Bounds on the effective energy density of a more general class of the Willis dielectric composites

Gaetano Tepedino Aranguren, Javier Quintero C.,
Eribel Marquina

Abstract. The authors Willis (see [**W**]) and Milkis (see [**MM**]) considered a composite formed by a periodic mixed in prescribed proportion of a two homogeneous dielectric materials whose respectively energy density are $W_1(Z) = \frac{\alpha_1}{2}|z|^2$, $W_2(Z) = \frac{\alpha_2}{2}|Z|^2 + \frac{\gamma}{4}|Z|^4$, being $0 < \alpha_1 < \alpha_2, \gamma > 0$. Willis gave a lower bound on the effective energy density, but his method failed to give an upper bound. The difference between Milkis work and ours is that Milkis gives a self-consistent asymptotic expansion for the effective dielectric constant when the microstructure geometry is fixed and the non-linear phase has very low volume fraction. By contrast, we will give bounds on the effective energy density for the same material with arbitrary geometry and volume fractions θ_1, θ_2, valid for any spatially periodic microstructure.

This work gives not only lower and upper bound of that composite but also of a more general class considering $W_1(Z) = \frac{\alpha_1}{2}|Z|^2$, $W_2(Z) = \frac{\alpha_2}{2}|Z|^2 + \frac{\gamma}{p}|Z|^p$ being $0 < \alpha_1 < \alpha_2, \gamma > 0$ and $p > 2$. Moreover, we will prove that our bounds converge, as $\gamma \to 0^+$, to the optimal bounds of the effective energy density $\widetilde{W}_L$ of the considered *Linear Composites*, that is when $\gamma = 0$. In the article [**L**.**C**] it has been proved that the optimal bounds of the **linear-isotropic** case (this is when $\gamma = 0$) are expressed in the form:

$$(\widetilde{W}_L - W_1)^*(\eta) \leq A(\eta)\,, \qquad (W_2 - \widetilde{W}_L)^*(\eta) \leq B(\eta)\,,$$

while in our composite, the bounds of the **isotropic** case will be expressed in the form

$$(\widetilde{W} - W_1)^*(\eta) \leq A(\eta) - \gamma\mathcal{L}(\eta) + o(\gamma^2)|\eta|^{2p-4}\,,$$

$$(W_0 + W_2 - \widetilde{W})^*(\eta) \leq B(\eta) - \gamma\mathcal{U}(\eta) + o(\gamma^2)|\eta|^{2p-4}$$

where $W_0(\xi) = \frac{\gamma 2^{p-1}}{p}|\xi|^p$.

Moreover, we will give bounds to the **anisotropic** case. This article is a generalization of the particular case $p = 4$, which is called the Willis Composite (this particular case was first studied by [**W**] and [**MM**] and later was completed by [**T**]).

Resumen. Los autores Willis (ver [**W**]) y Milkis (ver [**MM**]) consideraron un compuesto formado por una mezcla periódica y de proporción prescrita de dos materiales dieléctricos homogéneos, cuyas densidad de energía son, respectivamente, $W_1(Z) = \frac{\alpha_1}{2}|Z|^2$, $W_2(Z) = \frac{\alpha_2}{2}|Z|^2 + \frac{\gamma}{p}|Z|^p$, siendo $0 < \alpha_1 < \alpha_2, \gamma > 0$. Willis dio un límite inferior para la densidad de energía eficaz, pero su método no dio una cota superior. La diferencia entre el trabajo de Milkis y el nuestro es que Milkis da una expansión asintótica auto–consistente de la constante dieléctrica efectiva cuando la geometría micro estructura es fija y la fase no lineal tiene fracción de volumen muy bajo. Por el contrario, vamos a dar límites a la densidad de energía eficaz para el material propio con una geometría arbitraria y fracciones de volumen θ_1, θ_2 válido para cualquier micro estructura espacial periódica.

Este trabajo no sólo da cotas inferior y superior de ese compuesto pero también de una clase más general teniendo en cuenta $W_1(Z) = \frac{\alpha_1}{2}|Z|^2$ $W_2(Z) = \frac{\alpha_2}{2}|Z|^2 + \frac{\gamma}{p}|Z|^p$ es $0 < \alpha_1 < \alpha_2, \gamma > 0$ y $p > 2$. Por otra parte, vamos a probar que nuestros límites convergen cuando $\gamma \to 0^+$, a la cota óptima de la densidad efectiva de energía $\widetilde{W}_L$ del considerado *Composites lineales*, que es cuando $\gamma = 0$. En el artículo [**LC**] ha sido probado que los límites óptimos del caso **isótropo lineal** (esto es cuando $\gamma = 0$) se expresan en la forma:

$$(\widetilde{W}_L - W_1)^*(\eta) \leq A(\eta)\,, \qquad (W_2 - \widetilde{W}_L)^*(\eta) \leq B(\eta)\,,$$

y mientras en nuestro compuesto, los límites del caso **isotrópico** serán expresados en forma

$$(\widetilde{W} - W_1)^*(\eta) \leq A(\eta) - \gamma\mathcal{L}(\eta) + o(\gamma^2)|\eta|^{2p-4}\,,$$

$$(W_0 + W_2 - \widetilde{W})^*(\eta) \leq B(\eta) - \gamma\mathcal{U}(\eta) + o(\gamma^2)|\eta|^{2p-4}$$

donde $W_0(\xi) = \frac{\gamma 2^{p-1}}{p}|\xi|^p$.

Además, daremos límites en el caso **anisotrópico**. Este artículo es una generalización del caso particular $p = 4$, que se llama Willis compuesto (este caso fue estudiado por primera vez por [**W**] y [**MM**] y más tarde se completó con [**T**]).

1 Introduction

Given $\Omega \subset \mathbb{R}^n$ open-bounded the region occupied by a dielectric, then its *electrostatic potential* satisfies the constitutive equation field

$$\left\{ \begin{array}{rcll} -div_x \mathcal{E}(x,\nabla u)\nabla u & = & \rho & if \quad x \in \Omega \\ u & = & \varphi_0 & if \quad x \in \partial\Omega \end{array} \right. ,$$

$$\left\{ \begin{array}{rcll} -div_x \nabla_Z W(x,\nabla u) & = & \rho & if \quad x \in \Omega \\ u & = & \varphi_0 & if \quad x \in \partial\Omega \end{array} \right.$$

being ρ its *free charge density*, $E_0 = -\nabla\varphi_0$ its *electric field* at the boundary, and $\mathcal{E}$ is its *dielectric tensor*. If there is a function $W : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $\nabla_Z W(x,Z) = \mathcal{E}(x,Z)Z$, then we say that W is the *Energy Density* of this material. We will consider this class of material.

A dielectric is called *homogeneous* when W does not depend on the space variable, otherwise, it is called *heterogeneous or inhomogeneous*. If a material is inhomogeneous with energy density W, we say that $\widetilde{W} : \mathbb{R}^n \to \mathbb{R}$ is its *effective energy density* when $u_\epsilon \to u_0$, in some sense, being u_ϵ, u_0 solution of

$$\left\{ \begin{array}{rcll} -div_x \nabla_Z W(\frac{x}{\epsilon},\nabla u)\nabla u & = & \rho & if \quad x \in \Omega \\ u & = & \varphi_0 & if \quad x \in \partial\Omega \end{array} \right. ,$$

$$\left\{ \begin{array}{rcll} -div_x \nabla_Z \widetilde{W}(\nabla u) & = & \rho & if \quad x \in \Omega \\ u & = & \varphi_0 & if \quad x \in \partial\Omega \end{array} \right.$$

We will consider a composite formed by a periodic mixed of two homogeneous dielectric in prescribed proportions which energy densities are respectively W_1, W_2. The composite formed has energy density $W(x,\xi) = \chi_1(x)W_1(\xi) + \chi_2(x)W_2(\xi)$ in a cell Y, where χ_k is the characteristic function of Y_k, being $Y_1 \cap Y_2 = \emptyset$, $Y = Y_1 \cup Y_2 \subset \Omega$ a cell and $\theta_k = |Y_k|/|Y|$. We will suppose that the composite has a $Y-$periodic structure. That is, microscopically given $\epsilon > 0$, the energy density of the composite in a cell ϵY is $W_\epsilon(x,\xi) = W(x/\epsilon,\xi)$. Therefore, extending $Y-$periodically $W(.,\xi)$ to all $\mathbb{R}^n$, we find that

$$\left\{ \begin{array}{l} W(x,\xi) = \chi_1(x)W_1(\xi) + \chi_2(x)W_2(\xi) \\ \text{being } \; W_1(\xi) = \frac{\alpha_1}{2}|\xi|^2, \;\; W_2(\xi) = \frac{\alpha_2}{2}|\xi|^2 + \frac{\gamma}{p}|\xi|^p, \\ \text{where } \; < \alpha_1 < \alpha_2, \gamma > 0, p > 2\,. \end{array} \right. \tag{1.1}$$

The set Y is the open rectangle $\prod_{i=1}^{n}(0,a_i)$, being $\{a_1,\ldots,a_n\} \subset (0,\infty)$. We will prove that the *Effective Energy Density* is given by the variational principle

$$\widetilde{W}(\xi) = \inf_{v \in V_p} \fint_Y W(x, v+\xi)\, dx\,, \tag{1.2}$$

where V_p is the completion of $C^1_{per}(\overline{Y}, \mathbb{R}^n)$ under the L_p-norm. This and other spaces will be defined in the next section. The formula (1.2) will be proved together with the exact definition of the effective energy density.

2 Definitions and Notations

Definition 1 *Given $n \in \mathbb{N}$ and $\{a_1, \ldots, a_n\} \subset (0, \infty)$ we consider $Y = \prod_{i=1}^n (0, a_i)$. If $\emptyset \neq X$, a function $f : \mathbb{R}^n \to X$ is called $Y-$periodic when*

$$\forall x \in \mathbb{R}^n, \forall (\delta_1, \ldots, \delta_n) \in \mathbb{Z}^n : f(x_1, \ldots, x_n) = f(x_1 + \delta_1 a_1, \ldots, x_n + \delta_n a_n) \,. \tag{2.1}$$

The usual norm and the usual inner product in $\mathbb{R}^n$ will be denoted by $|.|$ and $\langle ., . \rangle$.

Definition 2 *The integral $\displaystyle{\int\!\!\!\!\!\!-}_{A} f$ means the average $\dfrac{1}{|A|}\displaystyle\int f$, where $|A|$ is the L-measure of A.*

Definition 3 *The space $C_{per}(Y)$ are the function $f : \mathbb{R}^n \to \mathbb{R}$ continues in $\overline{Y}$ and $Y-$periodic. In the same way we define $C^1_{per}(Y)$, $C_{per}(Y, \mathbb{R}^n)$, $C^1_{per}(Y, \mathbb{R}^n)$. If $1 \leq p \leq \infty$ the spaces $L^p_{per}(Y)$, $L^p_{per}(Y, \mathbb{R}^N)$ are defined in natural way and usually we will write L^p_{per} for both spaces. If $1 \leq p < \infty$ we will use the normalized norm $||f||_p = \left({\int\!\!\!\!\!\!-}_{Y} |f|^p \right)^{1/p}$. The usual inner product of L^2_{per} will be $\langle f, g \rangle_2 = {\int\!\!\!\!\!\!-}_{Y} fg$ and $\langle F, G \rangle_2 = {\int\!\!\!\!\!\!-}_{Y} \langle F, G \rangle$.*

Definition 4 *We will consider the following: natural spaces:*

$CV = \{$constants vector fields $\mathbb{R}^N \to \mathbb{R}^N\}$.

$M = M(Y) = \{\sigma \in C_{per}(Y, \mathbb{R}^N) : \sigma = \nabla u$ for some $u \in C^1_{per}(Y)\}$.

$N = N(Y) = \{\sigma \in C^1_{per}(Y, \mathbb{R}^N) : {\int\!\!\!\!\!\!-}_{Y} \sigma = \theta$ and $div(\sigma) = 0$ in $Y\}$.

Definition 5 *Given $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$ we will consider the spaces:*

$K_p = K_p(Y) = W^{1,p}_{per}(Y)$ the completion of $C^1_{per}(Y)$ under the norm $||u||_{1,p} = ||u||_p + ||\nabla u||_p$.

$V_p = V_p(Y) =$ the completion of M under the $||.||_p-$norm.

$S_q = S_q(Y) =$ the completion of N under the $||.||_q-$norm.

$X_q = \{\sigma \in L^q_{per}(Y, \mathbb{R}^N) : div(\sigma) = 0$ in $Y\}$. And given $\eta \in \mathbb{R}^N$ we have the space

$X_q(\eta) = \{\sigma \in X_q : {\int\!\!\!\!\!\!-}_{Y} \sigma = \eta\}$.

Definition 6 *Given* $N \in \mathbb{N}$, $\mathcal{A}_N = \{\emptyset \neq \Omega \subset \mathbb{R}^N : \Omega$ *is open and bounded*$\}$.

Definition 7 *Given* $(\mathcal{E}, \tau)$ *a topological vector space which satisfies the first axiom of countability,* $A \subset \mathcal{E}$, $S \subset \overline{\mathbb{R}}$, $a \in \overline{S}$, $\{F_s : A \to \overline{\mathbb{R}} \mid s \in \overline{S}\}$ *and* $u \in \overline{A}$, *we say* $\lambda = \Gamma(\tau) \lim\limits_{s \to a} F_s(u)$ *if and only if*

(**i**)$\forall \{s_n\} \subset S, \forall \{u_n\} \subset A$ *with* $s_n \to a$ *and* $u_n \overset{\tau}{\to} u$:
$$\lambda \leq \liminf_{n \to \infty} F_{s_n}(u_n).$$
(**ii**)$\forall \{s_n\} \subset S$ *with* $s_n \to a$ *there is* $\{u_n\} \subset A$ *with* $u_n \overset{\tau}{\to} u$ *such that*
$$\limsup_{n \to \infty} F_{S_n}(u_n) \leq \lambda.$$

In this work, given $N \in \mathbb{N}$, $1 \leq p < \infty$ *and* $\Omega \in \mathcal{A}_N$ *we will take* $\mathcal{E} = L^p(\Omega)$, $A = W^{1,p}(\Omega)$, τ *the topology induced by the* L^p-*norm and* τ^* *the weak*$^*-$*topology of* $W^{1,p}(\Omega)$.

Definition 8 *If* V *is a real reflexive topological vector space and* $f : V \to \overline{\mathbb{R}}$, *we define* $f^* : V^* \to \overline{\mathbb{R}}$ *and* $f^{**} : V \to \overline{\mathbb{R}}$ *as*

$$\forall l \in V^* : f^*(l) = \sup_{x \in V}\{l(x) - f(x)\}, \qquad \forall x \in V : f^{**}(x) = \sup_{l \in V}\{l(x) - f^*(x)\}$$

We will use the fact:

$f : V \to \overline{\mathbb{R}}$ is convex $\iff$ $f = f^{**}$, that is $\forall x \in V : f(x) = \sup\limits_{l \in V}\{l(x) - f^*(x)\}$.

3 Existence of $\widetilde{W}$, $\Gamma-$convergence and Homogenization

In this section we will prove the formula (1.2) and give general results for future considerations.

Lemma 1 *The function* $W : \mathbb{R}^n \to \mathbb{R}^N \to \mathbb{R}$ *defined by (1.1) satisfies:*

(**1**)$\forall z \in \mathbb{R}^N : W(., z)$ *is* $Y-$*periodic and measurable.*

(**2**)$\forall x \in \mathbb{R}^N : W(x, .)$ *is* $C^1(\mathbb{R}^n)$ *and strictly convex.*

(**3**) *There are* $\beta > 0$ *and* $\lambda \in L^1_{loc}(\mathbb{R}^N)$ *a* $Y-$*periodic positive function such that*

$$\forall x, z \in \mathbb{R}^n : 0 \leq \lambda^{-1}(x)|z|^p \leq W(x, z) \leq \beta(1 + |z|^p)) \,. \tag{3.1}$$

Lemma 2 *Given* $N \in \mathbb{N}, \Omega \in \mathcal{A}_N, 1 < p < \infty, p^{-1} + q^{-1} = 1, \rho \in L^q(\Omega)$, X *a closed linear subspace containing* $W_0^{1,p}(\Omega)$ *and* $W : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ *a function which satisfies the conditions* (**2**) *and* (**3**) *of lemma 1 (the periodicity is not necessary here), then the function*

$$T(\Omega, u) = \fint_{\Omega} (W(x, \nabla u) - \rho u)\, dx \,, \tag{3.2}$$

has an unique minimizer over X which satisfies

$$-div\nabla_z W(.,\nabla u)=\rho \qquad in\ \ \Omega,. \tag{3.3}$$

And reciprocally, the solution of (3.3) is the minimizer of (3.2).

Proof *This is a consequence of a more general statement proved in* [**D.A**] . □

Theorem 1 *$\forall N\in\mathbb{N},\Omega\in\mathcal{A}_N, 2\le p<\infty, p^{-1}+q^{-1}=1$ and W a function, which satisfies the conditions* (**1**) *to* (**3**) *of lemma 1. Then, the function $\widetilde{W}:\mathbb{R}^N\to\mathbb{R}$ defined as*

$$\widetilde{W}(\xi)=\inf_{u\in K_p}⨍_Y W(x,\nabla u+\xi)\,dx\,, \tag{3.4}$$

is well defined and has the following properties:
(**1**) *$\widetilde{W}$ satisfies the conditions* (**1**) *to* (**3**) *of lemma 1.*
(**2**) *Given $\varphi\in W^{1,p}(\mathbb{R}^N),\rho\in L^q_{per}$, $\epsilon>0$ and u_0,u_ϵ solutions of*

$$\begin{cases} -div_x\nabla_Z W(\frac{x}{\epsilon},\nabla u)\nabla u &= \rho \quad if \quad x\in\Omega\\ u &= \varphi \quad if \quad x\in\partial\Omega \end{cases},$$
$$\begin{cases} -div_x\nabla_Z \widetilde{W}(\nabla u) &= \rho \quad if \quad x\in\Omega\\ u &= \varphi \quad if \quad x\in\partial\Omega \end{cases} \tag{3.5}$$

then $u_\epsilon \overset{L^p(\Omega)}{\longrightarrow} u_0$ and $\nabla u_\epsilon \overset{L^p(\Omega)}{\rightharpoonup} \nabla u_0$.

Proof *This is a consequence of a more general statement proved in* [**AB**] . □

We conclude that $\forall\Omega\in A_N$ the operators $T_\epsilon(\Omega,u)=⨍_\Omega (W(x/\epsilon,\nabla u)-\rho u)\,dx$ is $\Gamma(\tau^*_\Omega)-$ convergent to the operators $T_0(\Omega,u)=⨍_\Omega \left[\widetilde{W}(\nabla u)-\rho u\right]\,dx$, as $\epsilon\to 0$.

We have compute $\widetilde{W}$ as a primal variational principle (3.4). The next section will prove a dual variational principal associate with $\widetilde{W}$. We suggest to our readers to review the definition of F^* the dual of the function $F:X\to\overline{\mathbb{R}}$, being X a real locally compact vector topological space, see for example [EK].

4 Dual and Hashin-Shtrikman Variational Principles

Lemma 3 *Let $1<p<\infty, p^{-1}+q^{-1}=1$ and ν the outer unit vector on ∂Y.*
(**1**) $\forall f\in C_{per}(Y):\displaystyle\int_{\partial Y} f=0.$

(**2**) $\forall u \in C^1_{per}(Y), \forall 1 \le i \le N : \int\limits_Y D_i u = 0.$

(**3**) $\sigma \in V_p \iff \sigma = \nabla u$ *some* $u \in K_p$.

(**4**) $\forall \xi \in \mathbb{R}^N, \forall v \in V_p : \fint\limits_Y \langle v, \xi \rangle = 0.$

(**5**) $\forall \xi \in \mathbb{R}^n, \forall \sigma \in S_q : \fint\limits_Y \langle \sigma, \xi \rangle = 0.$

(**6**) $\forall \sigma \in \mathcal{X}_q, \forall v \in V_p : \fint\limits_Y \langle \sigma, v \rangle = 0.$

(**7**) $V_p^{\perp} = S_q \oplus CV$, $\S_q^{\perp} = V_p \oplus CV$.

Theorem 2 *If $\widetilde{W}$ is the function defined by (3.4), then*

$$\forall \eta \in \mathbb{R}^n : \widehat{W}^*(\eta) = \inf_{\sigma \in S_q} \fint\limits_Y W^*(x, \sigma + \eta)\, dx\,. \tag{4.1}$$

Proof *From lemma 2 given $\xi \in \mathbb{R}^N$: $\widetilde{W}(\xi) = \fint\limits_Y W(x, \nabla u_\xi + \xi)\, dx$, where $u_\xi \in K_p$ and*

$$div \nabla_z W(., \nabla u_\xi + \xi) = 0 \;\; in \;\; Y\,. \tag{4.2}$$

Since $W(x,.)$ is convex, then $W(x,.) = W^{**}(x,.)$, therefore (see for example [**E**.**T**] *)*

$$\widetilde{W}(\xi) = \fint\limits_Y W(x, \nabla u_\xi + \xi)\, dx = \sup_{\sigma \in L^q_{per}} \fint\limits_Y \left[\langle \nabla u_\xi, \sigma \rangle - W^*(x, \sigma) \right] dx\,, \tag{4.3}$$

since the integrand is concave, thus the supreme is achieved at some $\sigma_\xi \in L^q_{per}$ which satisfies $\nabla W^*(., \sigma_\xi) = \nabla_\xi + \xi$ in Y, then (see [**E**.**T**] *) $\sigma_\xi = \nabla_z W(., \nabla u_\xi + \xi)$ in Y and by (4.3) $div(\sigma_\xi) = 0$ in Y, that is $\sigma_\xi \in X_q$. On the other hand, since $X_q \subset L^q_{per}$, then using the lemma 3, we have*

$$\widetilde{W}(\xi) \ge \sup_{\sigma \in X_q} \fint\limits_Y \left[\langle \nabla u_\xi + \xi, \sigma \rangle - W^*(x, \sigma) \right] dx = \sup_{\sigma \in X_q} \fint\limits_Y \left[\langle \xi, \sigma \rangle - W^*(x, \sigma) \right] dx\,. \tag{4.4}$$

Given $\sigma \in S_q$ and $\eta \in \mathbb{R}^N$ we have $\sigma + \eta \in X_q(\eta)$, then from (4.4)

$$\begin{aligned}&\forall\xi,\eta\in\mathbb{R}^N:\\&\widetilde{W}(\xi)\geq\sup_{\sigma\in S_q}\fint_Y\left[\langle\xi,\sigma+\eta\rangle-W^*(x,\sigma+\eta)\right]dx\\&=\langle\xi,\eta\rangle-\inf_{\sigma\in S_q}\fint_Y W^*(x,\sigma+\eta)\,dx\,,\end{aligned}\tag{4.5}$$

because $\fint_Y\langle\sigma,\xi\rangle=0$. *Subtracting* $\langle\xi,\eta\rangle$ *in both sides of (4.5), multiplying by* -1 *and taking supreme over* $\xi\in\mathbb{R}^N$, *we obtain*

$$\forall\eta\in\mathbb{R}^N:W^*(\eta)\leq\inf_{\sigma\in S_q}\fint_Y W^*(x,\sigma+\eta)\,dx\,.\tag{4.6}$$

On the other hand, since the supreme in (4.3) is achieved at $X_q\subset L^q_{per}$, *then*

$$\forall\xi\in\mathbb{R}^N:W^*(\xi)=\sup_{\sigma\in L^q_{per}}\fint_Y\left[\langle\xi,\sigma\rangle-W^*(x,\sigma)\right]dx\,,\tag{4.7}$$

subtracting $\langle\xi,\eta\rangle$ *from both sides of (4.7) with* $\eta=\fint_Y\sigma_\xi$ *we get* $\langle\xi,\eta\rangle-\widetilde{W}(\xi)=\inf_{\sigma\in X^q(\eta)}\fint_Y\left[W^*(x,\sigma)-\langle\sigma+\eta,\xi\rangle\right]dx$. *Since* $\widetilde{W}$ *is convex, we have* $\widetilde{W}(\xi)\geq\langle\xi,\eta\rangle-W^*(\eta)$, *therefore* $\widetilde{W}^*(\eta)\geq\inf_{\sigma\in X_q(\eta)}\fint_Y W^*(x,\sigma)$, *because* $\fint_Y(\sigma-\eta)\,dx=\theta$. *Hence*

$$\widetilde{W}^*(\eta)\geq\inf_{\sigma\in S_q}\fint_Y W^*(x,\sigma+\eta)\,dx,,\tag{4.8}$$

thus from (4.6), (4.7) and (4.8) we obtain (4.1). □

The arguments used in the proof of this theorem can be used to obtain, under certain conditions on W, other variational principles. Under a more general approach there is a method called (see [**H.S**]) Hashin-Shtrikman variational principles and improved in the article [**TW**] . We are going to present this result restricted to our particular case.

Theorem 3 (Talbot-Willis) *If* W *satisfies the hypothesis of lemma 1 and* $f_1,f_2,f_3,f_4:\mathbb{R}^N\to\mathbb{R}$ *are convex functions of class* C^1 *such that* $\forall x\in\mathbb{R}^N$: $W(x,.)-f_1,f_2-W(x,.),W^*(x,.)-f_3,f_4-W^*(x,.)$ *are convex, then*

$$\forall \xi \in \mathbb{R}^N : \widetilde{W}(\xi) = \sup_{\sigma \in L^q_{per}} \inf_{u \in K_p} \fint_Y \left[\langle \nabla u + \xi, \sigma \rangle - (W - f_1)^*(\sigma) + f_1(\nabla u + \xi) \right] dx \,. \tag{4.9}$$

$$\forall \xi \in \mathbb{R}^N : \widetilde{W}(\xi) = \inf_{\sigma \in L^q_{per}} \inf_{u \in K_p} \fint_Y \left[-\langle \nabla u + \xi, \sigma \rangle + (f_2 - W)^*(\sigma) + f_2(\nabla u + \xi) \right] dx \,. \tag{4.10}$$

$$\forall \eta \in \mathbb{R}^N : \widetilde{W}^*(\eta) = \sup_{v \in L^p_{per}} \inf_{\sigma \in S_q} \fint_Y \left[\langle \sigma + \eta, v \rangle - (W^* - f_3)^*(v) + f_3(\sigma + \eta) \right] dx \,. \tag{4.11}$$

$$\forall \eta \in \mathbb{R}^N : \widetilde{W}^*(\eta) = \inf_{v \in L^p_{per}} \inf_{\sigma \in S_q} \fint_Y \left[-\langle \sigma + \eta, v \rangle + (f_4 - W*)^*(v) + f_4(\sigma + \eta) \right] dx \,. \tag{4.12}$$

Proof *We have* $\widetilde{W}(\xi) = \inf_{u \in K_p} \fint_Y \left[W(x, \nabla u + \xi) - f_1(\nabla u + \xi) + f_1(\nabla u + \xi) \right] dx,$ *and since*

$W(x,.) - f_1$ *is convex, then, following the same approach of the proof of the theorems 1 and 2 we obtain*

$$\begin{aligned} \widetilde{W}(\xi) &= \inf_{u \in K_p} \sup_{\sigma \in L^q_{per}} \fint_Y \left[\langle \nabla u + \xi, \sigma \rangle - (W - f_1)^*(\sigma) + f_1(\nabla u + \xi) \right] dx \\ &= \inf_{u \in K_p} \sup_{\sigma \in L^q_{per}} T(u, \sigma) \,, \end{aligned}$$

since $\forall u \in K_p : T(u,.)$ *is concave on* L^q_{per} *and* $\forall \sigma \in L^q_{per} : T(.,\sigma)$ *is convex on* K_p*, then the usual Theorem of the Saddle Point (see for example*[**E.T**] *) gives the existence of* $(\hat{u}, \hat{\sigma}) \in K_p \times L^q_{per}$ *such that* $\inf_{u \in K_p} \sup_{\sigma \in L^q_{per}} T(u,\sigma) = T(\hat{u},\hat{\sigma}) = \sup_{\sigma \in L^q_{per}} \inf_{u \in K_p} T(u,\sigma)$*. Therefore, we can interchange sup and inf to obtain (4.9).*

The item (4.10) is easier because $W(x,.) - f_2$ *is a concave* C^1 *function, thus*

$$\widetilde{W}(\xi) = \inf_{u \in K_p} \inf_{\sigma \in L^q_{per}} \fint_Y \left[-\langle \nabla u + \xi, \sigma \rangle + (f_2 - W)^*(\sigma) + f_2(\nabla u + \xi) \right] dx \,,$$

and interchanging the order of the inf we obtain (4.10).

The items (4.11) and (4.12) are obtained by similar arguments using (4.1) of theorem 2. □

5 Some important results

Lemma 4 *Given $H = (h_{i,j}) \in \mathbb{R}(N,N)$ a symmetric real matrix with $\sigma(H) = \{\lambda_1, \ldots, \lambda_N\}$, then*

$$\forall r > 0 : \fint_{S_r} |H\eta|^2 = \frac{r^2}{N} \sum_{i=n}^{N} \lambda_i^2 = \frac{r^2}{N} tr(H^2)\,. \tag{5.1}$$

Proof *There is $P \in \mathbb{R}(N,N)$ such that $P^tP = I$ and $H = P^tDP$ where $D = diag\{\lambda_1, \ldots, \lambda_N\}$. Then $|H\eta|^2 = |DP\eta|^2$, $|det(P)| = 1$ and $|\eta|^2 = |P\eta|^2$. A change of variable gives $\fint_{S_r} |H\eta|^2 = \fint_{S_r} |Dz|^2 = \sum_{i=1}^{N} \lambda_i^2 \fint_{S_r} z_i^2$. Since $\fint_{S_r} |z_i|^2 = \fint_{S_r} |z_j|^2$, then $N \fint_{S_r} |z_i|^2 = \fint_{S_r} |z|^2 = r^2$ and $\fint_{S_r} |z_i|^2 = \frac{r^2}{N}$, thus we obtain (5.1).* □

Theorem 4 *If φ is a $Y-$periodic solution of $\Delta\varphi = \chi_k - \theta_k$ in Y, H its Hessian matrix, $\delta \in \mathbb{R}$, $\eta \in \mathbb{R}^N$, $r > 0$ and $u = \delta\langle \nabla\varphi, \eta\rangle$, then*

$$u \in K_p,\ \nabla u = \delta H\eta,\ \fint_Y H\chi_k = \fint_Y H^2,\ \fint_Y \langle \nabla u,\ \eta\rangle \chi_k = \delta \fint_Y |H\eta|^2\,, \tag{5.2}$$

$$\fint_{S_r} \fint_Y |H\eta|^2 = \frac{r^2}{N} \theta_1\theta_2\,. \tag{5.3}$$

Proof *Exists such a solution $\varphi \in W^{2,p}_{per}$ because $\chi_k - \theta_k \in L^t_{per}$ for all $1 \le t \le \infty$ and $\fint_Y (\chi_k - \theta_k) = 0$, then $u = \delta\langle \nabla u, \eta\rangle \in K_p$. Let $H = (h_{i,j})$ the Hessian matrix of φ, clearly H is real and symmetric, where $h_{i,j} = D_{i,j}\varphi$. We have $u = \delta \sum_{j=1}^{N} \eta_j D_j\varphi$, then $D_i u = \delta \sum_{I=1}^{N} \eta_j D_{i,j}\varphi$ and $\nabla u = H\eta$.*

Since φ is $Y-$periodic, then $\theta_k \fint_Y h_{i,j} = 0$, and $\fint_Y h_{i,j}\chi_k = \fint_Y (\chi_k - \theta_k) h_{i,j} = \fint_Y \Delta\varphi h_{i,j} = \sum_{t=1}^{N} \fint_Y h_{t,t} h_{i,h}$, integrating by parts, we get $\fint_Y h_{i,j}\chi_k = \sum_{t=1}^{N} \fint_Y h_{i,t} h_{t,j}$, then $\fint_Y H\chi_k = \fint_Y H^2$.

On the other hand $\displaystyle\fint_Y \langle \nabla u, \eta\rangle \chi_k = \delta \fint_Y \langle H\chi_k \eta, \eta\rangle = \delta \langle (\fint_Y H\chi_k)\eta, \eta\rangle =$
$\displaystyle\langle (\fint_Y H^2)\eta, \eta\rangle = \delta \fint_Y \langle H^2\eta, \eta\rangle = \delta \fint_Y \langle H\eta, H\eta\rangle = \delta \fint_Y |H\eta|^2.$

Using the Theorem of Fubini and lemma 4 we get
$\displaystyle\fint_{S_r}\fint_Y |H\eta|^2 = \fint_Y\fint_{S_r} |H\eta|^2 = \frac{r^2}{N}\fint_Y tr(H^2),$
and using integration by parts twice, we get

$$\begin{aligned}\fint_Y tr(H^2) &= \sum_{i=1}^N\sum_{j=1}^N \fint_Y h_{i,j}h_{j,i} = \sum_{i=1}^N\sum_{j=1}^N \fint_Y h_{i,i}h_{j,j}\\ &= \fint_Y |\Delta\varphi|^2 = \fint_Y |\chi_k - \theta_k|^2 = \theta_1\theta_2.\end{aligned}$$

□

Lemma 5 *Given* $p > 1$, *the function defined implicitly by*

$$\forall x \geq 0 : xG^{p-1}(x) + G(x) = 1\,, \tag{5.4}$$

is a well defined C^2 *function on* $[0,\infty)$, $G(0) = 1$, $G((0,\infty)) \subset (0,1)$ *and*

$$G(x) = 1 - x + o(x^2) \quad as \quad x \to 0^+\,. \tag{5.5}$$

Proof *Given* $a \geq 0$, *lets consider the function* $\varphi_a(x) = ax^{p-1} + x - 1$ *defined on* $[0,\infty)$. *If* $a = 0$ *this function has the unique real zero* $G(0) = 1$. *If* $a > 0$ *then* $\varphi_a' > 0$, $\varphi_a \in C[0\infty)$, $\varphi_a(0) < 0$ *and* $\varphi_a(1) > 0$, *then* φ_a *has an unique real zero* $G(a) \in (0,1)$. *Therefore, the function* $G : [0,\infty) \to \mathbb{R}$ *defined* $G(a)$ *to be the unique real zero of* φ_a *it is a well defined real function which satisfies* $\forall a \in [0,\infty) : aG^{p-1}(a) + G(a) = 1$, *thus clearly* $G(0) = 1$ *and* $G((0,\infty)) \subset (0,1)$.

Moreover, using the Implicit Function Theorem to $F : (0,\infty)\times(0,\infty) \to \mathbb{R}$ *given as* $F(x,y) = xy^{p-1} + y - 1$, *we obtain that* G *is differentiable and* $G' = G^{p-1}/(1 + (p-1)xG^{p-2})$, *then* G *is* C^1. *Using again the Implicit Function Theorem we get that* G *is* C^2.

On the other hand we have $G(0) = 1, G'(0) = -1, G''(0) = 2(p-1)$ *and using the L'Hopital Theorem we get* $\lim\limits_{x\to0^+} \frac{G(x)-1+x-(p-1)x^2}{x^2} = 0$ *and* $\lim\limits_{x\to0^+} \frac{G(x)-1+x}{x^2} = p - 1$. □

Lemma 6 *Given* $p > 2, \alpha > 0, \gamma > 0$ *and* $h : \mathbb{R}^N \to \mathbb{R}$ *given as*

$$\forall z \in \mathbb{R}^N : h(z) = \frac{\alpha}{2}|z|^2 + \frac{\gamma}{p}|z|^p\,, \tag{5.6}$$

then, h is a convex C^1 function which satisfies

$$\begin{aligned} &\forall\eta\in\mathbb{R}^N: h^*(\eta)=\left[\tfrac{1}{\alpha}(1-\tfrac{1}{p})G(b)+\tfrac{1}{\alpha}(\tfrac{1}{p}-\tfrac{1}{2})G^2(b)\right]|\eta|^2\,, \\ &\quad \textit{where } \ b=\tfrac{\gamma}{\alpha^{p-1}}|\eta|^{p-2}\,, \end{aligned} \tag{5.7}$$

$$\forall\eta\in\mathbb{R}^N: h^*(\eta)=\frac{1}{2\alpha}|\eta|^2-\frac{\gamma}{p\alpha^p}|\eta|^p+o(\gamma^2)|\eta|^{2p-4}, \ \ as \ \ \gamma\to 0^+\,. \tag{5.8}$$

Proof *We have $H(\eta)=f(|\eta|)$, where $f:[0,\infty)\to\mathbb{R}$ is given as $f(t)=\frac{\alpha}{2}t^2+\frac{\gamma}{p}t^p$. Then $h^*(\eta)=f^*(|\eta|)$, where $\forall s\geq 0: f^*(s)=\sup\{st-f(t):t\geq 0\}$. Clearly $f^*(0)=0$. If $s>0$, then $F^*(s)=s\widehat{t}-f(\widehat{t})$, where $s-\alpha\widehat{t}-\gamma\widehat{t}^{p-1}=0$, thus $\frac{\gamma}{s}\widehat{t}^{p-1}+\frac{\alpha}{s}\widehat{t}=1$. Taking $\widehat{z}=\frac{\alpha}{s}\widehat{t}$ we get $a\widehat{z}^{p-1}+\widehat{z}=1$, where $a=\gamma s^{p-2}/\alpha^{p-1}$. Therefore,*

$$\forall s\geq 0: f^*(s)=\frac{s^2}{\alpha}G(a)-\frac{s^2}{2\alpha}G^2(a)-\frac{\gamma s^p}{p\alpha^p}G^p(a), \ \ where \ \ a=\frac{\gamma s^{p-2}}{\alpha^{p-1}}\,. \tag{5.9}$$

Since $aG^{p-1}(a)+G(a)=1$, we get $\frac{s^p\gamma}{p\alpha^p}G^p(a)=\frac{s^2}{p\alpha}(G(a)-G^2(a))$, replacing this into (5.9) we get

$$\forall s\geq 0: f^*(s)=\left[\frac{1}{\alpha}(1-\frac{1}{p})G(a)+\frac{1}{\alpha}(\frac{1}{p}-\frac{1}{2})G^2(a)\right]s^2, \ \ where \ \ a=\frac{\gamma s^{p-2}}{\alpha^{p-1}}\,. \tag{5.10}$$

From (5.10) we obtain (5.7). On the other hand $G(a)=1-a+o(a^2)$ and $G^2(a)=1-2a+o(a^2)$, replacing this into (5.10) we get $f^*(s)=\frac{1}{2\alpha}s^2-\frac{1}{p\alpha^p}as^2+o(a^2)s^2$. Since $a=\gamma s^{p-2}/\alpha^{p-1}$, we get $\frac{1}{p\alpha}as^2=\frac{\gamma s^p}{p\alpha^p}$ and $o(a^2)s^2=o(\gamma^2)s^{2p-2}$, then $f^*(s)=\frac{1}{2\alpha}s^2-\frac{\gamma}{p\alpha^p}s^p+o(\gamma^2)s^{2p-4}$, from this we obtain (5.8) . □

6 A Lower Bound on $\widetilde{W}$

Theorem 5 *Given W by (1.1), (1.2) and $\widetilde{W}$ by (3.4), then $\forall r>0$:*

$$\fint_{S_r}(\widetilde{W}-W_1)^*(\eta)\leq\frac{1}{2\theta_2}\left(\frac{1}{\alpha_2-\alpha_1}+\frac{\theta_1}{N\alpha_1}\right)r^2-\frac{\gamma}{p(\alpha_2-\alpha_1)^p\theta_2^{p-1}}r^p+o(\gamma^2)r^{2p-4}\,. \tag{6.1}$$

Proof *Since $W(x,z)-W_1(z)=\chi_2(x)(W_2-W_1)(z)=\chi_2(x)h(z)$, where h is the convex C^1 function given by (5.6) with $\alpha=\alpha_2-\alpha_1$, then we can use (4.9) to obtain*

$$\forall\xi\in\mathbb{R}^N:\widetilde{W}(\xi)=\sup_{\sigma\in L^q_{per}}\inf_{u\in K_p}\fint_Y\left[\langle\nabla u+\xi,\sigma\rangle-\chi_2 h^*(\sigma)+W_1(\nabla u+\xi)\right]dx\,,$$

we choose $\sigma = \chi_2\eta$, *where* $\eta \in \mathbb{R}^N$ *and get*

$$\forall \xi, \eta \in \mathbb{R}^N : \widetilde{W}(\xi) \geq \theta_2\langle \xi, \eta\rangle - \theta_2 h^*(\eta) + \inf_{u\in K_p} \fint_Y \left[\langle \nabla u, \eta\rangle \chi_2 + W_1(\nabla u + \xi)\right] dx\,,$$

Since $W_1(\nabla u + \xi) = \dfrac{\alpha_1}{2}(|\nabla u|^2 + 2\langle \nabla u, \xi\rangle + |\xi|^2)$ *and* $\fint_Y \langle \nabla u, \xi\rangle = 0$, *then*

$$\forall \xi, \eta \in \mathbb{R}^N : \\ \widetilde{W}(\xi) \geq W_1(\xi) - \theta_2\langle \xi, \eta\rangle - \theta_2 h^*(\eta) + \inf_{u\in K_p} \fint_Y \left[\langle \nabla u, \eta\rangle \chi_2 + \frac{\alpha_1}{2}|\nabla u|^2\right] dx\,. \tag{6.2}$$

The inf in (6.2) is achieved by $u \in K_p$ *such that* $\alpha_1 \Delta u = -div(\eta\chi_2)$ *in* Y, *that is* $u = -\dfrac{1}{\alpha_1}\langle \nabla\varphi, \eta\rangle$ *where* $\varphi \in W^{2,p}(Y)$ *satisfies* $\Delta\varphi = \chi_2 - \theta_2$ *in* Y. *From (5.2) we get*

$$\forall \xi, \eta \in \mathbb{R}^N : (\widetilde{W} - W_1)(\xi) \geq \theta_2\langle \xi, \eta\rangle - \theta_2 h^*(\eta) - \frac{1}{2\alpha_1}\fint_Y |H\eta|^2\, dx\,, \tag{6.3}$$

subtracting $\langle \xi, \eta\rangle$, *multiplying by* (-1) *and taking sup over* $\xi \in \mathbb{R}^N$ *after having replaced* $\eta \rightleftharpoons \eta/\theta_2$ *on(6.3) we find*

$$\forall \eta \in \mathbb{R}^N : (\widetilde{W} - W_1)^*(\eta) \leq \theta_2 h^*(\eta/\theta_2) + \frac{1}{2\alpha_1\theta_2^2}\fint_Y |H\eta|^2\, dx\,, \tag{6.4}$$

where h^* *is given by (5.7). Integrating over* S_r *and using (5.3) we get*

$$\forall r > 0 : \fint_{S_r} (\widetilde{W} - W_1)^*(\eta) \leq \theta_2 f^*(r/\theta_2) + \frac{\theta_1}{2N\alpha_1\theta_2} r^2\,, \tag{6.5}$$

where f^* *is given by (5.10). Replacing (5.10) or (5.8) into (6.5) we obtain (6.1).* □

Corollary 1 *Under the same conditions of theorem 1, if* $\widetilde{W}$ *is isotropic, then*

$$\forall \eta \in \mathbb{R}^N : (\widetilde{W} - W_1)^*(\eta) \leq \\ \tfrac{1}{2\theta_2}\left(\tfrac{1}{\alpha_2 - \alpha_1} + \tfrac{\theta_1}{N\alpha_2}\right)|\eta|^2 - \tfrac{\gamma}{p(\alpha_2-\alpha_1)^p\theta_2^{p-1}}|\eta|^p + o(\gamma^2)|\eta|^{2p-4}\,. \tag{6.6}$$

Proof *Direct consequence of theorem 1.* □

Corollary 2 *Under the same condition of theorem 1, if $\widetilde{W}$ is isotropic, then the lower bound obtained by theorem 5 converges, as $\gamma \to 0$, to the optimal lower bound of the linear composite.*

Proof *The energy density of the linear composite is $W_L(x,z) = \chi_1(x)|\frac{\alpha_1}{2}|z|^2 + \chi_2(x)\frac{\alpha_2}{2}|z|^2$, let $\widetilde{W}_L$ its effective energy density, it has been proved, see for example* [**L.C**] *, that the optimal bound on $\widetilde{W}_L$ is given in the form $(\widetilde{W}_L - W_1)^*(\eta) \le A(\eta)$, while we have found*

$$(\widetilde{W} - W_1)^*(\eta) \le A(\eta) - \gamma\mathcal{L}(\eta) + o(\gamma^2)|\eta|^{2p-4}\,.$$

Moreover $W_L \le W$, then $\widetilde{W}_L - W_1 \le \widetilde{W}_L - W_1$ and $(\widetilde{W} - W_1)^*(\eta) \le (\widetilde{W}_L - W_1)^*(\eta) \le A(\eta)$. □

7 An Upper Bound on $\widetilde{W}$

Theorem 6 *Under the same hypothesis of theorem 5, for all $r > 0$:*

$$\fint_{S_r}(W_2-\widetilde{W})^*(\eta) \le \frac{1}{2\theta_1}\left(\frac{1}{\alpha_2-\alpha_1} - \frac{\theta_2}{N\alpha_2}\right)r^2 - \frac{\gamma}{p(\alpha_2-\alpha_1)^p\theta_1^{p-1}}r^p + o(\gamma^2)r^{2p-4}\,. \tag{7.1}$$

Proof *Since $W(x,z) - W_2(z) = \chi_1(x)(W_1 - W_2)(z)$, then $W_2(z) - W(x,z) = \chi_1(x)(W_2 - W_1)(z) = \chi_1()x)h(z)$, where h is the C^1 convex function given by (5.6) with $\alpha = \alpha_2 - \alpha_1$. Therefore, we can use (4.10) to get*

$$\forall\xi \in \mathbb{R}^N : \widetilde{W}(\xi) = \inf_{\sigma\in L^q_{per}} \inf_{u\in K_p} \fint_Y \left[-\langle\nabla u + \xi, \sigma\rangle + \chi_1 h^*(\sigma) + W_2(\nabla u + \xi)\right] dx\,,$$

given $\eta \in \mathbb{R}^N$, we can chose $\sigma = \eta\chi_1$ and obtain

$$\forall\xi,\eta \in \mathbb{R}^N : \widetilde{W} \le -\theta_1\langle\xi,\eta\rangle + \theta_1 h^*(\eta) + \inf_{u\in K_p}\fint_Y \left[-\langle\nabla u, \eta\rangle\chi_2 + W_2(\nabla u + \xi)\right] dx\,,$$

Since $W_2(\nabla u+\xi) = \frac{\alpha_2}{2}(|\nabla u|^2 + 2\langle\nabla u,\xi\rangle + |\xi|^2) + \frac{\gamma}{p}|\nabla u+\xi|^p$ and $\fint_Y \langle\nabla u,\xi\rangle = 0$, then

$$\forall\xi,\eta \in \mathbb{R}^N : \widetilde{W}(\xi) \le \frac{\alpha_2}{2}|\xi|^2 - \theta_1\langle\xi,\eta\rangle + \theta_1 h^*(\eta) + \inf_{u\in K_p}(T_0 + S)(u)\,, \tag{7.2}$$

where $T_0(u) = \fint_Y \left[-\langle \nabla u, \eta\rangle \chi_1 + \frac{\alpha_2}{2}|\nabla u|^2\right] dx$ *and* $S(u) = \frac{\gamma}{p}\fint_Y |\nabla u + \xi|^p\, dx$. *Since* $\inf_{u\in K_p}(T_o + S)(u) \le T_0(\hat{u}) + S(0)$, *where* $\hat{u}$ *is the minimizer of* T_0 *over* K_p, *we obtain* $\inf_{u\in K_p}(T_0 + S)(u) \le \inf_{K_p} T_0(u) + \dfrac{\gamma}{p}\|\xi|^p$, *then*

$$\forall \xi, \eta \in \mathbb{R}^N : \widetilde{W}(\xi) \le W_2(\xi) - \theta_1\langle \xi, \eta\rangle + \theta_1 h^*(\eta) + \inf_{u\in K_p} T_0(u)\,, \tag{7.3}$$

where the inf is achieved at $u \in K_p$ *such that* $\alpha_1 \Delta u = div(\eta\chi_1)$ *in* Y, *then* $u = \frac{1}{\alpha_1}\langle \nabla\varphi, \eta\rangle$ *and* φ *the solution of* $\Delta\varphi = \chi_1 - \theta_1$ *in* Y. *Therefore, using (5.2) we get*

$$\forall \xi, \eta \in \mathbb{R}^N : \widetilde{W}(\xi) \le W_2(\xi) - \theta_1\langle \xi, \eta\rangle + \theta_1 h^*(\eta) - \frac{1}{2\alpha_2}\fint_Y |H\eta|^2\, dx\,,$$

replacing $\eta \rightleftharpoons \eta/\theta_1$, *subtracting* $\langle \xi, \eta\rangle$ *and taking sup over* $\xi \in \mathbb{R}^N$ *we obtain*

$$(W_2 - \widetilde{W})^*(\eta) \le \theta_1 h^*(\eta/\theta_1) - \frac{1}{2\alpha_2\theta_1^2}\fint_Y |H\eta|^2\, dx\,, \tag{7.4}$$

integration over S_r *and using (5.3) we have*

$$\forall r > 0 : \fint_{S_r} (W_2 - \widetilde{W})^*(\eta) \le \theta_1 f^*(r/\theta_1) - \frac{\theta_2}{2N\alpha_2\theta_1} r^2\,, \tag{7.5}$$

where f^* *is the function given by (5.8). Replacing the inequality (5.8) into (7.5) we obtain (7.1).* □

Corollary 3 *If* $\widetilde{W}$ *is isotropic, then* $\forall \eta \in \mathbb{R}^N$:

$$(W_2 - \widetilde{W})^*(\eta) \le \frac{1}{2\theta_1}\left(\frac{1}{\alpha_2 - \alpha_1} - \frac{\theta_2}{N\alpha_2}\right)|\eta|^2 - \frac{\gamma}{p(\alpha_2 - \alpha_2)^p \theta_1^{p-1}}|\eta|^p + o(\gamma^2)|\eta|^{2p-4}\,. \tag{7.6}$$

Corollary 4 *Under the same condition of theorem 1, if* $\widetilde{W}$ *is isotropic, then the upper bound obtained by theorem 6 converges, as* $\gamma \to 0$, *to the optimal upper bound of the linear composite.*

Proof *Following the same notation of corollary 2, it has been proved, see for example* [**L.C**] , *that the optimal bound on* $\widetilde{W}_L$ *satisfies* $(W_2^0 - \widetilde{W}_L)^*(\eta) \le B(\eta)$, *being* $W_2^0(z) = \frac{\alpha_2}{2}|z|^2$, *while we have found*

$$(W_2 - \widetilde{W})^*(\eta) \le B(\eta) - \gamma\mathcal{U}_1(\eta) + o(\gamma^2)|\eta|^{2p-4}\,.$$

□

Theorem 7 *Under the same hypothesis of theorem 5 and* $0 < \theta_2 < \theta_1$*, then* $\forall r > 0$*:*

$$\fint_{S_r} (W_0 - \widetilde{W})^*(\eta) \quad \leq \quad \frac{1}{2\theta_1}\left(\frac{1}{\alpha_2 - \alpha_1} - \frac{\theta_2}{N\alpha_2}\right) r^2 - \frac{\gamma}{p(\alpha_2-\alpha_1)^p\theta_1^{p-1}} r^p \qquad (7.7)$$
$$+\frac{\gamma}{p}\frac{2^p N^{p/2}}{\alpha_2^p}\theta_2 \mathcal{Z}(p) r^p + o(\gamma^2) r^{2p-4}\,,$$

being $W_0(z) = \frac{\alpha_2}{2}|z|^2 + 2^{p-1}\frac{\gamma}{p}|z|^p$*, and* $\mathcal{Z}(p) = \mathcal{C}^p(p)$*, where* $\mathcal{C}(p)$ *is the Calderon-Zygmund-Stein constant given in* [**TG**] *.*

Proof *Following the procedure of the proof of the theorem 6 we had*

$$\forall \xi, \eta \in \mathbb{R}^N : \widetilde{W}(\xi) \leq \frac{\alpha_2}{2}|\xi|^2 - \theta_1\langle \xi, \eta\rangle + \theta_1 h^*(\eta) + \inf_{u\in K_p} T_\gamma(u) \qquad (7.8)$$

where $T_\gamma(u) = \fint_Y \left[-\langle \nabla u, \eta\rangle \chi_1 + \frac{\alpha_2}{2}|\nabla u|^2 + \frac{\gamma}{p}|\nabla u + \xi|^p\right] dx$*. Since* $|\nabla u + \xi|^p \leq 2^{p-1}|\nabla u|^p + 2^{p-1}|\xi|^p$*, we get*

$$\widetilde{W}(\xi) \leq W_0(\xi) - \theta_1\langle \xi, \eta\rangle + \theta_1 h^*(\eta) + \inf_{u\in K_p} S_\gamma(u)\,,$$

where $S_\gamma(u) = \fint_Y \left[-\langle \nabla u, \eta\rangle \chi_1 + \frac{\alpha_2}{2}|\nabla u|^2 + 2^{p-1}\frac{\gamma}{p}|\nabla u|^p\right] dx$*. We will estimate* $\inf_{u\in K_p} S_\gamma(u) \leq S_\gamma(u)$ *where* u *is the minimizer of* S_0*. Therefore following the notation of theorem 6 we get* $u = \frac{1}{\alpha_2}\langle \nabla\varphi, \eta\rangle$ *and* $\inf_{u\in K_p} S_\gamma(u) \leq -\frac{1}{2\alpha_1}\fint_Y |H\eta|^2 + 2^{p-1}\frac{\gamma}{p\alpha_2^p}\fint_Y |H\eta|^p$*, thus*

$$\forall \xi, \eta \in \mathbb{R}^N : (\widetilde{W} - W_0)^*(\xi) \leq -\theta_1\langle \xi, \eta\rangle + \theta_1 h^*(\eta) - \frac{1}{2\alpha_2}\fint_Y |H\eta|^2 + \frac{\gamma 2^{p-1}}{p\alpha_2^p}\fint_Y |H\eta 1^p\,,$$

replacing $\eta \rightleftharpoons \eta/\theta_1$*, adding* $\langle \xi, \eta\rangle$ *and taking sup over* $\xi \in \mathbb{R}^N$ *we get*

$$\forall \eta \in \mathbb{R}^N : (W_0 - \widetilde{W})(\eta) \leq \theta_1 h^*(\eta/\theta_1) - \frac{1}{2\alpha_2\theta_1^2}\fint_Y |H\eta|^2 + \frac{\gamma 2^{p-1}}{p\alpha_2^p\theta_1^p}\fint_Y |H\eta|^p\,. \qquad (7.9)$$

In [**T**] *has been found* $\mathcal{Z}(p) > 0$ *such that*

$$\fint_{S_r}\fint_Y |H\eta|^p \leq N^{p/2}\theta_2\theta_1(\theta_1^{p-1} + \theta_2^{p-1})\mathcal{Z}(p) r^p\,, \qquad (7.10)$$

replacing this inequality into (7.9) after having integrated over S_r, and using $0 < \theta_2 < \theta_1$, we finally get (7.7). □

Corollary 5 *Under the same hypothesis of theorem 5 and $0 < \theta_2 < \theta_1$. If $\widetilde{W}$ is isotropic, then*

$$\forall \eta \in \mathbb{R}^N : (W_0 - \widetilde{W})^*(\eta) \leq \frac{1}{2\theta_1}\left(\frac{1}{\alpha_2 - \alpha_1} - \frac{\theta_2}{N\alpha_2}\right)|\eta|^2 - \frac{\gamma}{p(\alpha_2 - \alpha_1)^p \theta_1^{p-1}}|\eta|^p \quad (7.11)$$
$$+ \frac{\gamma}{p}\frac{2^p N^{p/2}}{\alpha_2^p}\theta_2 \mathcal{Z}(p)|\eta|^p + o(\gamma^2)|\eta|^{2p-4},$$

being $W_0(z) = \frac{\alpha_2}{2}|z|^2 + 2^{p-1}\frac{\gamma}{p}|z|^p$, and $\mathcal{Z}(p) = \mathcal{C}^p(p)$, where $\mathcal{C}(p)$ is the Calderon-Zygmund-Stein constant given in [**TG**] .

Proof *Same proof of corollary 4.* □

Summary of Bounds

Let $\widetilde{W}$ be the effective energy density of the Willis-composite, $\widetilde{W}_L$ is the effective energy density of the linear composite, and B_l, B_u are the optimal lower and upper bounds respectively of the linear composite

It is known that

$$(\widetilde{W}_L - W_1)^*(\eta) \leq B_l(|\eta|)$$

and

$$(W_2 - \widetilde{W}_L)^*(\eta) \leq B_u(|\eta|),$$

where

$$B_l(t) = \frac{1}{2\theta_2}\left(\frac{1}{\alpha_2 - \alpha_1} + \frac{\theta_1}{N\alpha_1}\right)t^2 = a_1 t^2$$

and

$$B_u(t) = \frac{1}{2\theta_1}\left(\frac{1}{\alpha_2 - \alpha_1} - \frac{\theta_2}{N\alpha_2}\right)t^2 = a_2 t^2.$$

Then the bounds for the Willis-composite can be written as:

(a) In the anisitropic Willis-composite. For all $r > 0$:

$$\fint_{S_r} (\widetilde{W} - W_1)^*(\eta) ds \leq B_l(r) + \gamma M_1(r) + o(\gamma^2)$$

and

$$\fint_{S_r} (W^0 - \widetilde{W})^*(\eta) ds \leq B_u(r) + \gamma M_2(r) + o(\gamma^2),$$

where

$$W^0(Z) = \frac{\alpha_2}{2}|Z|^2 + \left(\frac{p-1}{p}\right)(\theta_1\theta_2)^{1/2}\gamma|p|^p.$$

(b) In the isotropic Willis-composite. For all $\eta \in \mathbb{R}^N$:

$$(\widetilde{W} - W_1)^*(\eta) \leq B_l(|\eta|) + \gamma M_1(|\eta|) + o(\gamma^2) \tag{7.12}$$

and

$$(W^0 - \widetilde{W})^*(\eta) \leq B_u(|\eta|) + \gamma M_2(|\eta|) + o(\gamma^2) \tag{7.13}$$

For both cases (a) and (b) : $M_1(t) = -b_1 t^p$ and $M_2(t) = b_2 t^p$ where

$$b_1 = \frac{1}{p}(\alpha_2 - \alpha_1)^{-p}\theta_2^{p-1}$$

$$b_2 = (p-1)\alpha_2^{-p}[C(p)]^p(\theta_1\theta_2)^{1/2}\theta_1^{-p}N^p - \frac{1}{p}\theta_1^{-p+1}(\alpha_2 - \alpha_1)^{-p}$$

Summary and Conclusions

In the isotropic case the bounds (7.11) and (7.12) implies the bounds:

$$\phi_l(|\eta|,\, \gamma,\, \theta_1) \leq \widetilde{W}(|\eta|,\, \gamma,\, \theta_1) \leq \phi_u(|\eta|,\, \gamma,\, \theta_1)$$

for all $r > 0$ and $\theta_1 \in [0,1]$.

We have that

$$\phi_l(t,\, \gamma,\, \theta_1) = \frac{\alpha_1}{2}t^2 - \overline{s}t - a_1\overline{s}^2 + \gamma b_1\overline{s}^p$$

where

$$\overline{s} = \begin{cases} 0, & \text{if} \quad \theta_1 = 1 \\ (\alpha_2 - \alpha_1)^{-\frac{1}{t}}, & \text{if} \quad \theta_1 = 0 \end{cases}$$

For all $\theta_1 \in (0,1)$: $\overline{s}$ satisfies

$$pb_1\gamma\overline{s}^{p-1} - 2a_1\overline{s} + t = 0.$$

On the other hand

$$\phi_u(t,\,\gamma,\,\theta_1) = \frac{\alpha_2}{2}t^2 - + \frac{p-1}{p}\gamma(\theta_1\theta_2)^{1/2}t^p - \overline{s}t + a_2\overline{s}^2 + \gamma b_2\overline{s}^p$$

where

$$\overline{s} = \begin{cases} 0, & \text{if } \theta_1 = 0 \\ (\alpha_2-\alpha_1)^{-1}t, & \text{if } \theta_1 = 1 \end{cases}$$

for all $\theta_1 \in (0,1)$: $\overline{s}$ satisfies

$$pb_2\gamma\overline{s}^{p-1} + 2a_2\overline{s} - t = 0.$$

Notice that $\widetilde{W}(\cdot,0,\theta_1) = \widetilde{W}_l(\cdot,\theta_1)$ and that $\phi_l(\cdot,0,\cdot)$, $\phi_u(\cdot,0,\cdot)$ are the optimal, respectively, lower and upper bounds of the isotropic linear composite.

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Gaetano Tepedino Aranguren,
Departamento de Matemáticas
Facultad de Ciencias,
Universidad de los Andes
Mérida, Venezuela.

Javier Quintero C.
Área de Matemática,
Universidad Nacional Abierta de Mérida.

Eribel Marquina.
Área de Matemática,
UNEFA, Mérida.