On the history of generalized quadrangles

J. W. P. Hirschfeld

Dedicated to J. A. Thas on his fiftieth birthday

Abstract

The ovoids of the generalized quadrangle of order (4,2) are derived from properties of the cubic surface with 27 lines over the complex numbers.

1 Introduction

A *generalized quadrangle of order* (s, t) is an incidence structure of points and lines such that:
(a) there is at most one line through two points;
(b) two lines intersect in at most one point:
(c) there are $s + 1$ points on every line where $s \geq 1$;
(d) there are $t + 1$ lines through every point where $t \geq 1$;
(e) for any point P and line ℓ not containing P there exists a unique line ℓ' through P meeting ℓ.

The only book devoted exclusively to this topic is Payne and Thas [6]. It is shown in Chapter 6 that there is a unique generalized quadrangle GQ$(4, 2)$ of order $(4, 2)$, which can be represented as the 45 points and 27 lines of the Hermitian surface $\mathcal{U}_{3,4}$ with equation $x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$ over GF(4). Its properties and analogy to the configuration of 27 lines of a cubic surface over the complex numbers or, for that matter, over any algebraically closed field of characteristic zero were noted by Freudenthal [3].

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An *ovoid* of a generalized quadrangle $\mathcal{S}$ is a set $\mathcal{O}$ of points such that every line of $\mathcal{S}$ contains precisely one point of $\mathcal{O}$. For surveys of the known ovoids, see [6, §3.4] and [5, Appendix VI]. In [1], Brouwer and Wilbrink classify all the ovoids of GQ$(4,2)$; there are precisely two non-isomorphic types. Here it is shown that this classification is implicit in the properties of the 27 lines of a non-singular, non-ruled cubic surface over the complex numbers as described by Steiner [9], [10] in 1856-1857. The 27 lines were discovered by Cayley and Salmon [2], [7] in 1849. The notation used below depends on the double-six configuration found by Schläfli [8] in 1858.

2 Review of properties of the complex cubic surface

Let $\mathcal{F}$ be a non-singular, non-ruled cubic surface over the complex numbers C. The 27 lines on $\mathcal{F}$ are

$$\begin{array}{ll} a_i, & i=1,\ldots,6, \\ b_i, & i=1,\ldots,6, \\ c_{ij}=c_{ji}, & i,j=1,\ldots,6,\ i\neq j. \end{array}$$

Each line meets 10 others:

$$\begin{array}{llll} a_i & \text{meets} & b_j, c_{ij}, & j\neq i; \\ b_i & \text{meets} & a_j, c_{ij}, & j\neq i; \\ c_{ij} & \text{meets} & a_i, a_j, b_i, b_j, c_{mn}, & m,n\neq i,j. \end{array}$$

They lie in threes in 45 planes:

$$\begin{array}{lll} 15 & a_i b_j c_{ij}, & j\neq i \\ 30 & c_{ij}c_{kl}c_{mn}, & \{i,j,k,l,m,n\}=\{1,2,3,4,5,6\}. \end{array}$$

Steiner showed how to partition the 27 lines into three sets of 9; in each set of 9, the lines are the intersections of two triads of planes, known as a Steiner trihedral pair. The trihedral pairs are typically as follows:

$$\begin{array}{ccccccccccc} & T_{123} & & & & T_{12,34} & & & & T_{123,456} & \\ c_{23} & a_3 & b_2 & & a_1 & b_4 & c_{14} & & c_{14} & c_{25} & c_{36} \\ b_3 & c_{13} & a_1 & & b_3 & a_2 & c_{23} & & c_{26} & c_{34} & c_{15} \\ a_2 & b_1 & c_{12} & & c_{13} & c_{24} & c_{56} & & c_{35} & c_{16} & c_{24}. \end{array}$$

There are 20 T_{ijk}, 90 $T_{ij,kl}$, 10 $T_{ijk,lmn}$. The 120 trihedral pairs form 40 triads, each giving a trichotomy of the 27 lines:

$$\begin{array}{lll} 10 & \text{like} & T_{123}, T_{456}, T_{123,456}, \\ 30 & \text{like} & T_{12,34}, T_{34,56}, T_{56,12}. \end{array}$$

These two triads are displayed:

T_{123}			T_{456}			$T_{123,456}$		
c_{23}	a_3	b_2	c_{56}	a_6	b_5	c_{14}	c_{25}	c_{36}
b_3	c_{13}	a_1	b_6	c_{46}	a_4	c_{26}	c_{34}	c_{15}
a_2	b_1	c_{12}	a_5	b_4	c_{45}	c_{35}	c_{16}	c_{24};

$T_{12,34}$			$T_{34,56}$			$T_{56,12}$		
a_1	b_4	c_{14}	a_3	b_6	c_{36}	a_5	b_2	c_{25}
b_3	a_2	c_{23}	b_5	a_4	c_{45}	b_1	a_6	c_{16}
c_{13}	c_{24}	c_{56}	c_{35}	c_{46}	c_{12}	c_{15}	c_{26}	c_{34}.

If three coplanar lines are concurrent, the point of intersection is an *Eckardt point* or *E-point* for short. Over **C**, the maximum number of E-points is 18 and this only occurs for the *equianharmonic* surface $\mathcal{E}$, which has canonical equation

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0. \tag{1}$$

The 27 lines on $\mathcal{E}$ are now simply described from the tetrahedron of reference $\mathcal{T}$. Each of the six edges of $\mathcal{T}$ meets $\mathcal{E}$ in three points. Take any one of these points and join it to the three points on the opposite edge; the 27 lines so formed are the lines of $\mathcal{E}$. The 18 points on the edges are the E-points. In fact, the points on the edge with equation $x_i = x_j = 0$ are

$$x_i = x_j = x_k^3 + x_l^3 = 0,$$

where $\{i, j, k, l\} = \{0, 1, 2, 3\}$.

Now, consider on $\mathcal{E}$, a set $\mathcal{S}$ of 9 points meeting all the lines. Then $\mathcal{S}$ can only be a set of 9 E-points on three edges of $\mathcal{T}$ such that no two of the three edges are opposite. Hence such a set of three edges is either the three edges through a vertex of $\mathcal{T}$ or the three edges in a face of $\mathcal{T}$. Hence there are 8 distinct sets $\mathcal{S}$ on $\mathcal{E}$.

3 Ovoids on GQ$(4, 2)$

Over GF(4), a cubic surface with 27 lines is Hermitian and has canonical form $\mathcal{U}_{3,4} = \mathcal{E}$, [4, §20.3]. It has 45 points and the tangent plane at a point meets the surface in three concurrent lines; that is, each point is an E-point. The 45 points and the 27 lines form the GQ$(4, 2)$ quadrangle. An ovoid of GQ$(4, 2)$ is a set of 9 points through which all 27 lines pass. By the polarity of $\mathcal{U}_{3,4}$ this becomes a set of 9 planes containing the 27 lines. This gives the following result.

Theorem 3.1 *An ovoid of GQ$(4, 2)$ is equivalent to choosing one trihedron from each pair in a triad of Steiner trihedral pairs.*

In other words, if a triad of Steiner trihedral pairs is written out as three 3×3 matrices of lines, choose the rows or the columns of each matrix.

Theorem 3.2 *For a complex cubic surface $\mathcal{F}$, the number of ways of choosing a set of* 9 *tritangent planes covering the* 27 *lines is* 320.

Proof. Each of the 40 triads of trihedral pairs gives 8 sets of tritangent planes.

To calculate the number of ovoids on GQ(4, 2), it is necessary to consider the last paragraph of §2 as it applies to $\mathcal{U}_{3,4}$. Consider the equation 1 for $\mathcal{U}_{3,4}$. The simplex of reference is a self-polar tetrahedron. Each edge contains three points apart from the vertices. As for $\mathcal{E}$, the joins of the three points on one edge to the three points on the opposite edge give 9 lines of the surface; the other pairs of opposite edges give the total of 27 lines. Thus each self-polar tetrahedron corresponds to a triad of trihedral pairs. Also, an ovoid is equivalent to a set of three edges of a tetrahedron, no two of which are opposite; that is, such a set of three edges is either the three edges through a vertex or the three edges in face of a tetrahedron.

Each plane section of $\mathcal{U}_{3,4}$ that is not a tangent plane is a Hermitian curve consisting of 9 points which, with the lines meeting three of the nine points, form a $(9_4, 12_3)$ configuration, equivalent to the affine plane $AG(2,3)$. There are four triangles partitioning the 9 points. This means that each plane set of 9 points on $\mathcal{U}_{3,4}$ giving an ovoid will occur for 4 tetrahedra. An ovoid from three concurrent edges of a tetrahedron is uniquely defined by the vertex on the three edges. So the number of ovoids corresponding to a face of a tetrahedron is $40 \times 4/4 = 40$, and the number of ovoids corresponding to a vertex of a tetrahedron is $40 \times 4 = 160$. This gives the conclusion.

Theorem 3.3 *The number of ovoids on* $\mathcal{U}_{3,4}$ *is* 200.

References

[1] **A. E. Brouwer and H. A. Wilbrink**. Ovoids and fans in the generalized quadrangle $GQ(4,2)$. *Geom. Dedicata*, 36, pp. 121–124, 1990.

[2] **A. Cayley**. On the triple tangent planes of surfaces of the third order. *The Cambridge and Dublin Mathematical Journal*, 4, pp. 118–132, 1849.

[3] **H. Freudenthal**. Une étude de quelques quadrangles généralisés. *Ann. Mat. Pura Appl.*, 102, pp. 109–133, 1975.

[4] **J. W. P. Hirschfeld**. *Finite Projective Spaces of Three Dimensions*. Oxford University Press, 1985.

[5] **J. W. P. Hirschfeld and J. A. Thas**. Sets with more than one representation as an algebraic curve of degree three. In *Finite Geometries and Combinatorial Designs*, pages 99–110. American Mathematical Society, 1990.

[6] **S. E. Payne and J. A. Thas**. *Finite Generalized Quadrangles*, volume 104 of *Research Notes in Mathematics*. Pitman, 1984.

[7] **G. Salmon**. On the triple tangent planes to a surface of the third order. *The Cambridge and Dublin Mathematical Journal*, 4, pp. 252–260, 1849.

[8] **L. Schläfli**. An attempt to determine the twenty-seven lines upon a surface of the third order and to divide such surfaces into species in reference to the reality of the lines upon the surface. *Quart. J. Math.*, 2, pp. 55–65,110–120, 1858.

[9] **J. Steiner**. The twenty-seven straight lines on the cubic surface. *Monatsberichte der Königlichen Preussischen Akademie der Wissenchaften zu Berlin*, pages 50–61, 1856.

[10] **J. Steiner**. Über die Flächen dritten Grades. *J. Reine Angew. Math.*, 53, pp. 133–141, 1857.

J. W. P. Hirschfeld
School of Mathematical and Physical Sciences
University of Sussex
Brighton BN1 9QH
United Kingdom