as Gauss Theorem, Cauchy Theorem and Cauchy-Pompeiu Formula [7]. To make it easier to read this paper, we have been included the most of details of computations.

2 Some region and boundary integrals

LEMMA 2.1. *For $|z| < 1$ and $|\tilde{\zeta}| < 1$*

$$i. \quad \frac{1}{\pi} \int_{|\zeta|<1} \frac{1 - |\zeta|^2}{\zeta(1 - \bar{z}\zeta)} d\xi d\eta = \frac{\bar{z}}{2}.$$

- ii. $\frac{1}{\pi} \int_{|\zeta|<1} \frac{(1-|\zeta|^2)(\tilde{\zeta}+\zeta)}{\zeta(1-\bar{z}\zeta)(\tilde{\zeta}-\zeta)} d\xi d\eta = 2\bar{\zeta} + \frac{\bar{z}}{2} + 2\bar{z} \frac{\bar{z}-\bar{\zeta}}{1-\bar{z}\bar{\zeta}} + \frac{\bar{z}-\bar{\zeta}|\tilde{\zeta}|^2}{1-\bar{z}\bar{\zeta}}.$
- iii. $\frac{1}{\pi} \int_{|\zeta|<1} \frac{(1-|\zeta|^2)(1+\zeta\bar{\zeta})}{\zeta(1-\bar{z}\zeta)(1-\zeta\bar{\zeta})} d\xi d\eta = \bar{\zeta} + \frac{\bar{z}}{2}.$
- iv. $\frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\bar{\zeta}}{\zeta(1-\bar{z}\zeta)} d\zeta = \bar{z}.$
- v. $\frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\bar{\zeta}(\tilde{\zeta}+\zeta)}{\zeta(\tilde{\zeta}-\zeta)(1-\bar{z}\zeta)} d\zeta = -\frac{\bar{z}(1+\bar{z}\tilde{\zeta})}{1-\bar{z}\tilde{\zeta}}.$
- vi. $\frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\bar{\zeta}(1+\zeta\bar{\zeta})}{\zeta(1-\bar{z}\zeta)(1-\zeta\bar{\zeta})} d\zeta = 2\bar{\zeta} + \bar{z}.$
- vii. $\frac{1}{\pi} \int_{|\zeta|<1} \frac{\bar{z}}{(1-\bar{z}\zeta)^2} d\xi d\eta = \bar{z}.$
- viii. $\frac{1}{\pi} \int_{|\zeta|<1} \frac{\bar{z}(\tilde{\zeta}+\zeta)}{(\tilde{\zeta}-\zeta)(1-\bar{z}\zeta)^2} d\xi d\eta = \bar{z} + \frac{2\bar{z}(|\tilde{\zeta}|^2-1)}{(1-\bar{z}\tilde{\zeta})^2}.$
- ix. $\frac{1}{\pi} \int_{|\zeta|<1} \frac{\bar{z}(1+\zeta\bar{\zeta})}{(1-\zeta\bar{z})^2(1-\zeta\bar{\zeta})} d\xi d\eta = \bar{z}.$

Proof. i. First we rewrite the given integral as

$$\begin{aligned} \frac{1}{\pi} \int_{|\zeta|<1} \frac{(1-|\zeta|^2)}{\zeta(1-\bar{z}\zeta)} d\xi d\eta &= \frac{1}{\pi} \int_{|\zeta|<1} \frac{1}{\zeta} d\xi d\eta + \frac{1}{\pi} \int_{|\zeta|<1} \frac{\bar{z}}{(1-\bar{z}\zeta)} d\xi d\eta \\ &\quad - \frac{1}{\pi} \int_{|\zeta|<1} \frac{\bar{\zeta}}{(1-\bar{z}\zeta)} d\xi d\eta. \quad (5) \end{aligned}$$

From the Cauchy-Pompeiu formula we have

$$\frac{1}{\pi} \int_{|\zeta|<1} \frac{1}{\zeta-z} d\xi d\eta = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\bar{\zeta}}{\zeta-z} d\zeta - \bar{z}.$$

Denoting the first integral on the right side as $h(z)$ we observe that it is an holomorphic function respect to z and

$$h^{(k)}(z) = \frac{k!}{2\pi i} \int_{|\zeta|=1} \frac{\bar{\zeta}}{(\zeta-z)^{k+1}} d\zeta.$$

Using the change $\zeta = e^{i\theta}$, $0 \leq \theta \leq 2\pi$, we can prove that $h^{(k)}(0) = 0$, $k = 0, 1, \dots$ which means $h(z) = \sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!} z^k = 0$. So we have

$$\frac{1}{\pi} \int_{|\zeta|<1} \frac{1}{\zeta - z} d\xi d\eta = -\bar{z} \text{ and if we make } z = 0 \text{ we get}$$

$$\frac{1}{\pi} \int_{|\zeta|<1} \frac{1}{\zeta} d\xi d\eta = 0. \quad (6)$$

From theorem's Gauss

$$\begin{aligned} \frac{1}{\pi} \int_{|\zeta|<1} \frac{\bar{z}}{1 - \bar{z}\zeta} d\xi d\eta &= \bar{z} \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\bar{\zeta}}{1 - \bar{z}\zeta} d\zeta = -\bar{z} \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\zeta}{1 - \bar{z}\zeta} d\bar{\zeta} \\ &= -\bar{z} \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{1}{\bar{\zeta} - \bar{z}} d\bar{\zeta} = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{1}{\zeta - z} d\zeta = \bar{z} \quad (7) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{\pi} \int_{|\zeta|<1} \frac{\bar{\zeta}}{1 - \bar{z}\zeta} d\xi d\eta &= \frac{1}{4\pi i} \int_{|\zeta|=1} \frac{\bar{\zeta}^2}{1 - \bar{z}\zeta} d\zeta = \\ \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{1}{2(1 - \bar{z}\zeta)\zeta^2} d\zeta &= \left(\frac{1}{2(1 - \bar{z}\zeta)} \right)' \Big|_{\zeta=0} = \frac{\bar{z}}{2}. \quad (8) \end{aligned}$$

Finally from (5)-(8) we obtain the desired result.

ii. We split the given integral in four integrals:

$$\begin{aligned} \frac{1}{\pi} \int_{|\zeta|<1} \frac{(1 - |\zeta|^2)(\tilde{\zeta} + \zeta)}{\zeta(1 - \bar{z}\zeta)(\tilde{\zeta} - \zeta)} d\xi d\eta &= \frac{1}{\pi} \int_{|\zeta|<1} \frac{1}{\zeta} d\xi d\eta + \frac{2}{\pi} \int_{|\zeta|<1} \frac{1}{\tilde{\zeta} - \zeta} d\xi d\eta \\ &+ \frac{1}{\pi} \int_{|\zeta|<1} \frac{\bar{z}(\tilde{\zeta} + \zeta)}{(1 - \bar{z}\zeta)(\tilde{\zeta} - \zeta)} d\xi d\eta - \frac{1}{\pi} \int_{|\zeta|<1} \frac{\bar{\zeta}(\tilde{\zeta} + \zeta)}{(1 - \bar{z}\zeta)(\tilde{\zeta} - \zeta)} d\xi d\eta. \end{aligned}$$

From proof of (i) we know that $\frac{1}{\pi} \int_{|\zeta|<1} \frac{1}{\zeta} d\xi d\eta = 0$.

Using Cauchy-Pompeiu Formula and Cauchy's theorem we get the value of the other integrals

$$\frac{1}{\pi} \int_{|\zeta|<1} \frac{1}{\zeta - \tilde{\zeta}} d\xi d\eta = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\bar{\zeta}}{\zeta - \tilde{\zeta}} d\zeta - \bar{\zeta} = -\frac{1}{2\pi i} \int_{|\zeta|=1} \frac{1}{1 - \tilde{\zeta}\bar{\zeta}} d\bar{\zeta} - \bar{\zeta} = -\bar{\zeta}$$

and

$$\begin{aligned}
\frac{1}{\pi} \int_{|\zeta| < 1} \frac{\tilde{\zeta} + \zeta}{(1 - \bar{z}\zeta)(\tilde{\zeta} - \zeta)} d\xi d\eta &= \frac{1}{\pi} \int_{|\zeta| < 1} \frac{\tilde{\zeta} - \zeta + 2\zeta}{(1 - \bar{z}\zeta)(\tilde{\zeta} - \zeta)} d\xi d\eta = \\
&\frac{1}{\pi} \int_{|\zeta| < 1} \frac{1}{1 - \bar{z}\zeta} d\xi d\eta + \frac{2}{\pi} \int_{|\zeta| < 1} \frac{\zeta}{(1 - \bar{z}\zeta)(\zeta - \tilde{\zeta})} d\xi d\eta = \\
&\frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\bar{\zeta}}{1 - \bar{z}\zeta} d\zeta + \frac{2}{2\pi i} \int_{|\zeta|=1} \frac{\bar{\zeta}}{(1 - \bar{z}\zeta)(\zeta - \tilde{\zeta})} d\zeta - \frac{2\bar{\zeta}}{1 - \bar{z}\tilde{\zeta}} = \\
&\frac{1}{2\pi i} \int_{|\zeta|=1} \frac{1}{(1 - \bar{z}\zeta)\zeta} d\zeta - \frac{2}{2\pi i} \int_{|\zeta|=1} \frac{\bar{\zeta}}{(\bar{\zeta} - \bar{z})(1 - \tilde{\zeta}\bar{\zeta})} d\bar{\zeta} - \frac{2\bar{\zeta}}{1 - \bar{z}\tilde{\zeta}} = \\
&1 + 2 \left(\frac{\zeta}{1 - \tilde{\zeta}\zeta} \right)_{\zeta=z} - \frac{2\bar{\zeta}}{1 - \bar{z}\tilde{\zeta}} = 1 + 2 \frac{\bar{z} - \tilde{\zeta}}{1 - \bar{z}\tilde{\zeta}}
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{\pi} \int_{|\zeta| < 1} \frac{\bar{\zeta}(\tilde{\zeta} + \zeta)}{(1 - \bar{z}\zeta)(\tilde{\zeta} - \zeta)} d\xi d\eta &= \frac{1}{\pi} \int_{|\zeta| < 1} \frac{\bar{\zeta}(\tilde{\zeta} - \zeta) + 2\bar{\zeta}\zeta}{(1 - \bar{z}\zeta)(\tilde{\zeta} - \zeta)} d\xi d\eta = \\
&\frac{1}{\pi} \int_{|\zeta| < 1} \frac{\bar{\zeta}}{1 - \bar{z}\zeta} d\xi d\eta + \frac{1}{\pi} \int_{|\zeta| < 1} \frac{2\bar{\zeta}\zeta}{(1 - \bar{z}\zeta)(\tilde{\zeta} - \zeta)} d\xi d\eta = \\
&\frac{\bar{z}}{2} + \frac{\bar{\zeta}^2 \tilde{\zeta}}{1 - \bar{z}\tilde{\zeta}} - \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\bar{\zeta}^2 \zeta}{(1 - \bar{z}\zeta)(\zeta - \tilde{\zeta})} d\zeta = \frac{\bar{z}}{2} + \frac{\bar{\zeta}|\tilde{\zeta}|^2 - \bar{z}}{1 - \bar{z}\tilde{\zeta}}.
\end{aligned}$$

Finally we have

$$\begin{aligned}
\frac{1}{\pi} \int_{|\zeta| < 1} \frac{(1 - |\zeta|^2)(\tilde{\zeta} + \zeta)}{\zeta(1 - \bar{z}\zeta)(\tilde{\zeta} - \zeta)} d\xi d\eta &= \\
2\bar{\zeta} + \bar{z} + 2\bar{z} \frac{\bar{z} - \tilde{\zeta}}{1 - \bar{z}\tilde{\zeta}} - \frac{\bar{z}}{2} - \frac{\bar{\zeta}|\tilde{\zeta}|^2 - \bar{z}}{1 - \bar{z}\tilde{\zeta}} &= 2\bar{\zeta} + \frac{\bar{z}}{2} + 2\bar{z} \frac{\bar{z} - \tilde{\zeta}}{1 - \bar{z}\tilde{\zeta}} + \frac{\bar{z} - \bar{\zeta}|\tilde{\zeta}|^2}{1 - \bar{z}\tilde{\zeta}}.
\end{aligned}$$

iii. Again we separate the given integral in four other

$$\begin{aligned} & \frac{1}{\pi} \int_{|\zeta|<1} \frac{(1-|\zeta|^2)(1+\zeta\bar{\zeta})}{\zeta(1-\bar{z}\zeta)(1-\zeta\bar{\zeta})} d\xi d\eta = \\ & \frac{1}{\pi} \int_{|\zeta|<1} \frac{1+\zeta\bar{\zeta}}{\zeta(1-\bar{z}\zeta)(1-\zeta\bar{\zeta})} d\xi d\eta - \frac{1}{\pi} \int_{|\zeta|<1} \frac{\bar{\zeta}(1+\zeta\bar{\zeta})}{(1-\bar{z}\zeta)(1-\zeta\bar{\zeta})} d\xi d\eta = \\ & \frac{1}{\pi} \int_{|\zeta|<1} \frac{1}{\zeta} d\xi d\eta + \frac{1}{\pi} \int_{|\zeta|<1} \frac{2\bar{\zeta}}{1-\zeta\bar{\zeta}} d\xi d\eta + \bar{z} \frac{1}{\pi} \int_{|\zeta|<1} \frac{(1+\zeta\bar{\zeta})}{(1-\bar{z}\zeta)(1-\zeta\bar{\zeta})} d\xi d\eta \\ & \quad - \frac{1}{\pi} \int_{|\zeta|<1} \frac{\bar{\zeta}(1+\zeta\bar{\zeta})}{(1-\bar{z}\zeta)(1-\zeta\bar{\zeta})} d\xi d\eta. \end{aligned}$$

$$\text{Since } \frac{1}{\pi} \int_{|\zeta|<1} \frac{1}{\zeta} d\xi d\eta = 0, \quad 2\bar{\zeta} \frac{1}{\pi} \int_{|\zeta|<1} \frac{1}{1-\zeta\bar{\zeta}} d\xi d\eta = 2\bar{\zeta},$$

$$\begin{aligned} \frac{1}{\pi} \int_{|\zeta|<1} \frac{1+\zeta\bar{\zeta}}{(1-\bar{z}\zeta)(1-\zeta\bar{\zeta})} d\xi d\eta &= \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\bar{\zeta}(1+\zeta\bar{\zeta})}{(1-\bar{z}\zeta)(1-\zeta\bar{\zeta})} d\zeta = \\ & \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{1+\zeta\bar{\zeta}}{(1-\bar{z}\zeta)(1-\zeta\bar{\zeta})} \frac{d\zeta}{\zeta} = \left. \frac{1+\zeta\bar{\zeta}}{(1-\bar{z}\zeta)(1-\zeta\bar{\zeta})} \right|_{\zeta=0} = 1 \end{aligned}$$

and

$$\begin{aligned} \frac{1}{\pi} \int_{|\zeta|<1} \frac{\bar{\zeta}(1+\zeta\bar{\zeta})}{(1-\bar{z}\zeta)(1-\zeta\bar{\zeta})} d\xi d\eta &= \frac{1}{2} \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\bar{\zeta}^2(1+\zeta\bar{\zeta})}{(1-\bar{z}\zeta)(1-\zeta\bar{\zeta})} d\zeta = \\ \frac{1}{2} \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{1+\zeta\bar{\zeta}}{(1-\bar{z}\zeta)(1-\zeta\bar{\zeta})\bar{\zeta}^2} d\zeta &= \frac{1}{2} \left(\frac{1+\zeta\bar{\zeta}}{(1-\bar{z}\zeta)(1-\zeta\bar{\zeta})} \right)'_{\zeta=0} = \bar{\zeta} + \frac{\bar{z}}{2}, \end{aligned}$$

where we used the Gauss'theorem, we get

$$\frac{1}{\pi} \int_{|\zeta|<1} \frac{(1-|\zeta|^2)(1+\zeta\bar{\zeta})}{\zeta(1-\bar{z}\zeta)(1-\zeta\bar{\zeta})} d\xi d\eta = 2\bar{\zeta} + \bar{z} - \bar{\zeta} - \frac{\bar{z}}{2} = \bar{\zeta} + \frac{\bar{z}}{2}.$$

iv. Follows from (8).

v. Follows from applying the Cauchy integral formula

$$\begin{aligned} & \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\bar{\zeta}(\tilde{\zeta} + \zeta)}{\zeta(\tilde{\zeta} - \zeta)(1 - \bar{z}\zeta)} d\zeta = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\bar{\zeta}(1 + \bar{\zeta}\tilde{\zeta})}{(\bar{\zeta} - \bar{z})(1 - \bar{\zeta}\tilde{\zeta})} d\bar{\zeta} \\ & = -\frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\zeta(1 + \zeta\tilde{\zeta})}{(\zeta - z)(1 - \zeta\tilde{\zeta})} d\zeta = -\left(\frac{\zeta(1 + \zeta\tilde{\zeta})}{1 - \zeta\tilde{\zeta}} \right) \Big|_{\zeta=z} = -\frac{\bar{z}(1 + \bar{z}\tilde{\zeta})}{1 - \bar{z}\tilde{\zeta}}. \end{aligned}$$

vi. From Cauchy integral formula we obtain

$$\frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\bar{\zeta}(1 + \zeta\tilde{\zeta})}{\zeta(1 - \bar{z}\zeta)(1 - \zeta\tilde{\zeta})} d\zeta = \left(\frac{1 + \zeta\tilde{\zeta}}{(1 - \zeta\tilde{\zeta})(1 - \zeta\tilde{\zeta})} \right)'_{\zeta=0} = 2\bar{\zeta} + \bar{z}.$$

vii. From Gauss' theorem and Cauchy integral formula we have

$$\frac{1}{\pi} \int_{|\zeta|<1} \frac{\bar{z}}{(1 - \bar{z}\zeta)^2} d\xi d\eta = \bar{z} \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{1}{(1 - \bar{z}\zeta)^2} \frac{d\zeta}{\zeta} = \bar{z} \left(\frac{1}{(1 - \bar{z}\zeta)^2} \right) \Big|_{\zeta=0} = \bar{z}.$$

viii. First we split the integral into two

$$\begin{aligned} & \frac{1}{\pi} \int_{|\zeta|<1} \frac{\bar{z}}{(1 - \bar{z}\zeta)^2} \frac{\tilde{\zeta} + \zeta}{\tilde{\zeta} - \zeta} d\xi d\eta = \\ & \quad \frac{\bar{z}}{\pi} \int_{|\zeta|<1} \frac{1}{(1 - \bar{z}\zeta)^2} d\xi d\eta + 2\bar{z} \frac{1}{\pi} \int_{|\zeta|<1} \frac{\zeta}{(1 - \bar{z}\zeta)^2(\tilde{\zeta} - \zeta)} d\xi d\eta. \end{aligned}$$

From Gauss' theorem we observe

$$\begin{aligned} & \frac{1}{\pi} \int_{|\zeta|<1} \frac{1}{(1 - \bar{z}\zeta)^2} d\xi d\eta = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\bar{\zeta}}{(1 - \bar{z}\zeta)^2} d\zeta = \\ & -\frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\bar{\zeta}}{(\bar{\zeta} - \bar{z})^2} d\bar{\zeta} = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\zeta}{(\zeta - z)^2} d\zeta = \overline{(\zeta)'_{\zeta=z}} = 1 \end{aligned}$$

and from Cauchy-Pompeiu formula and Gauss' theorem we obtain

$$\begin{aligned} & -\frac{1}{\pi} \int_{|\zeta|<1} \frac{\zeta}{(1 - \bar{z}\zeta)^2(\zeta - \tilde{\zeta})} d\xi d\eta = \\ & \quad \frac{\bar{\zeta}\tilde{\zeta}}{(1 - \bar{z}\tilde{\zeta})^2} - \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\bar{\zeta}\zeta}{(1 - \bar{z}\zeta)^2(\zeta - \tilde{\zeta})} d\zeta = \\ & \quad \left. \frac{|\tilde{\zeta}|^2}{(1 - \bar{z}\tilde{\zeta})^2} - \frac{1}{(1 - \bar{z}\zeta)^2} \right|_{\zeta=\tilde{\zeta}} = \frac{|\tilde{\zeta}|^2 - 1}{(1 - \bar{z}\tilde{\zeta})^2}. \end{aligned}$$

Therefore we have

$$\frac{1}{\pi} \int_{|\zeta| < 1} \frac{\bar{z}(\tilde{\zeta} + \zeta)}{(\tilde{\zeta} - \zeta)(1 - \bar{z}\zeta)^2} d\xi d\eta = \bar{z} + \frac{2\bar{z}(|\tilde{\zeta}|^2 - 1)}{(1 - \bar{z}\tilde{\zeta})^2}.$$

ix. From Gauss'theorem follows

$$\begin{aligned} \frac{1}{\pi} \int_{|\zeta| < 1} \frac{\bar{z}(1 + \zeta\bar{\zeta})}{(1 - \zeta\bar{z})^2(1 - \zeta\bar{\zeta})} d\xi d\eta &= \bar{z} \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\bar{\zeta}(1 + \zeta\bar{\zeta})}{(1 - \zeta\bar{z})^2(1 - \zeta\bar{\zeta})} d\zeta = \\ \bar{z} \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{(1 + \zeta\bar{\zeta})}{\zeta(1 - \zeta\bar{z})^2(1 - \zeta\bar{\zeta})} d\zeta &= \bar{z} \left(\frac{(1 + \zeta\bar{\zeta})}{(1 - \zeta\bar{z})^2(1 - \zeta\bar{\zeta})} \right) \Big|_{\zeta=0} = \bar{z}. \end{aligned}$$

□

LEMMA 2.2. For $|z| < 1$ and $|\tilde{\zeta}| < 1$

$$i. \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{(1 - |z|^2)\bar{\zeta}}{z\zeta} \log(1 - z\bar{\zeta}) d\zeta = 0.$$

$$ii. \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{(1 - |z|^2)\bar{\zeta}}{z\zeta} \log(1 - z\bar{\zeta}) \left(\frac{\tilde{\zeta} + \zeta}{\tilde{\zeta} - \zeta} \right) d\zeta = 0.$$

$$iii. \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{(1 - |z|^2)\bar{\zeta}}{z\zeta} \log(1 - z\bar{\zeta}) \left(\frac{1 + \zeta\bar{\zeta}}{1 - \zeta\bar{\zeta}} \right) d\zeta = 2\bar{\zeta} \frac{1 - |z|^2}{z} \log(1 - z\bar{\zeta}).$$

$$iv. \frac{1}{\pi} \int_{|\zeta| < 1} \frac{|\zeta|^2 - |z|^2}{\zeta(\zeta - z)} d\xi d\eta = \frac{\bar{z}^2}{2}.$$

$$v. \frac{1}{\pi} \int_{|\zeta| < 1} \frac{(|\zeta|^2 - |z|^2)\tilde{\zeta} + \zeta}{\zeta(\zeta - z)} \frac{\tilde{\zeta} + \zeta}{\tilde{\zeta} - \zeta} d\xi d\eta = 2\frac{\bar{z}|z|^2 - \tilde{\zeta}|\tilde{\zeta}|^2}{\tilde{\zeta}(z - \tilde{\zeta})} - 2|z|^2 \frac{\bar{z} - \tilde{\zeta}}{z - \tilde{\zeta}}.$$

$$vi. \frac{1}{\pi} \int_{|\zeta| < 1} \frac{(|\zeta|^2 - |z|^2)\tilde{\zeta}}{\zeta(\zeta - z)} \frac{1 + \zeta\bar{\zeta}}{1 - \zeta\bar{\zeta}} d\xi d\eta = \frac{1 + \bar{z}}{2} \frac{1 + z\bar{\zeta}}{1 - z\bar{\zeta}} - \frac{\tilde{\zeta}(z - \bar{z})}{2(1 - z\bar{\zeta})}.$$

Proof. i. Applying the Cauchy'theorem

$$\begin{aligned} \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\bar{\zeta}(1 - |z|^2)\log(1 - z\bar{\zeta})}{z\zeta} d\zeta &= \\ \frac{(1 - |z|^2)}{z} \frac{1}{2\pi i} \int_{|\zeta|=1} \log(1 - \bar{z}\zeta) d\zeta &= 0, \end{aligned}$$

ii. As before we separate the integral into two

$$\begin{aligned} & \frac{1}{2\pi i} \int_{|\zeta|=1} \bar{\zeta} \frac{1-|z|^2}{z} \log(1-z\bar{\zeta}) \frac{\tilde{\zeta}+\zeta}{\tilde{\zeta}-\zeta} \frac{d\zeta}{\zeta} = \\ & \frac{1-|z|^2}{z} \left[\frac{1}{2\pi i} \int_{|\zeta|=1} \bar{\zeta} \log(1-z\bar{\zeta}) \frac{\tilde{\zeta}-\zeta}{\tilde{\zeta}-\zeta} \frac{d\zeta}{\zeta} + \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{2\zeta\bar{\zeta} \log(1-z\bar{\zeta})}{\zeta(\tilde{\zeta}-\zeta)} d\zeta \right] = \\ & \frac{1-|z|^2}{z} \left[\frac{1}{2\pi i} \int_{|\zeta|=1} \log(1-\bar{z}\zeta) d\zeta + \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\log(1-\bar{z}\zeta)}{\bar{\zeta}\zeta-1} d\zeta \right] = 0. \end{aligned}$$

iii. Follows from Cauchy integral formula

$$\begin{aligned} & \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\bar{\zeta}(1-|z|^2) \log(1-z\bar{\zeta})}{z} \left(\frac{1+\bar{\zeta}\zeta}{1-\bar{\zeta}\zeta} \right) \frac{d\zeta}{\zeta} = \\ & - \frac{1-|z|^2}{z} \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\bar{\zeta} \log(1-z\bar{\zeta}) \zeta(\bar{\zeta}+\bar{\zeta})}{\zeta(\bar{\zeta}-\bar{\zeta})} \frac{d\bar{\zeta}}{\bar{\zeta}} = \\ & \frac{1-|z|^2}{z} \left(- \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\log(1-z\bar{\zeta})(\bar{\zeta}-\zeta+2\zeta)}{\bar{\zeta}-\zeta} d\zeta \right) = \\ & = \frac{1-|z|^2}{z} (\log(1-\bar{z}\zeta)) 2\zeta \Big|_{\zeta=\bar{\zeta}} = \frac{1-|z|^2}{z} \log(1-z\bar{\zeta}) 2\bar{\zeta}. \end{aligned}$$

iv. Separating the integral into two we have

$$\begin{aligned} & \frac{1}{\pi} \int_{|\zeta|<1} \frac{|\zeta|^2-|z|^2}{\zeta(\zeta-z)} d\xi d\eta = \\ & \frac{1}{\pi} \int_{|\zeta|<1} \frac{\bar{\zeta}}{(\zeta-z)} d\xi d\eta - \frac{1}{\pi} \int_{|\zeta|<1} \frac{|z|^2}{\zeta(\zeta-z)} d\xi d\eta. \quad (9) \end{aligned}$$

For the first integral on the right side, Cauchy-Pompeiu formula yields

$$\frac{1}{2\pi} \int_{|\zeta|<1} \frac{2\bar{\zeta}}{\zeta-z} d\xi d\eta = \frac{1}{2} \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\bar{\zeta}^2}{\zeta-z} d\zeta - \frac{\bar{z}^2}{2}.$$

Denoting the first integral on the right side as $h(z)$ we observe that it is an holomorphic function respect to z and

$$h^{(k)}(z) = \frac{k!}{2\pi i} \int_{|\zeta|=1} \frac{\bar{\zeta}^2}{(\zeta-z)^{k+1}} d\zeta.$$

Using the change $\zeta = e^{i\theta}$, $0 \leq \theta \leq 2\pi$, we can prove that $h^{(k)}(0) = 0$, $k = 0, 1, \dots$ which means $h(z) = \sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!} z^k = 0$. So we have

$$\frac{1}{\pi} \int_{|\zeta| < 1} \frac{\bar{\zeta}}{\zeta - z} d\xi d\eta = -\frac{\bar{z}^2}{2}.$$

Now the second integral on the right side of (9) becomes

$$\frac{1}{\pi} \int_{|\zeta| < 1} \frac{|z|^2}{\zeta(\zeta - z)} d\xi d\eta = \bar{z} \frac{1}{\pi} \int_{|\zeta| < 1} \frac{1}{\zeta} - \frac{1}{\zeta - z} d\xi d\eta = \bar{z}(-\bar{z}) = -\bar{z}^2.$$

Here we have used the proof of (i) in Lemma (2.1). Consequently, the integral on the left side in (9) is equal to $\frac{\bar{z}^2}{2}$.

v. Observing that

$$\frac{\tilde{\zeta} + \zeta}{\zeta(\tilde{\zeta} - \zeta)} = \frac{1}{\zeta} + \frac{2}{\tilde{\zeta} - \zeta}$$

and applying Cauchy-Pompeiu formula we obtain the identity.

vi. Follows similar to (v). □

3 A combined boundary value problem

Using an iterative method (see [2]) we solve our combined boundary problem.

THEOREM 3.1. *The (Dirichlet-Neumann)-Schwarz problem for the inhomogeneous tri-analytic equation in the unit disc*

$$\begin{aligned} \partial_{\bar{z}}^3 \omega &= f \text{ in } \mathbb{D}, \quad \omega = \gamma_0 \text{ on } \partial\mathbb{D}, \quad \partial_{\nu} \partial_{\bar{z}} \omega = \gamma_1 \text{ on } \partial\mathbb{D}, \quad \partial_{\bar{z}} \omega(0) = c, \\ \operatorname{Re}(\partial_{\bar{z}} \partial_{\bar{z}} \omega) &= \gamma_2 \text{ on } \partial\mathbb{D}, \quad \operatorname{Im}(\partial_{\bar{z}} \partial_{\bar{z}} \omega(0)) = c_1 \end{aligned}$$

for $f \in L_1(\mathbb{D}, \mathbb{C}) \cap C(\partial\mathbb{D}, \mathbb{C})$, $\gamma_0, \gamma_1 \in C(\partial\mathbb{D}, \mathbb{C})$, $\gamma_2 \in C(\partial\mathbb{D}, \mathbb{R})$, $c \in \mathbb{C}$, $c_1 \in \mathbb{R}$, is uniquely solvable if and only if for $z \in \mathbb{D}$,

$$\begin{aligned} c + ic \frac{\bar{z}}{2} + \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\gamma_0(\zeta)}{1 - \bar{z}\zeta} d\zeta + \frac{1}{2\pi i} \int_{|\zeta|=1} \gamma_2(\zeta) [2\zeta + \frac{\bar{z}}{2} - (2\bar{z} + 1)\bar{\zeta}] \frac{d\zeta}{\zeta} \\ - \frac{1}{2\pi} \int_{|\zeta| < 1} \frac{f(\zeta)}{\zeta} \left[2\zeta + \frac{\bar{z}}{2} + (2\bar{z} + |\zeta|^2) \frac{\bar{z} - \bar{\zeta}}{1 - \bar{z}\zeta} \right] d\xi d\eta \\ + \frac{1}{2\pi} \int_{|\zeta| < 1} \frac{\overline{f(\zeta)}}{\bar{\zeta}} \left[\frac{\bar{z}}{2} + \bar{\zeta} \right] d\xi d\eta = 0 \quad (10) \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\gamma_1(\zeta)}{1-\bar{z}\zeta} \frac{d\zeta}{\zeta} + \bar{z} \frac{1}{2\pi i} \int_{|\zeta|=1} \gamma_2(\zeta) \frac{1+\bar{z}\zeta}{1-\bar{z}\zeta} \frac{d\zeta}{\zeta} \\ & - \bar{z} \frac{1}{2\pi} \int_{|\zeta|<1} \frac{f(\zeta)}{\zeta} \frac{1+\bar{z}\zeta}{1-\bar{z}\zeta} d\xi d\eta - \frac{1}{2\pi} \int_{|\zeta|<1} \frac{\overline{f(\zeta)}}{\bar{\zeta}} (\bar{z}+2\bar{\zeta}) d\xi d\eta \\ & - \frac{1}{2\pi} \int_{|\zeta|<1} \frac{f(\zeta)}{\zeta} \left[\bar{z} + \frac{2\bar{z}(|\zeta|^2-1)}{(1-\bar{z}\zeta)^2} \right] d\xi d\eta + \bar{z} \frac{1}{2\pi} \int_{|\zeta|<1} \frac{\overline{f(\zeta)}}{\bar{\zeta}} d\xi d\eta = 0. \quad (11) \end{aligned}$$

The solution then is

$$\begin{aligned} \omega(z) &= c\bar{z} + ic\frac{\bar{z}}{2} + \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\gamma_0(\zeta)}{\zeta-z} d\zeta \\ & + \frac{1}{2\pi i} \int_{|\zeta|=1} \gamma_1(\zeta) \frac{1-|z|^2}{z} \log(1-z\bar{\zeta}) \frac{d\zeta}{\zeta} \\ & - \frac{1}{\pi} \int_{|\zeta|<1} \frac{\overline{f(\zeta)}}{\bar{\zeta}} \frac{1-|z|^2}{z} \log(1-z\bar{\zeta}) d\xi d\eta \\ & + \frac{1}{2\pi i} \int_{|\zeta|=1} 2\gamma_2(\zeta) \left[\frac{\bar{z}|z|^2-\bar{\zeta}}{\zeta(z-\zeta)} - |z|^2 \frac{\bar{z}-\bar{\zeta}}{z-\zeta} \right] \frac{d\zeta}{\zeta} \\ & - \frac{1}{\pi} \int_{|\zeta|<1} \frac{f(\zeta)}{\zeta} \left[\frac{\bar{z}|z|^2-\bar{\zeta}|\zeta|^2}{\zeta(z-\zeta)} - |z|^2 \frac{\bar{z}-\bar{\zeta}}{z-\zeta} \right] d\xi d\eta \\ & + \frac{1}{2\pi} \int_{|\zeta|<1} \frac{\overline{f(\zeta)}}{\bar{\zeta}} \left[\frac{1+\bar{z}+z\bar{\zeta}}{2} \frac{1}{1-z\bar{\zeta}} - \frac{\bar{\zeta}(z-\bar{z})}{\bar{z}(1-z\bar{\zeta})} \right] d\xi d\eta. \quad (12) \end{aligned}$$

Proof. We consider de following problems, the Dirichlet-Neumann problem:

$$\partial_{\bar{z}}^2 \omega = \Psi \text{ in } \mathbb{D}, \quad \omega = \gamma_0 \text{ on } \partial\mathbb{D}, \quad \partial_\nu \partial_{\bar{z}} \omega = \gamma_1 \text{ on } \partial\mathbb{D}, \quad \partial_{\bar{z}} \omega(0) = c,$$

where $\Psi \in L_1(\mathbb{D}, \mathbb{C}) \cap C^2(\partial\mathbb{D}, \mathbb{C})$, and the Schwarz problem:

$$\partial_{\bar{z}} \Psi = f \text{ in } \mathbb{D}, \quad \operatorname{Re}(\Psi) = \gamma_2 \text{ on } \partial\mathbb{D}, \quad \operatorname{Im}(\Psi)(0) = c_1.$$

Because of Theorem (1.2) the solvability conditions for the Dirichlet-Neumann problem are

$$c - \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\gamma_0(\zeta)}{1-\bar{z}\zeta} d\zeta + \frac{1}{\pi} \int_{|\zeta|<1} \frac{\Psi(\zeta)}{\zeta} \frac{1-|\zeta|^2}{1-\bar{z}\zeta} d\xi d\eta = 0 \quad (13)$$

and

$$\frac{1}{2\pi i} \int_{|\zeta|=1} (\gamma_1(\zeta) - \bar{\zeta}\Psi(\zeta)) \frac{1}{\zeta(1-\bar{z}\zeta)} d\zeta + \frac{1}{\pi} \int_{|\zeta|<1} \frac{\bar{z}\Psi(\zeta)}{(1-\bar{z}\zeta)^2} d\xi d\eta = 0, \quad (14)$$

which yield the unique solution

$$\begin{aligned} \omega(z) = & c\bar{z} + \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\gamma_0(\zeta)}{\zeta - z} d\zeta \\ & + \frac{1}{2\pi i} \int_{|\zeta|=1} [\gamma_1(\zeta) - \bar{\zeta}\Psi(\zeta)] \frac{\log(1 - z\bar{\zeta})}{\zeta} \frac{1 - |z|^2}{z} \frac{d\zeta}{\zeta} \\ & + \frac{1}{\pi} \int_{|\zeta|<1} \frac{|\zeta|^2 - |z|^2}{(\zeta - z)} \frac{\Psi(\zeta)}{\zeta} d\xi d\eta. \end{aligned} \quad (15)$$

After Theorem (1.1) the unique solution for the Schwarz problem arrive from the Cauchy-Schwarz-Pompeiu formula

$$\begin{aligned} \Psi(z) = & ic_1 + \frac{1}{2\pi i} \int_{|\zeta|=1} \gamma(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} \\ & - \frac{1}{2\pi} \int_{|\zeta|<1} \left[\frac{f(\zeta)}{\zeta} \frac{\zeta + z}{\zeta - z} + \frac{\overline{f(\zeta)}}{\bar{\zeta}} \frac{1 + z\bar{\zeta}}{1 - z\bar{\zeta}} \right] d\xi d\eta. \end{aligned} \quad (16)$$

Replacing (16) into (13) and (14) we obtain

$$\begin{aligned} 0 = & c - \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\gamma_0(\zeta)}{1 - \bar{z}\zeta} d\zeta + \frac{1}{\pi} \int_{|\zeta|<1} \frac{1 - |\zeta|^2}{1 - \bar{z}\zeta} \frac{1}{\zeta} [ic_1 \\ & + \frac{1}{2\pi i} \int_{|\tilde{\zeta}|=1} \gamma_2(\tilde{\zeta}) \frac{\tilde{\zeta} + \zeta}{\tilde{\zeta} - \zeta} \frac{d\tilde{\zeta}}{\tilde{\zeta}} - \frac{1}{2\pi} \int_{|\zeta|<1} \left[\frac{f(\tilde{\zeta})}{\tilde{\zeta}} \frac{\tilde{\zeta} + \zeta}{\tilde{\zeta} - \zeta} - \frac{\overline{f(\tilde{\zeta})}}{\bar{\tilde{\zeta}}} \frac{1 + \zeta\bar{\tilde{\zeta}}}{1 - \zeta\bar{\tilde{\zeta}}} \right] d\tilde{\xi} d\tilde{\eta}] d\xi d\eta. \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\gamma_1(\zeta)}{\zeta(1 - \bar{z}\zeta)} d\zeta - \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\bar{\zeta}}{\zeta(1 - \bar{z}\zeta)} [ic_1 \\ & + \frac{1}{2\pi i} \int_{|\tilde{\zeta}|=1} \gamma_2(\tilde{\zeta}) \frac{\tilde{\zeta} + \zeta}{\tilde{\zeta} - \zeta} \frac{d\tilde{\zeta}}{\tilde{\zeta}} - \frac{1}{2\pi} \int_{|\zeta|<1} \left[\frac{f(\tilde{\zeta})}{\tilde{\zeta}} \frac{\tilde{\zeta} + \zeta}{\tilde{\zeta} - \zeta} - \frac{\overline{f(\tilde{\zeta})}}{\bar{\tilde{\zeta}}} \frac{1 + \zeta\bar{\tilde{\zeta}}}{1 - \zeta\bar{\tilde{\zeta}}} \right] d\tilde{\xi} d\tilde{\eta}] d\zeta \\ & + \frac{1}{\pi} \int_{|\zeta|<1} \frac{\bar{z}}{(1 - \bar{z}\zeta)^2} \left[ic_1 + \frac{1}{2\pi i} \int_{|\tilde{\zeta}|=1} \gamma_2(\tilde{\zeta}) \frac{\tilde{\zeta} + \zeta}{\tilde{\zeta} - \zeta} \frac{d\tilde{\zeta}}{\tilde{\zeta}} \right. \\ & \left. - \frac{1}{2\pi} \int_{|\zeta|<1} \left[\frac{f(\tilde{\zeta})}{\tilde{\zeta}} \frac{\tilde{\zeta} + \zeta}{\tilde{\zeta} - \zeta} - \frac{\overline{f(\tilde{\zeta})}}{\bar{\tilde{\zeta}}} \frac{1 + \zeta\bar{\tilde{\zeta}}}{1 - \zeta\bar{\tilde{\zeta}}} \right] d\tilde{\xi} d\tilde{\eta} \right] d\xi d\eta = 0. \end{aligned}$$

respectively. Using the identities of Lemma 2.1 we reach the solvability conditions (10) and (11).

Now we look for the solution (12). For it we substitute (16) into the solution (15), so we have

$$\begin{aligned} \omega(z) = & c\bar{z} + \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\gamma_0(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\gamma_1(\zeta) \log(1 - z\bar{\zeta})}{\zeta} \frac{1 - |z|^2}{z} d\zeta \\ & - \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\bar{\zeta} \log(1 - z\bar{\zeta})}{\zeta} \frac{1 - |z|^2}{z} \left[ic_1 + \frac{1}{2\pi i} \int_{|\tilde{\zeta}|=1} \gamma_2(\tilde{\zeta}) \frac{\tilde{\zeta} + \zeta}{\tilde{\zeta} - \zeta} \frac{d\tilde{\zeta}}{\tilde{\zeta}} \right. \\ & \left. - \frac{1}{2\pi} \int_{|\zeta|<1} \left[\frac{f(\tilde{\zeta})}{\tilde{\zeta}} \frac{\tilde{\zeta} + \zeta}{\tilde{\zeta} - \zeta} - \frac{\overline{f(\tilde{\zeta})}}{\bar{\tilde{\zeta}}} \frac{1 + \zeta\bar{\tilde{\zeta}}}{1 - \zeta\bar{\tilde{\zeta}}} \right] d\tilde{\xi}d\tilde{\eta} \right] d\zeta \\ & + \frac{1}{\pi} \int_{|\zeta|<1} \frac{|\zeta|^2 - |z|^2}{\zeta(\zeta - z)} \left[ic_1 + \frac{1}{2\pi i} \int_{|\tilde{\zeta}|=1} \gamma_2(\tilde{\zeta}) \frac{\tilde{\zeta} + \zeta}{\tilde{\zeta} - \zeta} \frac{d\tilde{\zeta}}{\tilde{\zeta}} \right. \\ & \left. - \frac{1}{2\pi} \int_{|\zeta|<1} \left[\frac{f(\tilde{\zeta})}{\tilde{\zeta}} \frac{\tilde{\zeta} + \zeta}{\tilde{\zeta} - \zeta} - \frac{\overline{f(\tilde{\zeta})}}{\bar{\tilde{\zeta}}} \frac{1 + \zeta\bar{\tilde{\zeta}}}{1 - \zeta\bar{\tilde{\zeta}}} \right] d\tilde{\xi}d\tilde{\eta} \right] d\xi d\eta. \end{aligned}$$

The solution (12) follows from Lemma 2.2. \square

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