On q-clan geometry, $q = 2^e$

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Dedicated to J. A. Thas on his fiftieth birthday

Abstract

Let **C** be a q-clan, $q = 2^e$, and let GQ(**C**) be the associated generalized quadrangle. Using a result from S. E. Payne and L. A. Rogers [14], we prove that there are exactly $q+1$ flocks of the quadratic cone associated with GQ(**C**), and that two of these flocks are projectively equivalent if and only if a special collineation of GQ(**C**) exists.

Moreover, the collineation group of the generalized quadrangle associated with any generalized Subiaco q-clan is investigated, and it is completely determined for a special class of these q-clans.

1 Introduction and review

For q any power of 2, W. Cherowitzo, T. Penttila, I. Pinneri and G. Royle [2] have given a most interesting construction of new infinite families of q-clans. These provide many new examples of each of the following: generalized quadrangles (GQ) with order (q^2, q) having subquadrangles of order (q, q); ovals in $\mathrm{PG}(2, q)$; flocks of a quadratic cone in $\mathrm{PG}(3, q)$; line spreads of $\mathrm{PG}(3, q)$; translation planes with dimension 2 over their kernel. In [2] the name Subiaco was given to all these objects. Apart from the Lunelli–Sce oval in $\mathrm{PG}(2, 16)$ (cf. [7]), the Subiaco ovals are the first nontranslation ovals ever found with q a square, and the Lunelli–Sce oval is obtained as a special case (cf. [2]). The Subiaco translation planes are especially interesting in that they have no Baer involutions and their elation groups have order $2q$.

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Received by the editors in February 1994

AMS Mathematics Subject Classification : Primary 51E12, Secondary 05B25

Keywords : Generalized quadrangles, q-clan geometry, flocks, translation planes

Throughout this article $q = 2^e$ and $F = GF(q)$. As this work is both a continuation and a generalization of [12], we repeat only those definitions and results needed to clarify the exact context in which we work. And since all our constructions are based on fields of characteristic 2, the definitions, notations, etc., that we use are for the most part valid **as given** only in this case.

Recall that a q-clan $\mathbf{C}$ is a set $\mathbf{C}=\left\{A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix} : t \in F\right\}$ of 2×2 matrices over F such that $A_s - A_t$ is anisotropic whenever $s, t \in F$ with $s \neq t$. Given a q-clan $\mathbf{C}$, there is a standard (cf. [6], [9]) construction of a generalized quadrangle GQ($\mathbf{C}$), associated with $\mathbf{C}$. This construction is reviewed below.

We begin with a *Fundamental Theorem* for GQ($\mathbf{C}$), so-called because of its obvious analogy with the Fundamental Theorem of projective geometry. As a consequence of the Fundamental Theorem we are able to assign to each line through the special point (∞) of GQ($\mathbf{C}$), a projective equivalence class of flocks of a quadratic cone in PG$(3, q)$ in such a way that two such lines belong to the same orbit of the collineation group of GQ($\mathbf{C}$) if and only if these two lines are assigned the same class of flocks. This general theory is then applied to a study of the full collineation group of the Subiaco GQ of order (q^2, q) and, when possible, to determine the orbits of this group on the subquadrangles of order q and their associated ovals. In [12] this project was completed for a special family with $q = 2^e$, e odd. Here we succeed for a special case with $q = 2^{2r}, r$ odd, 5 does not divide r. And we provide a great deal of information in the general case. The results obtained so far suggest that probably there is always just one orbit on the lines through the point (∞), and hence only one class of flocks associated with GQ($\mathbf{C}$). For the case studied here with $q = 2^{2r}, r$ odd, at least when 5 does not divide r we can say that there are exactly two orbits on the associated ovals, one of size $(q + 1)/5$ and one of size $4(q + 1)/5$.

This article is organized as follows. After concluding section 1 with a review of the construction of GQ($\mathbf{C}$) and its associated subquadrangles, ovals and translation planes, we devote section 2 to the Fundamental Theorem and a description of the collineation group of GQ($\mathbf{C}$). This section gives for characteristic 2 an analogue of "derivation" of flocks given in [1] for q odd. Section 3 gives the Subiaco q-clans as a specialization of a formally more general version of the type whose study was begun in [12]. In sections 4, 5 and 6 the collineation group of GQ($\mathbf{C}$) for this generalized Subiaco $\mathbf{C}$ is studied in great detail. In section 7 a special Subiaco construction with $q = 4^r, r$ odd, is studied. Finally, in section 8 we prove that the Subiaco translation planes have no Baer involutions and have elation groups of order $2q$.

Let $\mathbf{C}=\ \{A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix} : t \in F\}$ be a q-clan. Let K be the cone $K = \{(x_0, x_1, x_2, x_3) \in \mathrm{PG}(3, q) : x_1^2 = x_0x_2\}$ with vertex $P = (0, 0, 0, 1)$. Then the flock associated with $\mathbf{C}$ (cf. [18]) is the partition of $K\backslash\{P\}$ by the set of q disjoint conics that are the intersections of K with the planes in

$$\mathcal{F}(\mathbf{C}) = \{\pi_t = [x_t, y_t, z_t, 1]^T : t \in F\} \tag{1}$$

For convenience, we also refer to $\mathcal{F}(\mathbf{C})$ as the "flock of $\mathbf{C}$". To construct the generalized quadrangle GQ($\mathbf{C}$) associated with $\mathbf{C}$ we use the group $G = F^2 \times F \times F^2$

with binary operation (cf. [9], [13])

$$(\alpha, c, \beta) \cdot (\alpha', c', \beta') = (\alpha + \alpha', c + c' + \beta \circ \alpha', \beta + \beta'). \tag{2}$$

where

$$\alpha \circ \beta = \sqrt{\alpha P \beta^T} \tag{3}$$

for $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\alpha,\ \beta \in F^2$. Note that $\alpha \circ \beta = 0$ if and only if $\{\alpha, \beta\}$ is F–dependent.

Let $\tilde{F} = F \cup \{\infty\}$. The associated 4–gonal family $\mathcal{Q} = \mathcal{Q}(\mathbf{C}) = \{A(t) : t \in \tilde{F}\}$ is given by

$$\begin{aligned} A(\infty) &= \{(0, 0, \beta) \in G : \beta \in F^2\} \\ A(t) &= \{(\alpha, \sqrt{\alpha A_t \alpha^T}, y_t \alpha) : \alpha \in F^2\}\ ,\ t \in F. \end{aligned} \tag{4}$$

The center of G is $Z = \{(0, c, 0) \in G : c \in F\}$. And for $t \in \tilde{F}$, the *tangent space* of $\mathbf{C}$ at $A(t)$ is $A^*(t) = A(t)Z$. Then the standard construction of $\mathrm{GQ}(\mathbf{C})$ is as follows:

Points of $\mathrm{GQ}(\mathbf{C})$ are of three types:
(i) Elements $g = (\alpha, c, \beta)$ of G.
(ii) Cosets $A^*(t)g, \quad t \in \tilde{F}, g \in G$.
(iii) The symbol (∞).

Lines of $\mathrm{GQ}(\mathbf{C})$ are of two types:
(a) Cosets $A(t)g, \quad t \in \tilde{F}, g \in G$.
(b) Symbols $[A(t)], \quad t \in \tilde{F}$.

Incidence is defined by: the point (∞) is on the $q+1$ lines $[A(t)]$ of type (b). The point $A^*(t)g$ is on the line $[A(t)]$ and on the q lines of type (a) contained in $A^*(t)g$. The point g of type (i) is on the $q+1$ lines $A^*(t)g$ of type (a) that contain it. There are no other incidences.

The resulting point–line geometry $\mathrm{GQ}(\mathbf{C})$ is a GQ of order (q^2, q) precisely because $\mathbf{C}$ is a q-clan (cf. [6], [8], [9]). Since all the GQ considered in this work are nonclassical and derived from a q-clan, the point (∞) is the unique point fixed by all collineations (cf. [14], [16]). Moreover, right multiplication by elements of G induces a group of collineations of $\mathrm{GQ}(\mathbf{C})$ acting regularly on those points of $\mathrm{GQ}(\mathbf{C})$ not collinear with (∞), and fixing each line through (∞). Hence to determine the full collineation group $\mathcal{G}$ of $\mathrm{GQ}(\mathbf{C})$ it suffices to determine the subgroup $\mathcal{G}_0$ fixing $(0, 0, 0)$ (and of course fixing (∞)).

Recall that for $0 \neq \alpha \in F^2, \quad G_\alpha = \{a\alpha, c, b\alpha) : a, b, c \in F\}$ is a subgroup of G associated with a subquadrangle $\mathrm{GQ}(\alpha)$ of order q (cf. [12]), and hence with an oval $\mathcal{O}_\alpha$. If $\alpha = (a_1, a_2) \neq (0, 0)$, then $\mathcal{O}_\alpha$ is given by

$$\mathcal{O}_\alpha = \left\{(1, \sqrt{a_1^2 x_t + a_1 a_2 y_t + a_2^2 z_t}, y_t) : t \in F\right\} \cup \{(0, 0, 1)\} \tag{5}$$

as a set of points of $\mathrm{PG}(2, q)$.

Clearly each element of $\mathcal{G}_0$ must permute the G_α (and hence the GQ(α), $\mathcal{O}_\alpha$, respectively) among themselves, and no multiplication by a nonidentity element of G can do so. Hence $\mathcal{G}_0$ is the full group of collineations of GQ($\mathbf{C}$) that act on the set of G_α. Note: $G_\alpha = G_\beta$ iff $\{\alpha, \beta\}$ is F–dependent.

We wish to thank Tim Penttila for helpful conversations that led to the elimination of a significant error in section 7.

2 The Fundamental Theorem for GQ(C)

The theorem referred to by the title of this section consists of proposition 2.1 and 2.2 taken together. Moreover, the proof of IV.1 of [14], with only the most trivial changes in notation, just enough to reflect the change in point of view, yields a proof of the correct version for all prime powers q. However, we state it here only for $q = 2^e$ since we want to use the specific group binary operation given by equation (2). Moreover, without loss of generality, we may assume that each q-clan $\mathbf{C}=\{A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix}$: $t \in F\}$ has been normalized and indexed so that $y_t = t^{\frac{1}{2}}$ and $A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

If $\mathbf{C}' = \{A'_t = \begin{pmatrix} x'_t & y'_t \\ 0 & z'_t \end{pmatrix} : t \in F\}$ is a second (normalized!) q-clan, the same group G is used to construct both GQ($\mathbf{C}$) and GQ($\mathbf{C}'$). So points of type (i) and (iii) are denoted by $(\alpha, c, \beta) \in G$ and (∞) for both GQ. But lines of GQ($\mathbf{C}'$) are denoted $[A'(t)]$ and $A'(t)$ in the obvious manner, and points of type (ii) of GQ($\mathbf{C}'$) are denoted by $(A')^*(t)g$.

Proposition 2.1 *Let* $\theta : \mathrm{GQ}(\mathbf{C}) \to \mathrm{GQ}(\mathbf{C}')$ *be an isomorphism with* $\theta : (\infty) \mapsto (\infty)$, $\theta : [A(\infty)] \mapsto [A'(\infty)]$, $\theta : (0,0,0) \mapsto (0,0,0)$. *Then the following exist:*

(i) $\sigma \in$ *Aut*(F)

(ii) $D = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in GL(2, q)$

(iii) $0 \neq \lambda \in F$

(iv) a permutation $\pi : F \to F : t \mapsto \bar{t}$ *satisfying* $A'_{\bar{t}} \equiv \lambda D^T A_t^\sigma D + A'_{\bar{0}}$, *for all* $t \in F$ [1].

Then $\theta : \mathrm{GQ}(\mathbf{C}) \to \mathrm{GQ}(\mathbf{C}')$ *is induced by an automorphism (also denoted by θ) of G of the following form:*

$$\theta = \theta(\sigma, D, \lambda, \pi) : (\alpha, c, \beta) \mapsto \tag{6}$$
$$(\lambda^{-1}\alpha^\sigma D^{-T}, \lambda^{-\frac{1}{2}}c^\sigma + \lambda^{-1}\sqrt{\alpha^\sigma D^{-T} A'_{\bar{0}} D^{-1} (\alpha^\sigma)^T}\ , \beta^\sigma PDP + \bar{0}^{\frac{1}{2}}\lambda^{-1}\alpha^\sigma D^{-T}).$$

Conversely, given σ, D, λ, π *as described above, the map* $\theta = \theta(\sigma, D, \lambda, \pi)$ *of equation (6) induces an isomorphism from* GQ*(*$\mathbf{C}$*) to* GQ*(*$\mathbf{C}'$*) mapping* (∞), $[A(\infty)]$, $(0,0,0)$, *respectively, to* (∞), $[A'(\infty)]$, $(0,0,0)$.

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[1]Recall that $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \equiv \begin{pmatrix} r & s \\ t & u \end{pmatrix}$ means $x = r, w = u$ and $y + z = s + t$

Proof. Similar to that of IV.1 of [14]. □

Proposition 2.2 *If* $\mathbf{C}$, $\mathbf{C}'$ *are two normalized q-clans, then the flocks* $\mathcal{F}(\mathbf{C})$ *and* $\mathcal{F}(\mathbf{C}')$ *are projectively equivalent if and only if there is an isomorphism* $\theta : \mathrm{GQ}(\mathbf{C}) \to \mathrm{GQ}(\mathbf{C}')$ *mapping* (∞), $[A(\infty)]$, $(0,0,0)$, *respectively, to* (∞), $[A'(\infty)]$, $(0,0,0)$. *And any such isomorphism must be of the form given in equation (6).*

Proof. According to [5], the general semilinear transformation of $\mathrm{PG}(3,q)$ (defined as a map on planes!) which leaves invariant the cone $K : x_1^2 = x_0x_2$ is given (for planes not containing the vertex P) by

$$T_\theta : \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{pmatrix} \lambda a^2 & \lambda ab & \lambda b^2 & x_0 \\ 0 & \lambda(ad+bc) & 0 & y_0 \\ \lambda c^2 & \lambda cd & \lambda d^2 & z_0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x^\sigma \\ y^\sigma \\ z^\sigma \\ 1 \end{bmatrix}, \tag{7}$$

for arbitrary $D = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in GL(2,q)$, $\sigma \in \mathrm{Aut}(F)$, $0 \neq \lambda \in F$, $x_0, y_0, z_0 \in F$.

Suppose $\mathbf{C}$ and $\mathbf{C}'$ are two (normalized!) q-clans with associated flocks $\mathcal{F}(\mathbf{C})$ and $\mathcal{F}(\mathbf{C}')$ respectively. Then equation (7) may be interpreted to say that $\mathcal{F}(\mathbf{C})$ and $\mathcal{F}(\mathbf{C}')$ are projectively equivalent iff there are σ, D, λ, π as described above satisfying

$$\begin{bmatrix} x'_{\bar t} & y'_{\bar t} \\ 0 & z'_{\bar t} \end{bmatrix} \equiv \lambda D^T \begin{bmatrix} x_t^\sigma & y_t^\sigma \\ 0 & z_t^\sigma \end{bmatrix} D + \begin{bmatrix} x'_{\bar 0} & y'_{\bar 0} \\ 0 & z'_{\bar 0} \end{bmatrix} \forall t \in F. \tag{8}$$

□

Note that a consequence of equation (8) is that $y'_{\bar t} = \lambda\, det(D)(y_t)^\sigma + y'_{\bar 0}$, so that for $\mathbf{C}$, $\mathbf{C}'$ both normalized,

$$\bar t = [\lambda\, det(D)]^2 t^\sigma + \bar 0. \tag{9}$$

Remark. The parameters used in equations (6) and (9) are carried over from [12], where they were used because the condition (iv) of proposition 2.1 or equation (8) appears in [5]. But we might prefer to change parameters by putting $\lambda = \mu^{-1}, D = \mu B^{-T}, \Delta = \det(B)$. Then the important revised relationships become:

$$\begin{aligned} &(i) \quad A'_{\bar t} \equiv \mu B^{-1} A_t^\sigma B^{-T} + A'_{\bar 0} \\ &(ii) \quad (\alpha, c, \beta)^\theta = (\alpha^\sigma B, \mu^{\frac12} c^\sigma + \sqrt{\alpha^\sigma B A'_{\bar 0} B^T (\alpha^\sigma)^T}\ , (\mu\Delta^{-1}\beta^\sigma + \bar 0^{\frac12}\alpha^\sigma)B) \\ &(iii) \quad \bar t = (\mu\Delta^{-1})^2 t^\sigma + \bar 0. \end{aligned} \tag{10}$$

The description in equation (10) seems a little simpler to use than the traditional form in equation (6), so we use it in section 3 to give a description of all collineations of $\mathrm{GQ}(\mathbf{C})$. Unfortunately all the previously published work on collineations of $\mathrm{GQ}(\mathbf{C})$ (for both even and odd q), as well as the myriad computations we have done for $\mathbf{C}$ of generalized Subiaco type, have been based on the form given in equation (6). Hence in section 4 we revert to it.

In fact any automorphism θ of G replaces the 4–gonal family $\mathcal{Q} = \mathcal{Q}(\mathbf{C})$ with some 4–gonal family $\mathcal{Q}^\theta$. But we are especially interested in certain types of automorphisms of G that produce new 4–gonal families that can easily be seen to have associated q-clans. For the first type we revert to the general notation $A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix}$.

Shift by $s,\ s \in F$. Let

$$\tau_s : (\alpha, c, \beta) \mapsto (\alpha, c + \sqrt{\alpha A_s \alpha^T}, \beta + y_s\alpha). \tag{11}$$

The important thing is that shifting by s produces a projectively equivalent flock. The new q-clan $\mathbf{C}'$ has $A'_{\bar{t}} = A'_{t+s} = A_t + A_s$, i.e, $A'_x = A_{x+s} + A_s$. And if $\mathbf{C}$ is normalized so that $y_t = t^{\frac{1}{2}}$, then also $\mathbf{C}'$ has $y'_x = x^{\frac{1}{2}}$. Here we write $\mathbf{C}' = \mathbf{C}^{\tau_s}$ and $A'_t = A_t^{\tau_s}$. In equation (10) put $\bar{t} = t + s, \mu = 1, B = I, A'_{\bar{0}} = A_s$ to see that $\mathcal{F}(\mathbf{C})$ and $\mathcal{F}(\mathbf{C}^{\tau_s})$ are projectively equivalent.

For the next two types we really do want to assume that $\mathbf{C}$ is normalized.

Scale by $a,\ 0 \neq a \in F$. Let

$$\sigma_a : (\alpha, c, \beta) \mapsto (\alpha, a^{\frac{1}{4}}c, a^{\frac{1}{2}}c). \tag{12}$$

Here σ_a leaves $A(\infty)$ and $A(0)$ invariant, and for $t \in F$ maps

$$(\alpha, \sqrt{\alpha A_t \alpha^T}, t^{\frac{1}{2}}\alpha) \mapsto (\alpha, \sqrt{\alpha(a^{\frac{1}{2}}A_t)\alpha^T}\ , (at)^{\frac{1}{2}}\alpha),$$

so that A_t is replaced with $A'_{\bar{t}} = \begin{pmatrix} a^{\frac{1}{2}}x_t & (at)^{\frac{1}{2}} \\ 0 & a^{\frac{1}{2}}z_t \end{pmatrix} = A'_{at}$. In equation (10) put $\mu = a^{\frac{1}{2}}, B = I, \sigma = id, A'_{\bar{0}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \bar{t} = at$ to see that $\mathcal{F}(\mathbf{C})$ and $\mathcal{F}(\mathbf{C}^{\sigma_a})$ are projectively equivalent.

The flip. Let

$$\varphi : (\alpha, c, \beta) \mapsto (\beta, c + \alpha \circ \beta, \alpha). \tag{13}$$

Here $\varphi : A(\infty) \leftrightarrow A(0)$, and for $0 \neq t \in F$,

$$\varphi : (\alpha, \sqrt{\alpha A_t \alpha^T}, t^{\frac{1}{2}}\alpha) \mapsto (\gamma, \sqrt{\gamma(t^{-1}A_t)\gamma^T}\ , (t^{-1})^{\frac{1}{2}}\gamma),$$

where $\gamma = t^{\frac{1}{2}}\alpha$. So $A_t = \begin{pmatrix} x_t & t^{\frac{1}{2}} \\ 0 & z_t \end{pmatrix}$ is replaced with $A'_{\bar{t}} = \begin{pmatrix} t^{-1}x_t & (t^{-1})^{\frac{1}{2}} \\ 0 & t^{-1}z_t \end{pmatrix}$, where $\bar{t} = t^{-1}$. And the new q-clan $\mathbf{C}' = \mathbf{C}^{\varphi}$ is clearly normalized.

Flipping is the first type of automorphism of G we have considered that moves $A(\infty)$. It is clear that flipping replaces a q-clan $\mathbf{C}$ with a new q-clan $\mathbf{C}^\varphi$, but it is not clear in general whether or not $\mathcal{F}(\mathbf{C})$ and $\mathcal{F}(\mathbf{C}^\varphi)$ are equivalent. By III.3 of [11] they are equivalent for the previously known q-clans (with $q = 2^e$), and we show later that they are equivalent for all the Subiaco q-clans (cf. section 3).

Shifting, flipping and scaling provide recoordinatizations of a given generalized quadrangle GQ($\mathbf{C}$). As permutations of the indices of the lines through (∞), these recoordinatizations have the following description as linear fractional maps on $\tilde{F}$: $\tau_s : t \mapsto t + s$; $\varphi : t \mapsto t^{-1}$; $\sigma_a : t \mapsto at$. This is for all $t \in \tilde{F}$ with the usual conventions for arithmetics with ∞. Shifting, flipping (or not), shifting and scaling provide all of these Möbius transformations. Hence we recognize PGL$(2, q)$ acting on $\tilde{F} \simeq$ PG$(1, q)$. And PGL$(2, q)$ is sharply triply transitive on PG$(1, q)$. Suppose θ_1 and θ_2 are two different sequences of shifts, flips and scales that effect the same permutation on $\tilde{F}$ and replace $\mathcal{Q}(\mathbf{C})$ with $\mathcal{Q}(\mathbf{C}^{\theta_1})$ and $\mathcal{Q}(\mathbf{C}^{\theta_2})$, respectively. It would be nice to know that $\mathcal{F}(\mathbf{C}^{\theta_1})$ and $\mathcal{F}(\mathbf{C}^{\theta_2})$ are projectively equivalent. That this is so is an immediate corollary of the next theorem.

Theorem 2.3 *Let $\theta : G \mapsto G$ be an automorphism of G obtained as a finite sequence of shifts, flips and scales. Moreover, suppose $\mathbf{C}$ and $\mathbf{C}'$ are two (normalized) $q-$clans for which θ maps the $4-$gonal family $\mathcal{Q}(\mathbf{C})$ to the $4-$gonal family $\mathcal{Q}(\mathbf{C}')$ in such a way that it effects the identity permutation on $\tilde{F}$. Then θ must have the form $\theta : (\alpha, c, \beta) \mapsto (a\alpha, ac, a\beta)$ for some non zero a in F, and hence $\mathcal{Q}(\mathbf{C}) = \mathcal{Q}(\mathbf{C}')$.*

Proof. It is clear that any finite sequence θ of shifts, flips and scales leads to an automorphism of G of the form

$$\theta : (\alpha, c, \beta) \mapsto (a\alpha + b\beta, u^{\frac{1}{2}}c + \sqrt{\alpha A\alpha^T + \alpha D\beta^T + \beta C\beta^T}\ , v\alpha + w\beta).$$

Suppose also that θ fixes each element of $\tilde{F}$. Then $\theta :$ GQ$(\mathbf{C}) \mapsto$ GQ$(\mathbf{C}')$; $[A(\infty)] \mapsto [A'(\infty)]$, $(\infty) \mapsto (\infty)$ and $(0,0,0) \mapsto (0,0,0)$, so must have the form prescribed by the Fundamental Theorem, as given for example by equation (10). Then in equation (10) $\sigma = id, \bar{0} = 0$, so clearly $b = 0 = v$, $A \equiv D \equiv C \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and $B = aI$. So θ must have the form $\theta : (\alpha, c, \beta) \mapsto (a\alpha, u^{\frac{1}{2}}c, a^{-1}u\beta)$. Then $\theta : (\alpha, \sqrt{\alpha A_t\alpha^T}, t^{\frac{1}{2}}\alpha) \mapsto (a\alpha, u^{\frac{1}{2}}\sqrt{\alpha A_t\alpha^T}, a^{-1}ut^{\frac{1}{2}}\alpha)$. Put $\gamma = a\alpha$ to see that this image, which must be in $A'(\bar{t}) = A'(t)$, is $(\gamma, \sqrt{a^{-2}u\gamma A_t\gamma^T}, a^{-2}ut^{\frac{1}{2}}\gamma)$. And this is in $A'(t)$ if and only if $u = a^2$. □

Put $N = \{\theta_a : G \to G : (\alpha, c, \beta) \mapsto (a\alpha, ac, a\beta) \mid 0 \neq a \in F\}$. We define N to be the *kernel of* GQ($\mathbf{C}$).

Note. In the notation of equations (6) and (7) there is a homomorphism $T : \theta \mapsto T_\theta$ from the group $\mathcal{H}$ of collineations of GQ($\mathbf{C}$) fixing (∞), $[A(\infty)]$ and $(0,0,0)$ to the subgroup of PΓL$(4, q)$ leaving invariant the cone K and the flock $\mathcal{F}(\mathbf{C})$. The kernel of T is also N. Moreover, for *nonlinear* flocks, i.e., nonclassical $q-$clans $\mathbf{C}$, there is a result related to the preceding theorem which shows that the kernel of GQ($\mathbf{C}$) plays a role similar to one played by the kernel of a translation GQ (cf.8.5 of [15]).

Theorem 2.4 *The kernel N for a nonlinear normalized $q-$clan $\mathbf{C}$, is the group of collineations of GQ($\mathbf{C}$) fixing (∞) and $(0,0,0)$ linewise.*

Proof. Clearly each element of N fixes (∞) and $(0,0,0)$ linewise. So let θ be any collineation of GQ($\mathbf{C}$) that does so. Then for the Fundamental Theorem given in equation (10), $\sigma = id$, $\bar{0} = 0$, $u\Delta^{-1} = 1$, and $A_t \equiv uB^{-1}A_tB^{-T}$ for all $t \in F$. Suppose $B^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\Delta^{-1} = ad + bc$ and this last relation is equivalent to

$$\left(I - u \begin{pmatrix} a^2 & ab & b^2 \\ 0 & \Delta^{-1} & 0 \\ c^2 & cd & d^2 \end{pmatrix} \right) \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We claim that this implies that

$$I = u \begin{pmatrix} a^2 & ab & b^2 \\ 0 & \Delta^{-1} & 0 \\ c^2 & cd & d^2 \end{pmatrix}.$$

For if not, then there are elements $u, v, w \in F$, not all zero, for which the point $(u, v, w, 0) \in \mathrm{PG}(3, q)$ lies in each plane $\pi_t = [x_t, y_t, z_t, 1]^T$ of the flock $\mathcal{F}(\mathbf{C})$. In this case by a result of J. A. Thas [18], the flock $\mathcal{F}(\mathbf{C})$ must be linear. So we have $B^{-1} = aI$, with $u\Delta^{-1} = 1$, $u = a^{-2}$ and $\theta : (\alpha, c, \beta) \mapsto (a^{-1}\alpha, a^{-1}c, a^{-1}\beta)$. So $\theta \in N$. $\square$

We are now able to assign to each line through (∞) in GQ($\mathbf{C}$) its own class of projectively equivalent flocks. For each $s \in F$, let $i_s = \tau_s \circ \varphi$, a shift by s followed by a flip. And put $i_\infty = id$. Start with a normalized q-clan $\mathbf{C}$. For each $s \in \tilde{F}$, applying i_s to G yields a normalized q-clan $\mathbf{C}^{i_s}$. We assign to the line $[A(s)]$ the class of flocks projectively equivalent to $\mathcal{F}(\mathbf{C}^{i_s})$. One obvious goal of this section is the following basic result.

Theorem 2.5 *Let $\mathbf{C}$ be a normalized q-clan. Then there is an automorphism of GQ($\mathbf{C}$) mapping $[A(s_1)]$ to $[A(s_2)]$, $s_1, s_2 \in \tilde{F}$, if and only if the flocks $\mathcal{F}(\mathbf{C}^{i_{s_1}})$ and $\mathcal{F}(\mathbf{C}^{i_{s_2}})$ are projectively equivalent.*

Proof. If θ is an automorphism of GQ($\mathbf{C}$) mapping $[A(s_1)]$ to $[A(s_2)]$, without loss of generality we may assume θ fixes $(0,0,0)$ (we recall that we only discuss collineations fixing (∞)). Then apply proposition 2.2 to $i_{s_1}^{-1} \circ \theta \circ i_{s_2} : \mathrm{GQ}(\mathbf{C}^{i_{s_1}}) \to \mathrm{GQ}(\mathbf{C}^{i_{s_2}})$ to see that $\mathcal{F}(\mathbf{C}^{i_{s_1}})$ and $\mathcal{F}(\mathbf{C}^{i_{s_2}})$ are projectively equivalent. Conversely, if $\mathcal{F}(\mathbf{C}^{i_{s_1}})$ and $\mathcal{F}(\mathbf{C}^{i_{s_2}})$ are projectively equivalent, there is an isomorphism $\bar{\theta} : \mathrm{GQ}(\mathbf{C}^{i_{s_1}}) \to \mathrm{GQ}(\mathbf{C}^{i_{s_2}})$ of the type described in proposition 2.2. Then

$$\theta = i_{s_1} \circ \bar{\theta} \circ i_{s_2}^{-1} : \mathrm{GQ}(\mathbf{C}^{i_{s_1}}) \to \mathrm{GQ}(\mathbf{C}^{i_{s_2}}) : [A(s_1)] \mapsto [A(s_2)].$$

$\square$

Throughout the remainder of this section **C** denotes a fixed, normalized q-clan. To fix the notation, $\mathcal{G}_0$ denotes the group of all collineations of GQ(**C**) fixing the point $(0,0,0)$ (and of course the point (∞)). $\mathcal{H}$ is the subgroup of $\mathcal{G}_0$ fixing $[A(\infty)]$, and $\mathcal{M}$ the subgroup of $\mathcal{H}$ fixing $[A(0)]$. From proposition 2.1 and equation (10) we have

$$\mathcal{H} = \{\bar{\theta} = \bar{\theta}(\mu, B, \sigma, \pi) : A_{\bar{t}} \equiv \mu B^{-1} A_t^{\sigma} B^{-T} + A_{\bar{0}},\ t \in F\}, \tag{14}$$

where

$$\bar{\theta}(\mu, B, \sigma, \pi) : (\alpha, c, \beta) \mapsto (\alpha^{\sigma} B, \mu^{\frac{1}{2}} c^{\sigma} + \sqrt{\alpha^{\sigma} B A_{\bar{0}} B^T (\alpha^{\sigma})^T}, (\mu \Delta^{-1} \beta^{\sigma} + \bar{0}^{\frac{1}{2}} \alpha^{\sigma}) B)$$

and $\pi : t \mapsto \bar{t} = (\mu\Delta^{-1})^2 t^{\sigma} + \bar{0}$, $\Delta = \det(B)$.

So it is easy to write down $\mathcal{M}$.

$$\mathcal{M} = \{\bar{\theta} = \bar{\theta}(\mu, B, \sigma, \pi) : A_{\bar{t}} \equiv \mu B^{-1} A_t^{\sigma} B^{-T},\ t \in F\}, \tag{15}$$

where

$$\bar{\theta}(\mu, B, \sigma, \pi) : (\alpha, c, \beta) \mapsto (\alpha^{\sigma} B, \mu^{\frac{1}{2}} c^{\sigma}, \mu \Delta^{-1} \beta^{\sigma} B)$$

and $\pi : t \mapsto \bar{t} = (\mu\Delta^{-1})^2 t^{\sigma}$.

Fix $s \in F$. We now determine the most general collineation θ of GQ(**C**) mapping $[A(\infty)]$ to $[A(s)]$. Given such a θ, then

$$\bar{\theta} = i_{\infty}^{-1} \circ \theta \circ i_s = \bar{\theta}(\mu, B, \sigma, \pi) : \mathrm{GQ}(\mathbf{C}) \mapsto \mathrm{GQ}(\mathbf{C}^{i_s})$$

for some $\bar{\theta}(\mu, B, \sigma, \pi)$ of the type given in equation (10).

So $\theta = \bar{\theta} \circ i_s^{-1}$. Here

$$\begin{aligned} &(i) \quad i_s : (\alpha, c, \beta) \mapsto (\beta + s^{\frac{1}{2}}\alpha, c + \sqrt{\alpha A_s \alpha^T} + \alpha \circ \beta, \alpha) \\ &(ii) \quad i_s^{-1} : (\alpha, c, \beta) \mapsto (\beta, c + \sqrt{\beta A_s \beta^T} + \alpha \circ \beta, \alpha + s^{\frac{1}{2}}\beta). \end{aligned} \tag{16}$$

We have

$$\begin{aligned} \bar{\theta} = \bar{\theta}(\mu, B, \sigma, \pi) &: (\alpha, c, \beta) \mapsto \\ &(\alpha^{\sigma} B, \mu^{\frac{1}{2}} c^{\sigma} + \sqrt{\alpha^{\sigma} B A_{\bar{0}}^{i_s} B^T (\alpha^{\sigma})^T}\ , (\mu \Delta^{-1} \beta^{\sigma} + \bar{0}^{\frac{1}{2}} \alpha^{\sigma}) B). \end{aligned} \tag{17}$$

where the following hold:

$$\begin{aligned} &(i) \quad A_{\bar{t}}^{i_s} \equiv \mu B^{-1} A_t^{\sigma} B^{-T} + A_{\bar{0}}^{i_s}, \forall t \in F. \\ &(ii) \quad \pi : t \mapsto \bar{t} = (\mu\Delta^{-1})^2 t^{\sigma} + \bar{0}, \forall t \in F. \\ &(iii) \quad A_x^{i_s} = x[A_{x^{-1}+s} + A_s], \text{for } \ 0 \neq x \in F. \end{aligned}$$

Write $g_s(\alpha) = \sqrt{\alpha A_s \alpha^T}$, so $g_s(c\alpha) = c g_s(\alpha)$ and $g_s(\alpha + \beta) = g_s(\alpha) + g_s(\beta) + s^{\frac{1}{4}} \alpha \circ \beta$.

Then using equations (16) and (17) and massaging a bit, we obtain

Theorem 2.6 *There is a collineation $\theta : \mathrm{GQ}(\mathbf{C}) \to \mathrm{GQ}(\mathbf{C}) : [A(\infty)] \mapsto [A(s)]$, for a given $s \in F$, precisely when there is a $\bar{\theta} = \bar{\theta}(\mu, B, \sigma, \pi) : \mathrm{GQ}(\mathbf{C}) \mapsto \mathrm{GQ}(\mathbf{C}^{i_s})$ as in equation (17), in which case $\theta = \bar{\theta} \circ i_s^{-1}$ acts on G as*

$$\theta = (\alpha, c, \beta) \mapsto (\alpha', c', \beta') \tag{18}$$

where

$$\begin{aligned}
\alpha' &= \bar{0}^{\frac{1}{2}}\alpha^\sigma B + \mu\Delta^{-1}\beta^\sigma B, \\
c' &= \mu^{\frac{1}{2}}c^\sigma + \sqrt{\alpha^\sigma B A_{\bar{0}}^{i_s} B^T (\alpha^\sigma)^T} + (\mu\Delta^{-1}g_s(\beta^\sigma B) + \bar{0}^{\frac{1}{2}}g_s(\alpha^\sigma B) + \\
&\qquad +(1+s\bar{0})^{\frac{1}{4}}(\mu/\Delta)^{\frac{1}{2}}(\alpha^\sigma \circ \beta^\sigma), \\
\beta' &= \mu\Delta^{-1}s^{\frac{1}{2}}\beta^\sigma B + (1+s\bar{0})^{\frac{1}{2}}\alpha^\sigma B.
\end{aligned}$$

3 Generalized Subiaco form

Each Subiaco q-clan $\mathbf{C}$ consists of matrices A_t that have the following special form.

$$A_t = \begin{pmatrix} F(t) & t^{\frac{1}{2}} \\ 0 & G(t) \end{pmatrix} (t \in F), \tag{19}$$

where

$$\begin{aligned}
F(t) &= f(t)/k(t) + Ht^{\frac{1}{2}},\ 0 \neq H \in F, \\
G(t) &= g(t)/k(t) + Kt^{\frac{1}{2}},\ 0 \neq K \in F, \\
f(t) &= \sum_{i=1}^{4} a_i t^i;\ g(t) = \sum_{i=1}^{4} b_i t^i; \\
k(t) &= t^4 + c_2 t^2 + c_0.
\end{aligned}$$

Note that $\mathbf{C}$ is normalized. And $t^2 + \sqrt{c_2}t + \sqrt{c_0}$ must be irreducible over F so that $k(t) \neq 0$ for all $t \in F$. Hence $tr(c_0/c_2^2) = 1$, where $tr(x)$ denotes the absolute trace of x for $x \in F$.

Using this notation we can now give in our notation the Subiaco GQ presented in [2]:

Construction I. Let $q = 2^e$ with e odd (so $t^2 + t + 1 \neq 0$ for all $t \in F$) and $e \geq 5$ (to obtain new GQ). Put $f(t) = t^2 + t$, $g(t) = t^4 + t^3$, $k(t) = t^4 + t^2 + 1$, $H = K = 1$.

Construction II. Let $e = 2r \geq 6$, r odd. Then F contains an element w for which $\omega^2 + \omega + 1 = 0$. Put $f(t) = t^4 + \omega t^3 + \omega t^2$, $g(t) = \omega^2 t^3 + \omega^2 t^2 + \omega t$, $k(t) = t^4 + \omega^2 t^2 + 1$, $H = \omega^2$, $K = 1$.

Construction III. Let $e \geq 4$ and choose $\delta \in F$ for which both $\delta^2 + \delta + 1 \neq 0$ and $tr(1/\delta) = 1$ (so that $t^2 + \delta t + 1 \neq 0$ for all $t \in F$). Then put

$$\begin{array}{rcl} f(t) & = & \delta^2 t^4 + (\delta^2 + \delta^3 + \delta^4)t^3 + (\delta^2 + \delta^3 + \delta^4)t^2 + \delta^2 t \ , \ H = 1 \\ g(t) & = & \frac{\delta^3}{\delta^2+\delta+1} t^4 + (\delta^2 + \delta^3 + \delta^4)t^3 + (\frac{\delta^2+\delta^4}{\delta^2+\delta+1})t, \\ & & K = (\delta^{\frac{1}{2}} + \delta^{\frac{3}{2}} + \delta^{\frac{5}{2}})^{-1} \\ k(t) & = & t^4 + \delta^2 t^2 + 1. \end{array} \tag{20}$$

A q-clan $\mathbf{C}$ will be called a GS q-clan (for *Generalized Subiaco*) provided it has the form given in equation (19), and a *Subiaco q-clan* if it has the form given by any of constructions I, II and III. According to the authors of [2], constructions I and II may be transformed into special cases of construction III. Nevertheless we have found it quite helpful to consider all three, especially as we have yet to complete our study of construction III. In [12] construction I was investigated and $\mathcal{G}_0$ was determined to be transitive on the lines through (∞) as well as on the subquadrangles GQ(α). Here we determine $\mathcal{G}_0$ for construction II. As a first step we establish the claim made in section 2 that flipping preserves flocks for Subiaco GQ.

Theorem 3.1 *For each of the Subiaco GQ, the flip produces a new flock projectively equivalent to the original. The corresponding involutory collineation of GQ($\mathbf{C}$) that interchanges $[A(t)]$ and $[A(t^{-1})]$ for $t \in \tilde{F}$, for each of the constructions* I, II *and* III, *resp., is as follows :*

$$(i) \qquad (\alpha, c, \beta) \mapsto (\beta P, c + \alpha \circ \beta, \alpha P)$$

$$(ii) \qquad (\alpha, c, \beta) \mapsto (\beta \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}, c + \alpha \circ \beta, \alpha \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix})$$

$$(iii) \qquad (\alpha, c, \beta) \mapsto (\beta \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, c + \alpha \circ \beta, \alpha \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix})$$

Proof. For construction I an easy computation shows that $t^{-1}F(t) = G(t^{-1})$ and $t^{-1}G(t) = F(t^{-1})$. Hence $A'_t = A'_{t^{-1}} = \begin{pmatrix} G(t^{-1}) & (t^{-1})^{\frac{1}{2}} \\ 0 & F(t^{-1}) \end{pmatrix}$. In equation (10) put $u = 1, B = P, \sigma = id, A'_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ to see that the flocks are equivalent.

In construction II we have $t^{-1}F(t) = \omega^2 G(t^{-1})$ and $t^{-1}G(t) = \omega F(t^{-1})$. Use $\begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}\begin{pmatrix} \omega^2 G(t^{-1}) & (t^{-1})^{\frac{1}{2}} \\ 0 & \omega F(t^{-1}) \end{pmatrix}\begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix} = \begin{pmatrix} F(t^{-1}) & (t^{-1})^{\frac{1}{2}} \\ 0 & G(t^{-1}) \end{pmatrix}$ to complete the proof.

In construction III, (just for this proof), we adopt the notation $F(t) = f(t) + t^{\frac{1}{2}}, G(t) = g(t) + Kt^{\frac{1}{2}}$. Then for $0 \neq t \in F$ it is a straightforward exercise to show that $t^{-1}f(t) = g(t^{-1}) + t^{-1}g(t)$.

Then $t^{-1}F(t) = t^{-1}(f(t)+t^{\frac{1}{2}}) = g(t^{-1})+t^{-1}g(t)+(t^{-1})^{\frac{1}{2}} = f(t^{-1})+t^{-\frac{1}{2}} = F(t^{-1})$. Similarly, $t^{-1}G(t) = t^{-1}(g(t) + Kt^{\frac{1}{2}}) = f(t^{-1}) + g(t^{-1}) + g(t^{-1}) + Kt^{\frac{1}{2}} = F(t^{-1})t^{\frac{1}{2}} +$

$G(t^{-1})$. Now put $u = 1, B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \sigma = id, A'_{\bar{0}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ to complete the proof. □

4 Collineations of GS q-clans.

Throughout sections 4, 5 and 6 we assume that $\mathbf{C}$ is a GS q-clan with the notation of equation (19). Then by proposition 2.1 with $\mathbf{C} = \mathbf{C}'$,

$$\mathcal{H} = \{\theta(\sigma, D, \lambda, \pi) : A_{t^\pi} = \lambda D^T A_t^\sigma D + A_{0^\pi}\ ,\ t \in F\}. \tag{21}$$

With $D = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, 3.4 of [12] says that $\theta(\sigma, D, \lambda, \pi)$ is a collineation of GQ($\mathbf{C}$) iff the following hold for all $t \in F$

$$\begin{array}{rl} (i) & (a^2H^\sigma + ab + b^2K^\sigma)/H = ad + bc = (c^2H^\sigma + cd + d^2K^\sigma)/K. \\ (ii) & \pi : t \mapsto \bar{t} = \lambda^2(ad+bc)^2 t^\sigma + \bar{0}. \\ (iii) & f(\bar{t})k(t)^\sigma k(\bar{0}) + \lambda[a^2 f(t)^\sigma + b^2 g(t)^\sigma]k(\bar{t})k(\bar{0}) = k(\bar{t})k(t)^\sigma f(\bar{0}). \\ (iv) & g(\bar{t})k(t)^\sigma k(\bar{0}) + \lambda[c^2 f(t)^\sigma + d^2 g(t)^\sigma]k(\bar{t})k(\bar{0}) = k(\bar{t})k(t)^\sigma g(\bar{0}). \end{array} \tag{22}$$

By substituting the expression for $\bar{t}$ of equation (22) (ii) into equation (22) (iii) and (iv) we obtain two polynominal equations in t^σ having degree at most 8. Since these equations must hold for all $t \in F$, by assuming that $e \geq 5$ (the necessity of equation (22) (i) as proved in [12] required $e \geq 5$) we may compute the coefficients on $(t^\sigma)^i, 0 \leq i \leq 8$, and know that equation (22) holds iff each such coefficient is zero.

Certain expressions occur repeatedly in the coefficients of $(t^\sigma)^i$, so we adopt the following notation:

$$\begin{array}{rl} (i) & \Delta = \det(D) = ad + bc \neq 0. \\ (ii) & T = \lambda^2 \Delta^2 \neq 0. \\ (iii) & A_i = a^2 a_i^\sigma + b^2 b_i^\sigma,\ 1 \leq i \leq 4. \\ (iv) & B_i = c^2 a_i^\sigma + d^2 b_i^\sigma,\ 1 \leq i \leq 4. \end{array} \tag{23}$$

Then we compute the coefficients on $(t^\sigma)^i$ in equation (23) (iii) and (iv), respectively.

Coefficients on $(t^\sigma)^8$:

$$\begin{array}{rl} (i) & T^4(f(\bar{0}) + (a_4 + \lambda A_4)k(\bar{0})) \\ (ii) & T^4(g(\bar{0}) + (b_4 + \lambda B_4)k(\bar{0})). \end{array} \tag{24}$$

Coefficients on $(t^\sigma)^7$:

$$\begin{array}{rl} (i) & T^3 k(\bar{0})(a_3 + \lambda A_3 T) \\ (ii) & T^3 k(\bar{0})(b_3 + \lambda B_3 T). \end{array} \tag{25}$$

Coefficients on $(t^\sigma)^6$:

$$\begin{aligned} &(i) \quad T^2\{k(\bar{0})[T^2(a_4c_2^\sigma + \lambda A_2) + a_2 + \bar{0}a_3 + \lambda c_2A_4] + f(\bar{0})(T^2c_2^\sigma + c_2)\} \\ &(ii) \quad T^2\{k(\bar{0})[T^2(b_4c_2^\sigma + \lambda B_2) + b_2 + \bar{0}b_3 + \lambda c_2B_4] + g(\bar{0})(T^2c_2^\sigma + c_2)\}. \end{aligned} \tag{26}$$

Coefficients on $(t^\sigma)^5$:

$$\begin{aligned} &(i) \quad Tk(\bar{0})[\lambda A_1T^3 + a_3c_2^\sigma T^2 + \lambda c_2A_3T + a_1 + a_3\bar{0}^2] \\ &(ii) \quad Tk(\bar{0})[\lambda B_1T^3 + b_3c_2^\sigma T^2 + \lambda c_2B_3T + b_1 + b_3\bar{0}^2]. \end{aligned} \tag{27}$$

Coefficients on $(t^\sigma)^4$:

$$\begin{aligned} &(i) \quad k(\bar{0})[a_4c_0^\sigma T^4 + (\lambda c_2A_2 + c_2^\sigma(a_2 + a_3\bar{0}))T^2 + \lambda A_4k(\bar{0})] + \\ &\qquad\qquad + f(\bar{0})(T^4c_0^\sigma + c_2^{\sigma+1}T^2) \\ &(ii) \quad k(\bar{0})[b_4c_0^\sigma T^4 + (\lambda c_2B_2 + c_2^\sigma(b_2 + b_3\bar{0}))T^2 + \lambda B_4k(\bar{0})] + \\ &\qquad\qquad + g(\bar{0})(T^4c_0^\sigma + c_2^{\sigma+1}T^2). \end{aligned} \tag{28}$$

Coefficients on $(t^\sigma)^3$:

$$\begin{aligned} &(i) \quad k(\bar{0})[a_3c_0^\sigma T^3 + \lambda A_1c_2T^2 + c_2^\sigma(a_1 + a_3\bar{0}^2)T + \lambda A_3k(\bar{0})] \\ &(ii) \quad k(\bar{0})[b_3c_0^\sigma T^3 + \lambda B_1c_2T^2 + c_2^\sigma(b_1 + b_3\bar{0}^2)T + \lambda A_3k(\bar{0})]. \end{aligned} \tag{29}$$

Coefficients on $(t^\sigma)^2$:

$$\begin{aligned} &(i) \quad k(\bar{0})[c_0^\sigma(a_2 + a_3\bar{0})T^2 + \lambda A_2k(\bar{0})] + f(\bar{0})c_2c_0^\sigma T^2 \\ &(ii) \quad k(\bar{0})[c_0^\sigma(b_2 + b_3\bar{0})T^2 + \lambda B_2k(\bar{0})] + g(\bar{0})c_2c_0^\sigma T^2. \end{aligned} \tag{30}$$

Coefficients on (t^σ):

$$\begin{aligned} &(i) \quad k(\bar{0})[c_0^\sigma(a_1 + a_3\bar{0}^2)T^2 + \lambda A_1k(\bar{0})] \\ &(ii) \quad k(\bar{0})[c_0^\sigma(b_1 + b_3\bar{0}^2)T^2 + \lambda B_1k(\bar{0})]. \end{aligned} \tag{31}$$

Finally, the constant term is identically zero.

First we concentrate on the coefficients of the odd powers of t^σ. From equation (25) we have

$$\begin{aligned} &(i) \quad a_3 = \lambda A_3T \\ &(ii) \quad b_3 = \lambda B_3T \\ &(iii) \quad a_1b_3 + b_1a_3 = \lambda T(a_1B_3 + b_1A_3). \end{aligned} \tag{32}$$

In equation (27) cancel $Tk(\bar{0})$, compute $b_3(i) + a_3(ii)$, and use equation (32) to obtain

$$a_1b_3 + b_1a_3 = \lambda T^3(a_3B_1 + b_3A_1). \tag{33}$$

In equation (29) cancel $k(\bar{0})$, compute $b_3(i)+a_3(ii)$, and use equations (32) and (33) to obtain

$$(a_1b_3 + a_3b_1)(c_2^\sigma T^2 + c_2) = 0. \tag{34}$$

In equation (31) cancel $k(\bar{0})$ and compute $b_3(i) + a_3(ii)$, multiply by T^3 and use equation (33) to obtain

$$(a_1b_3 + a_3b_1)(c_0^\sigma T^4 + k(\bar{0})) = 0. \tag{35}$$

From equations (34) and (35) it is clear that it would be convenient to know that $\Delta_{13} = a_1b_3 + a_3b_1 \neq 0$. Moreover, this condition does hold for all the Subiaco q-clans. So we pause to consider this condition a little more closely.

5 Ordinary GS q-clans

Let $\mathbf{C}$ be a GS q-clan and continue to use notation adopted in section 4. For $1 \leq i < j \leq 4$, put $\Delta_{ij} = a_ib_j + a_jb_i$. We say $\mathbf{C}$ is *nonsingular* provided $\Delta_{13} \neq 0$, and *singular* otherwise. Recall that any finite sequence of shifts, flips and scales that leaves invariant $A(t)$ for three values of $t \in \tilde{F}$, must leave them all invariant and must give back the original q-clan. This means that two sequences of shifts, flips and scales that give the same Möbius transformation on $\tilde{F}$ yield the same q-clan. Hence in order to see what happens to the Δ_{ij} under such a sequence, it suffices to consider the shift τ_s, the shift–flip i_s, and the scale σ_a.

Theorem 5.1 *If* $\mathbf{C}$ *is nonsingular, then shifting and scaling each return a nonsingular* q*-clan.*

Proof. The scale σ_a fixes $A(\infty)$ and $A(0)$, and replaces the matrix

$$A_t = \begin{pmatrix} F(t) & t^{1/2} \\ 0 & G(t) \end{pmatrix} \quad \text{with} \quad A'_{at} = \begin{pmatrix} a^{\frac{1}{2}}F(t) & (at)^{1/2} \\ 0 & a^{\frac{1}{2}}G(t) \end{pmatrix},$$

where

$$a^{\frac{1}{2}}F(t) = F'(at) = \left[\sum_1^4 a_i a^{\frac{9}{2}-i}(at)^i\right] / \left[(at)^4 + a^2c_2(at)^2 + a^4c_0\right] + H(at)^{\frac{1}{2}}.$$

So with $F'(x) = f'(x)/k'(x) + Hx^{\frac{1}{2}}, f'(x) = \sum_1^4 a'_i x^i, k'(x) = x^4 + a^2c_2x^2 + a^4c_0 = x^4 + c'_2x^2 + c'_0$, and similarly for $g'(x) = \sum_1^4 b'_i x^i$, we have $a'_i = a_i a^{\frac{9}{2}-i}$, $b'_i = b_i a^{\frac{9}{2}-i}$, $c'_2 = a^2c_2, c'_0 = a^4c_0$, and

$$\Delta'_{ij} = \Delta_{ij}^{\sigma_a} = a'_ib'_j + a'_jb'_i = a^{9-i-j}\Delta_{ij}. \tag{36}$$

Then $\Delta'_{ij} \neq 0$ iff $\Delta_{ij} \neq 0$, as $a \neq 0$.

Now compute the q-clan $\mathbf{C}^{\tau_s}$ obtained from the shift τ_s, which replaces A_t with $A'_{t+s} = A_t + A_s$. Substitute $t = x + s$ in $F(t) + F(s)$ and $G(t) + G(s)$, respectively, to obtain

$$\begin{aligned} &(i) \quad (f(t)/k(t) + Ht^{\frac{1}{2}}) + (f(s)/k(s) + Hs^{\frac{1}{2}}) = f^{\tau_s}(x)/k^{\tau_s}(x) + Hx^{\frac{1}{2}} \qquad (37) \\ &(ii) \quad (g(t)/k(t) + Kt^{\frac{1}{2}}) + (g(s)/k(s) + Ks^{\frac{1}{2}}) = g^{\tau_s}(x)/k^{\tau_s}(x) + Kx^{\frac{1}{2}} \\ &(iii) \quad k^{\tau_s}(x) = x^4 + c_2^{\tau_s}x^2 + c_0^{\tau_s} \text{ with } c_2^{\tau_s} = c_2\ ,\ c_0^{\tau_s} = k(s), \end{aligned}$$

where

$$\begin{aligned}
f^{\tau_s}(x) &= x^4(a_4 + f(s)/k(s)) + a_3x^3 + x^2(a_2 + a_3s + c_2f(s)/k(s)) + x(a_1 + a_3s^2)\\
g^{\tau_s}(x) &= x^4(b_4 + g(s)/k(s)) + b_3x^3 + x^2(b_2 + b_3s + c_2g(s)/k(s)) + x(b_1 + b_3s^2)\\
k^{\tau_s}(x) &= x^4 + c_2x^2 + k(s).
\end{aligned}$$

So with a now obvious choice of notation we have

$$\begin{aligned}
&(i) \quad \Delta_{13}^{\tau_s} = \Delta_{13}, \qquad (38)\\
&(ii) \quad \Delta_{24}^{\tau_s} = \Delta_{24} + s\Delta_{34} + \frac{f(s)}{k(s)}(b_2 + b_3s + c_2b_4) + \frac{g(s)}{k(s)}(a_2 + a_3s + c_2a_4),
\end{aligned}$$

which with a little effort can be rewritten as

$$(iii)\ k(s)\Delta_{24}^{\tau_s} = c_0\Delta_{24} + s(\Delta_{12} + c_0\Delta_{34} + c_2\Delta_{14}) + s^2\Delta_{13}.$$

□

Next, compute the q-clan $\mathbf{C}^{i_s}$ obtained from the shift–flip i_s, which replaces A_t with $A^{i_s}_{(t+s)^{-1}} = (t+s)^{-1}(A_t + A_s)$. After making the substitution $t = x^{-1} + s$ and clearing terms, we find

$$\begin{aligned}
&(i) \quad a_4^{i_s} = \frac{a_1 + a_3s^2}{k(s)}, b_4^{i_s} = \frac{b_1 + b_3s^2}{k(s)}\\
&(ii) \quad a_3^{i_s} = \frac{k(s)(a_2 + a_3s) + c_2f(s)}{k(s)^2}, b_3^{i_s} = \frac{k(s)(b_2 + b_3s) + c_2g(s)}{k(s)^2}\\
&(iii) \quad a_2^{i_s} = \frac{a_3}{k(s)}, b_2^{i_s} = \frac{b_3}{k(s)} \qquad (39)\\
&(iv) \quad a_1^{i_s} = \frac{k(s)a_4 + f(s)}{k(s)^2}, b_1^{i_s} = \frac{k(s)b_4 + g(s)}{k(s)^2}\\
&(v) \quad c_0^{i_s} = k(s)^{-1}, c_2^{i_s} = c_2k(s)^{-1}\\
&(vi) \quad k(s)^3\Delta_{13}^{i_s} = c_0\Delta_{24} + s(\Delta_{12} + c_0\Delta_{34} + c_2\Delta_{14}) + s^2\Delta_{13}\\
&(vii) \quad \Delta_{24}^{i_s} = \tfrac{\Delta_{13}}{k(s)^2}.
\end{aligned}$$

From equation (39) (vi) it is not clear whether the shift–flip, applied to a nonsingular q-clan, returns a nonsingular q-clan. Define two q-clans to be *equivalent* provided their associated flocks are equivalent. So when we assigned to a line $[A(s)]$ the class of flocks equivalent to $\mathcal{F}(\mathbf{C}^{i_s})$, we could also have assigned to it the class of q-clans equivalent to $\mathbf{C}^{i_s}$. Since each Möbius transformation on $\tilde{F}$ is obtained by a unique sequence either of the form a shift, a flip, a shift and scale, or just a shift and scale, and since shifting and scaling return equivalent q-clans, and return nonsingular q-clans if the original is nonsingular, the following definition makes sense. A line $[A(s)], s \in \tilde{F}$, is *nonsingular* iff the q-clan $\mathbf{C}^{i_s}$ is nonsingular, in which case all equivalent q-clans are nonsingular. And we say that $\mathbf{C}$ is *ordinary* provided all lines through (∞) are nonsingular.

It will turn out that all Subiaco q-clans are ordinary.

For the present, however, it suffices to assume that the GS q-clan $\mathbf{C}$ has both $[A(\infty)]$ and $[A(0)]$ nonsingular. From now on we make the following

Basic assumption: $\Delta_{13} \neq 0 \neq \Delta_{24}$.

It is clear from equation (39)(vi) with $s = 0$ that this is equivalent to assuming that both $[A(\infty)]$ and $[A(0)]$ are nonsingular.

Then from equations (34) and (35) we conclude

$$\begin{array}{ll} (i) & c_2^\sigma T^2 = c_2 \text{ (or } T = c_2^{\frac{1-\sigma}{2}}) \\ (ii) & k(\bar{0}) = c_0^\sigma T^4 = c_2^2\,(c_0/c_2^2)^\sigma. \end{array} \tag{40}$$

Equation (40) (ii) is equivalent to

$$(\bar{0}^2/c_2)^2 + (\bar{0}^2/c_2) = (c_0/c_2^2)^\sigma + (c_0/c_2^2), \tag{41}$$

from which it follows that either

$$(c_0/c_2^2)^\sigma = c_0/c_2^2, \quad \text{and} \quad \bar{0} = 0 \quad \text{or} \quad \bar{0} = \sqrt{c_2}, \tag{42}$$

or

$$(c_0/c_2^2)^\sigma \neq c_0/c_2^2, \text{ and } \bar{0}^2/c_2 = (\frac{c_0}{c_2^2})^{2^0} + (\frac{c_0}{c_2^2})^{2^1} + \ldots + (\frac{c_0}{c_2^2})^{\sigma/2} + \varepsilon \tag{43}$$

where $\varepsilon = 0$ or $\varepsilon = 1$.

From equations (25), (27) and (40) we have

$$\begin{array}{ll} (i) & \lambda T^3 A_1 = a_1 + a_3\bar{0}^2 \\ (ii) & \lambda T^3 B_1 = b_1 + b_3\bar{0}^2 \\ (iii) & \lambda T A_3 = a_3 \\ (iv) & \lambda T B_3 = b_3. \end{array} \tag{44}$$

Write these out in detail and put $A = \begin{pmatrix} T^2 a_1^\sigma & T^2 b_1^\sigma \\ a_3^\sigma & b_3^\sigma \end{pmatrix}$. Then equations (25) and (27) (with equation (40) valid) are equivalent to

$$\lambda T A \begin{pmatrix} a^2 & c^2 \\ b^2 & d^2 \end{pmatrix} = \begin{pmatrix} a_1 + a_3\bar{0}^2 & b_1 + b_3\bar{0}^2 \\ a_3 & b_3 \end{pmatrix}, \tag{45}$$

which is solved by

$$\lambda \begin{pmatrix} a^2 & c^2 \\ b^2 & d^2 \end{pmatrix} = \frac{1}{T^3\Delta_{13}^\sigma} \begin{pmatrix} b_3^\sigma(a_1 + a_3\bar{0}^2) + T^2 b_1^\sigma a_3 & b_3^\sigma(b_1 + b_3\bar{0}^2) + T^2 b_1^\sigma b_3 \\ a_3^\sigma(a_1 + a_3\bar{0}^2) + T^2 a_1^\sigma a_3 & a_3^\sigma(b_1 + b_3\bar{0}^2) + T^2 a_1^\sigma b_3 \end{pmatrix}. \tag{46}$$

And it is now straightforward to verify that equations (40) and (44) imply all of equations (25), (27), (29) and (31). But we must still deal with equations (24), (26), (28) and (30). Equation (24) says

$$\begin{array}{ll} (i) & f(\bar{0}) = (a_4 + \lambda A_4)k(\bar{0}) \\ (ii) & g(\bar{0}) = (b_4 + \lambda B_4)k(\bar{0}). \end{array} \tag{47}$$

And equation (26), in the presence of equation (40), is equivalent to

$$\begin{aligned} (i) \quad & \lambda T^2 A_2 = a_2 + a_3\bar{0} + c_2(a_4 + \lambda A_4) \\ (ii) \quad & \lambda T^2 B_2 = b_2 + b_3\bar{0} + c_2(b_4 + \lambda B_4). \end{aligned} \tag{48}$$

And it is now straightforward to check that equations (28) and (30) follow from equations (40), (47) and (48). Now write out equations (47) and (48) to see that they are equivalent to

$$\begin{aligned} & \lambda \begin{pmatrix} T^2 a_2^\sigma + c_2 a_4^\sigma & T^2 b_2^\sigma + c_2 b_4^\sigma \\ c_0^\sigma T^4 a_4^\sigma & c_0^\sigma T^4 b_4^\sigma \end{pmatrix} \begin{pmatrix} a^2 & c^2 \\ b^2 & d^2 \end{pmatrix} \\ = & \begin{pmatrix} a_2 + a_3\bar{0} + c_2 a_4 & b_2 + b_3\bar{0} + c_2 b_4 \\ a_4 c_0^\sigma T^4 + f(\bar{0}) & b_4 c_0^\sigma T^4 + g(\bar{0}) \end{pmatrix}. \end{aligned} \tag{49}$$

This can be solved to yield a form that we give as part of the following theorem, whose proof is thereby complete.

Theorem 5.2 *Let* $\mathbf{C}$ *be a GS q-clan* ($q = 2^e, e \geq 5$) *given by equation (19) with* $\Delta_{13} \neq 0 \neq \Delta_{24}$. *Let* $\sigma \in Aut(F), D = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{GL}(2,q), 0 \neq \lambda \in F$. *Let* $\pi : F \to F$ *be a permutation satisfying* $\pi : t \mapsto \bar{t} = Tt^\sigma + \bar{0}$, *where* $T = \lambda^2(ad+bc)^2$. *Then the map*

$$\begin{aligned} \theta &= \theta(\sigma, D, \lambda, \pi) : G \to G : (\alpha, c, \beta) \mapsto (\lambda^{-1}\alpha^\sigma D^{-T}, \\ & \lambda^{-\frac{1}{2}} c^\sigma + \lambda^{-1}\sqrt{\alpha^\sigma D^{-T} A_{\bar{0}} D^{-1} (\alpha^\sigma)^T},\ \beta^\sigma PDP + \bar{0}^{\frac{1}{2}} \lambda^{-1} \alpha^\sigma D^{-T}) \end{aligned} \tag{50}$$

induces a collineation of $\mathrm{GQ}(\mathbf{C})$ *fixing* $(0,0,0)$, *fixing* $[A(\infty)]$, *and mapping* $[A(t)]$ *to* $[A(\bar{t})]$, *for* $t \in F$, *iff the following conditions are satisfied:*

(i) $(a^2 H^\sigma + ab + b^2 K^\sigma)/H = ad + bc = (c^2 H^\sigma + cd + d^2 K^\sigma)/K$.

(ii) $c_2^\sigma T^2 = c_2$

(iii) $k(\bar{0}) = c_0^\sigma T^4$

$$(iv) \quad \lambda \begin{pmatrix} a^2 & c^2 \\ b^2 & d^2 \end{pmatrix} \frac{1}{T^3 \Delta_{13}^\sigma} \begin{pmatrix} b_3^\sigma(a_1 + a_3\bar{0}^2) + T^2 b_1^\sigma a_3 & b_3^\sigma(b_1 + b_3\bar{0}^2) + T^2 b_1^\sigma b_3 \\ a_3^\sigma(a_1 + a_3\bar{0}^2) + T^2 a_1^\sigma a_3 & a_3^\sigma(b_1 + b_3\bar{0}^2) + T^2 a_1^\sigma b_3 \end{pmatrix} = \frac{1}{T^6 c_0^\sigma \Delta_{24}^\sigma} \begin{pmatrix} X & Y \\ W & Z \end{pmatrix}$$

where

$$\begin{aligned} X &= f(\bar{0})(T^2 b_2^\sigma + c_2 b_4^\sigma) + c_0^\sigma T^4 (T^2 a_4 b_2^\sigma + (a_2 + a_3\bar{0}) b_4^\sigma), \\ Y &= g(\bar{0})(T^2 b_2^\sigma + c_2 b_4^\sigma) + c_0^\sigma T^4 (T^2 b_4 b_2^\sigma + (b_2 + b_3\bar{0}) b_4^\sigma), \\ W &= f(\bar{0})(T^2 a_2^\sigma + c_2 a_4^\sigma) + c_0^\sigma T^4 (T^2 a_4 a_2^\sigma + (a_2 + a_3\bar{0}) a_4^\sigma), \\ Z &= g(\bar{0})(T^2 a_2^\sigma + c_2 a_4^\sigma) + c_0^\sigma T^4 (T^2 b_4 a_2^\sigma + (b_2 + b_3\bar{0}) a_4^\sigma). \end{aligned}$$

Conversely, each collineation of $\mathrm{GQ}(\mathbf{C})$ *fixing* $(0,0,0)$ *and* $[A(\infty)]$ *(and necessarily* (∞)*) must be of this form.*

It is convenient to have at hand computational information about $\mathcal{H}$.

Let $\theta_i = \theta(\sigma_i, D_i, \lambda_i, \pi_i), i = 1, 2$, be two elements of $\mathcal{H}$ (as described in equation (50). It is an easy exercise to compute the following, where $\Delta_i = \det(D_i)$, so $t^{\pi_i} = \lambda_i^2 \Delta_i^2 t^{\sigma_i} + 0^{\pi_i}, i = 1, 2$.

$$\begin{array}{ll} (i) & \theta_1 \circ \theta_2 = \theta(\sigma_1 \circ \sigma_2, D_1^{\sigma_2} D_2, \lambda_1^{\sigma_2} \lambda_2, \pi_3), \text{ where} \\ & t^{\pi_3} = (\lambda_1^{\sigma_2}\lambda_2)^2 (\Delta_1^{\sigma_2}\Delta_2)^2 t^{\sigma_1 \circ \sigma_2} + \lambda_2^2 \Delta_2^2 0^{\pi_1 \circ \sigma_2} + 0^{\pi_2} \\ (ii) & \theta(\sigma, D, \lambda, \pi)^{-1} = \theta(\sigma^{-1}, D^{-\sigma^{-1}}, \lambda^{-\sigma^{-1}}, \bar{\pi}), \text{ where} \\ & t^{\bar{\pi}} = (\lambda\Delta)^{-2\sigma^{-1}} (t^{\sigma^{-1}} + 0^{\pi\sigma^{-1}}). \end{array} \tag{51}$$

In previously published work (cf. [11], [12], [17]) the notation $\theta(\sigma, D, \lambda)$ was used to denote a collineation of the type given in equation (50). The permutation π was clearly always understood to be present. But this notation that ignores π, is really satisfactory only when π is uniquely determined by σ, D, λ. The only examples we have studied where this is not the case have $t \mapsto A_t$ an additive map, i.e., the point-line dual GQ is a TGQ (cf. [10]). In this case there are collineations fixing $[A(\infty)]$ and mapping $[A(t)] \mapsto [A(t + x)]$, for each fixed $x \in F$ and for all $t \in \tilde{F}$. And it turned out that in the work cited above it was always sufficient to determine these $\theta(\sigma, D, \lambda)$ with $0^\pi = 0$, i.e., π was always uniquely determined by σ, D and λ. We now show that for nonsingular q-clans it is always the case that π is uniquely determined by σ, D and λ whenever $\theta(\sigma, D, \lambda, \pi)$ exists.

Theorem 5.3 *For the nonsingular* GS *q-clan* $\mathbf{C}$,*if $\theta(\sigma, D, \lambda, \pi) \in \mathcal{H}$, then π is uniquely determined by σ, D and λ.*

Proof. Suppose there are two collineations $\theta(\sigma, D, \lambda, \pi_i), i = 1, 2$, with $\pi_1 \neq \pi_2$. Since $t^{\pi_i} = \lambda^2 \Delta^2 t^\sigma + 0^{\pi_i}$, clearly $0^{\pi_1} \neq 0^{\pi_2}$. Then

$$\begin{aligned} \theta(\sigma, D, \lambda, \pi_1) \circ \theta(\sigma, D, \lambda, \pi_2)^{-1} &= \theta(\sigma, D, \lambda, \pi_1) \circ \theta(\sigma^{-1}, D^{-\sigma^{-1}}, \lambda^{-\sigma^{-1}}, \bar{\pi}_2) \\ &= \theta(id, I, 1, \pi_3), \end{aligned}$$

where $t^{\bar{\pi}_2} = \lambda^{-2\sigma^{-1}} \Delta^{-2\sigma^{-1}} (t^{\sigma^{-1}} + 0^{\pi_2 \sigma^{-1}})$, so that $t^{\pi_3} = t + \lambda^{-2\sigma^{-1}} \Delta^{-2\sigma^{-1}} (0^{\pi_1} + 0^{\pi_2})^{\sigma^{-1}}$. Hence it suffices to show that if $\theta(id, I, 1, \pi)$ is a collineation, then $\pi = id$. Here we have $\pi : t \mapsto \bar{t} = t + 0^\pi = t + \bar{0}$, and $A_{\bar{t}} = A_{t+\bar{0}} \equiv A_t + A_{\bar{0}}$.

Since all these matrices are upper triangular, this means that $A_{t+\bar{0}} = A_t + A_{\bar{0}}$, which must hold for all $t \in F$. Writing out what this means for the (1,1) entries and clearing denominators, we obtain $f(t + \bar{0})k(t)k(\bar{0}) = k(t + \bar{0})f(t)k(\bar{0}) + k(t + \bar{0})k(t)f(\bar{0})$. In this last equation the coefficient on t^8 is $f(\bar{0})$, forcing $f(\bar{0}) = 0$, this implies $f(t + \bar{0})k(t) = k(t + \bar{0})f(t)$, i.e. $[f(t) + f(\bar{0}) + a_3(t^2\bar{0} + t\bar{0}^2)]k(t) = [k(t) + k(\bar{0}) + c_0]f(t)$, or $a_3(t^2\bar{0} + t\bar{0}^2)k(t) = (k(\bar{0}) + c_0)f(t)$. Here the coefficient on t^6 is $a_3\bar{0}$. So if $\bar{0} \neq 0$, then $a_3 = 0$. Similarly, if $\bar{0} \neq 0$ then $b_3 = 0$, this contradicts the assumption that $\Delta_{13} \neq 0$. Hence $\bar{0} = 0$ and $\pi = id$. $\square$

6 Collineations of special interest

Still assuming that $\mathbf{C}$ is a GS q-clan with $\Delta_{13} \neq 0 \neq \Delta_{24}$, suppose that $\theta(\sigma, D, \lambda)$ is a collineation defined by equation (50), so the conditions of theorem 5.2 are satisfied for some uniquely determined $\bar{0}$.

Lemma 6.1 *Suppose that $\bar{0} = 0$. From equation (41) we have $(c_0/c_2^2)^\sigma = c_2/c_2^2$, so $c_0^{\sigma-1} = c_2^{\frac{\sigma-1}{2}}$. Then $c_0 = k(0) = k(\bar{0}) = c_0^\sigma T^4$, so $c_0^{\frac{1-\sigma}{4}} = T = c_2^{\frac{1-\sigma}{2}}$ (by equation (50) (ii)), and $f(\bar{0}) = g(\bar{0}) = 0$. In this case the remaining conditions of theorem 5.2 become*

$$(i) \quad (a^2H^\sigma + ab + b^2K^\sigma)/H = ad + bc = (c^2H^\sigma + cd + d^2K^\sigma)/K.$$

$$(ii) \quad \lambda \begin{pmatrix} a^2 & c^2 \\ b^2 & d^2 \end{pmatrix} = \frac{c_2^{\frac{\sigma-1}{2}}}{\Delta_{13}^\sigma} \begin{pmatrix} c_2^{\sigma-1}a_1b_3^\sigma + a_3b_1^\sigma & c_2^{\sigma-1}b_1b_3^\sigma + b_3b_1^\sigma \\ c_2^{\sigma-1}a_1a_3^\sigma + a_3a_1^\sigma & c_2^{\sigma-1}b_1a_3^\sigma + b_3a_1^\sigma \end{pmatrix}$$
$$= \frac{1}{\Delta_{24}^\sigma} \begin{pmatrix} c_2^{\sigma-1}a_2b_4^\sigma + a_4b_2^\sigma & c_2^{\sigma-1}b_2b_4^\sigma + b_4b_2^\sigma \\ c_2^{\sigma-1}a_2a_4^\sigma + a_4a_2^\sigma & c_2^{\sigma-1}b_2a_4^\sigma + b_4a_2^\sigma \end{pmatrix}.$$

Lemma 6.2 *Suppose σ fixes all coefficients of f, g, k, as well as H and K. Then the following hold.*

(i) $T = 1$

(ii) If $\bar{0} = 0$, then all the conditions of theorem 5.2 are satisfied iff $b = c = 0, a = d$, and $\lambda a^2 = 1$.

(iii) If $\bar{0} \neq 0$, then $\bar{0} = \sqrt{c_2}, a = d$, and $a = bK + cH$.

Proof. From condition (ii) in theorem 5.2 we have $T = 1$. Condition (iv) in that theorem becomes

$$\lambda \begin{pmatrix} a^2 & c^2 \\ b^2 & d^2 \end{pmatrix} = \begin{pmatrix} 1 + \dfrac{a_3b_3\bar{0}^2}{\Delta_{13}} & \dfrac{b_3^2\bar{0}^2}{\Delta_{13}} \\ \dfrac{a_3^2\bar{0}^2}{\Delta_{13}} & 1 + \dfrac{a_3b_3\bar{0}^2}{\Delta_{13}} \end{pmatrix} \tag{52}$$
$$= \begin{pmatrix} 1 + \dfrac{f(\bar{0})(b_2 + c_2b_4) + c_0\sqrt{c_2}a_3b_4}{c_0\Delta_{24}} & \dfrac{g(\bar{0})(b_2 + c_2b_4) + c_0\sqrt{c_2}b_3b_4}{c_0\Delta_{24}} \\ \dfrac{f(\bar{0})(a_2 + c_2a_4) + c_0\sqrt{c_2}a_3a_4}{c_0\Delta_{24}} & 1 + \dfrac{g(\bar{0})(a_2 + c_2a_4) + c_0\sqrt{c_2}b_3a_4}{c_0\Delta_{24}} \end{pmatrix}.$$

Hence in any case $a = d$. Then condition (i) in theorem 5.2 is equivalent to $b(a + bK + cH) = 0 = c(a + bK + cH)$. If both $b = 0$ and $c = 0$, from the first equality of equation (52), then $\bar{0} = 0$, since $\Delta_{13} \neq 0$. If not, then $a = bK + cH$. And from equation (42) $\bar{0} = 0$ or $\bar{0} = \sqrt{c_2}$. □

Theorem 6.3 *If $\theta = \theta(\sigma, D, \lambda)$ is a nonidentity involution, then $\lambda = T = 1, \bar{0} = \sqrt{c_2}$ and $\sigma = id$. So lemma 6.2 applies and $\theta = \theta(id, D, 1)$ is uniquely determined, if it exists.*

Proof. Suppose that $\theta = \theta(\sigma, D, \lambda, \pi)$ is a nonidentity involution. Since $id = \theta^2 = \theta(\sigma^2, D^\sigma D, \lambda^{\sigma+1}), \sigma^2 = id$. So either $\sigma = id$ or $\sigma : x \mapsto \bar{x}$ is conjugation with respect to a subfield L of index 2 in F. First suppose $\sigma \neq id$ and put $L = \mathrm{GF}(2^r) \subseteq F = \mathrm{GF}(2^{2r})$. Put $x = (0^\pi)^2/c_2$ and $y = c_0/c_2^2$. By equation (43) we have $x = y^{2^0} + y^{2^1} + \cdots + y^{2^{r+1}} + \varepsilon$, where $\varepsilon = 0$ or $\varepsilon = 1$. So $x + \bar{x} = (y^{2^0} + y^{2^1} + \cdots + y^{2^{r-1}}) + (y^{2^r} + y^{2^{r+1}} + \cdots + y^{2^{2r-1}}) = \mathrm{tr}(y)$, since $\varepsilon + \bar{\varepsilon} = 0$. By hypothesis $\mathrm{tr}(y) = 1$, so $x + \bar{x} = 1$. From $\theta^2 = id$ we also have $D^\sigma D = I$, so $\Delta^{\sigma+1} = 1$ and $\lambda^{\sigma+1} = 1$. Hence $T^{\sigma+1} = 1$. Then $t^\pi = Tt^\sigma + 0^\pi$ and $t = (t^\pi)^\pi = T(Tt^\sigma + 0^\pi)^\sigma + 0^\pi = T^{\sigma+1}t^{\sigma^2} + T(0^\pi)^\sigma + 0^\pi$, so that $0^\pi = T(0^\pi)^\sigma$, or $T = (0^\pi)^{1-\sigma}$. By equation (40)(i), $(0^\pi)^{1-\sigma} = c_2^{\frac{1-\sigma}{2}}$, or $((0^\pi)^2/c_2)^{1-\sigma} = 1$, i.e., $x = x^\sigma = \bar{x}$, contradicting $x + \bar{x} = 1$. Hence $\sigma = id$, implying $\Delta = 1 = \lambda$ and also lemma 6.2 applies. If $0^\pi = 0$, by 6.2 (ii) $D = I$, forcing $\theta = id$. Hence $0^\pi = \sqrt{c_2}$. □

The preceding two results have interesting consequences for the Subiaco GQ.

Theorem 6.4 *Let* **C** *be a Subiaco q-clan as given by construction* I, II *or* III. *Let L be the smallest subfield of F containing c_2, and put $r = [F : L]$ so r is odd. Then the subgroup $\mathcal{M}$ of $\mathcal{G}_0$ fixing $[A(\infty)]$ and $[A(0)]$ is given by*

$$\begin{aligned} \mathcal{M} = \ & \{\theta(\sigma, aI, a^{-2}) : (\alpha, c, \beta) \mapsto (a\alpha^\sigma, ac^\sigma, a\beta^\sigma) : \\ & 0 \neq a \in F, \sigma \in \mathrm{Gal}(F/L)\}. \end{aligned} \tag{53}$$

So the order of $\mathcal{M}$ is $(q-1)r$. In construction I, $q = 2^r$; *in construction* II, $q = 4^r$; *in construction* III *the most we can say is that $q = 2^{ir}$ with r odd.*

Proof. Let $\theta \in \mathcal{G}_0$ fix $[A(\infty)]$ and $[A(0)]$, so θ is of the type covered by lemma 6.1. Since $c_0 = 1$ for all Subiaco GQ, $\bar{0} = 0$ implies $T = 1$ and $c_2^\sigma = c_2$. But for all Subiaco q-clans, $c_2^\sigma = c_2$ implies that σ fixes all coefficients of f, g, k as well as H and K, so that lemma 6.2 also applies. And lemma 6.2 (ii) explicitly gives the form of $\theta \in \mathcal{M}$. Since $\mathrm{tr}(c_0/c_2^2) = \mathrm{tr}(1/c_2) = 1, r = [F : L]$ must be odd. □

At this point we have nothing to add to the results presented in [12] for construction I, however, using the results of this section they are now rather easy to obtain.

7 Construction II

Here $q = 2^e, e = 2r, r$ odd, $r \geq 3$. And $w \in F$ satisfies $w^2 + w + 1 = 0$.

$$\left\{ \begin{array}{llll} a_4 = 1 & b_4 = 0 & c_0 = 1 & \Delta_{13} = \Delta_{24} = w^2 \\ a_3 = w & b_3 = w^2 & c_2 = w^2 & \Delta_{12} = \Delta_{34} = w^2 \\ a_2 = w & b_2 = w^2 & H = w^2 & \Delta_{14} = w \\ a_1 = 0 & b_1 = w & K = 1 & \end{array} \right. \tag{54}$$

Lemma 7.1 *The group $\mathcal{H}$ consists of four cosets of $\mathcal{M}$, where $\mathcal{M}$ is given by equation* (53). *So $|\mathcal{H}| = 4(q-1)r = 2e(q-1)$.*

Proof. Suppose $\theta(\sigma, D, \lambda) \in \mathcal{H}$. There are two cases.

Case 1. $w^\sigma = w$, i.e., $\sigma \in \mathrm{Gal}(F/L)$. Here $T = c_2^{\frac{1-\sigma}{2}} = 1$, $k(\bar{0}) = c_0^\sigma T^4 = 1$. So $\bar{0} = 0$ or $\bar{0} = \sqrt{c_2} = w$, by lemma 6.2. The case $\bar{0} = 0$ was finished by theorem 6.4, giving exactly the group $\mathcal{M}$. And if $\bar{0} = w$, then $f(0) = 1$, $g(0) = w$, and we can use equation (52) to finish off Case 1. The unique nonidentity involution fixing $[A(\infty)]$ is $I_\infty = \theta(id, \begin{pmatrix} 0 & w^2 \\ w & 0 \end{pmatrix}, 1)$, and the coset $\mathcal{M}I_\infty$ consists of all collineations in $\mathcal{H}$ mapping $[A(0)]$ to $[A(w)]$.

Case 2. $w^\sigma = w^2$, so $t^\sigma = t^2$ for all $t \in L$. Here $T = c_2^{\frac{1-\sigma}{2}} = w^2$, $\Delta_{13}^\sigma = w$, $\Delta_{24}^\sigma = w$. Then $k(\bar{0}) = c_0^\sigma T^4 = w^2$ implies $\bar{0} = w^2 + w\varepsilon$ for $\varepsilon \in \{0, 1\}$. And also $f(\bar{0}) = w + w^2\varepsilon$, $g(0) = w^2 + \varepsilon$. According to both matrices in condition (iv) in theorem 5.2 we have

$$\lambda \begin{pmatrix} a^2 & c^2 \\ b^2 & d^2 \end{pmatrix} = \begin{pmatrix} w + \varepsilon & 1 + w\varepsilon \\ 1 + w\varepsilon & 1 + w^2\varepsilon \end{pmatrix}. \tag{55}$$

Hence $b = c$ and $\lambda a^2 = w\lambda d^2$, implying $a = w^2 d$. The equalities of condition (i) in lemma 6.2 imply that $a^2 w + ab + b^2 = w^2(ad + bc) = w^2(c^2 w + cd + d^2)$. These equalities with $b = c$ and $a = w^2 d$ are equivalent to $d^2 + w^2 db + wb^2 = 0$, or $(wd/b)^2 + (wd/b) + 1 = 0$. So $wd/b = w$ or $wd/b = w^2$. Since $T = \lambda^2(ad + bc)^2 = w^2$, we can determine two coset representatives as follows: $\theta(2, \begin{pmatrix} w & w^2 \\ w^2 & w^2 \end{pmatrix}, w^2)$ maps $[A(t)]$ to $[A(\bar{t})]$ where $\bar{t} = w^2 t^2 + w^2$, and $\theta(2, \begin{pmatrix} 1 & 1 \\ 1 & w \end{pmatrix}, w^2)$ maps $[A(t)]$ to $[A(\bar{t})]$ where $\bar{t} = w^2 t^2 + 1$.

Put $\psi = \theta(2, \begin{pmatrix} w^2 & 1 \\ 1 & 1 \end{pmatrix}, 1) \in \mathcal{M}\theta(2, \begin{pmatrix} w & w^2 \\ w^2 & w^2 \end{pmatrix}, w^2)$. Then $I_\infty \psi = \psi I_\infty \in \mathcal{M}\theta(2, \begin{pmatrix} 1 & 1 \\ 1 & w \end{pmatrix}, w^2)$, and we have the following coset decomposition of $\mathcal{H}$.

$$\mathcal{H} = \mathcal{M} \cup \mathcal{M}I_\infty \cup \mathcal{M}\psi \cup \mathcal{M}I_\infty \psi. \tag{56}$$

□

Corollary 7.2 *The stabilizer $\mathcal{H}$ of $[A(\infty)]$ in $\mathcal{G}_0$ has an orbit of length 4 consisting of $[A(t)]$ with $t \in \mathrm{GF}(4)$.*

We now want to determine the unique involution (if there is one) fixing the line $[A(s)]$ for each $s \in F$. Note that the unique (nonidentity) involution fixing $[A(\infty)]$ is obtained by putting $\sigma = id$ and $b = w$ in equation (56). After shifting by s and flipping we obtain the following new parameters for $\mathbf{C}^{i_s}$ using equation (39) (write $x^{i_s} = x'$ for simplicity).

$$
\begin{aligned}
&a_4' = \frac{ws^2}{k(s)},\ b_4' = \frac{(w^2s^2+w)}{k(s)},\ H' = w^2 \\
&a_3' = \frac{(ws^5+s^4+ws+w)}{k(s)^2},\ b_3' = \frac{(w^2s^5+w^2s^4+ws+w^2)}{k(s)^2},\ K' = 1 \\
&a_2' = \frac{w}{k(s)},\ b_2' = \frac{w^2}{k(s)},\ c_2' = \frac{w^2}{k(s)} \\
&a_1' = \frac{(ws^3+s^2+1)}{k(s)^2},\ b_1' = \frac{(w^2s^3+w^2s^2+ws)}{k(s)^2},\ c_0' = \frac{1}{k(s)} \\
&\Delta_{13}' = \frac{w^2}{k(s)^{5/2}},\ \Delta_{24}' = \frac{w^2}{k(s)^2}
\end{aligned}
\tag{57}
$$

Put $v = \sqrt{k(s)} = s^2+ws+1$. Then the unique (nonidentity) involution θ_s of $\mathrm{GQ}(\mathbf{C}^{i_s})$ fixing $[A^{i_s}(w)]$ and $(0,0,0)$ can be determined and shown to exist by a routine application of lemma 6.2 and theorem 6.3, and we get:

Lemma 7.3 *$\theta_s = \theta(id, D, 1)$, with $\bar{0} = \sqrt{c_2'} = w/v$, and*

$$
D = D_s = \frac{1}{v^{5/2}} \begin{pmatrix} w^2s^4+w^2s & w^2s^5+w^2s^4+ws+w^2 \\ ws^5+s^4+ws+w & w^2s^4+w^2s \end{pmatrix}. \tag{58}
$$

Since $a = d, D = D^{-1},\ \det(D) = 1$ and $PDP = D^{-T}$, it follows that

$$
\theta_s : (\alpha, c, \beta) \mapsto (\alpha D^T, c + \sqrt{\alpha D^T A_{a'} D \alpha^T}, (\beta + \sqrt{\bar{0}}\alpha)D^T)\ ,\ \bar{0} = w/v. \tag{59}
$$

Now using equations (16) and (59) we may compute the involution I_s in $\mathcal{G}_0$ that fixes $[A(s)]$:

$$
\begin{aligned}
I_s \ &= \ i_s \circ \theta_s \circ i_s^{-1} : (\alpha, c, \beta) \mapsto \\
&([(1+\bar{0}\sqrt{s})\alpha + \bar{0}\beta]D^T, -, [s\sqrt{\bar{0}}\alpha + (1+\sqrt{s\bar{0}})\beta]D^T)\ ,\ \bar{0} = w/v.
\end{aligned}
\tag{60}
$$

where we could compute the middle coordinate if we wanted to.

Described as a Möbius transformation on the elements of $\tilde{F}$ as indices of the lines through ∞, we have

$$
I_s : t \mapsto ((t+s)^{-1} + w/v)^{-1} + s = \frac{(s^2+1)t + ws^2}{wt + s^2 + 1}. \tag{61}
$$

In particular,

$$
I_s : \infty \mapsto (s^2+1)w^2. \tag{62}
$$

This completes a proof of the following:

Theorem 7.4 *The group $\mathcal{G}_0$ of collineations of the* GQ(**C**) *of construction* II *that fix $(0,0,0)$ is transitive on the lines through (∞) and has order $2e(q^2-1)$ ($q = 2^e, e = 2r, r$ odd). This implies that only one class of flocks arises from the q-clan* **C**.

However, we shall now show that $\mathcal{G}_0$ (and hence $\mathcal{G}$) is not transitive on the subquadrangles $\mathrm{GQ}(\alpha)$ of order q. (Recall the description of $\mathrm{GQ}(\alpha)$ given near the end of section 1.) To indicate the action of $\mathcal{G}_0$ on the subgroups G_α it suffices to indicate the action of $\mathcal{G}_0$ on the α considered as points in $\mathrm{PG}(1,q)$. If $\sigma \in \mathrm{Aut}(F)$, and if $x^\sigma = x^2$ for all $x \in F$, we write $\alpha^{(2)}$ to indicate α^σ. And since $\mathcal{G}_0 = \bigcup\{\mathcal{H}I_t \mid t \in \tilde{F}\}$, in light of equation (56) it suffices to consider the action on $\alpha \in \mathrm{PG}(1,q)$ of $\mathcal{M}$ along with the action on α of each coset representative.

$$
\begin{array}{ll}
(i) & \mathcal{M} \ni \theta(\sigma, aI, a^{-2}) : \alpha \mapsto \alpha^\sigma \ (0 \neq a \in F; \sigma \in \mathrm{Gal}(F/L)) \\
(ii) & I_\infty : \alpha \mapsto \alpha \begin{pmatrix} 0 & w \\ w^2 & 0 \end{pmatrix} \equiv \alpha \begin{pmatrix} 0 & w^2 \\ 1 & 0 \end{pmatrix} \\
(iii) & \psi : \alpha \mapsto \alpha^{(2)} \begin{pmatrix} w^2 & w^2 \\ w^2 & w \end{pmatrix} \\
(iv) & I_\infty\psi : \alpha \mapsto [\alpha \begin{pmatrix} 0 & w^2 \\ 1 & 0 \end{pmatrix}]^{(2)} \begin{pmatrix} w^2 & w^2 \\ w^2 & w \end{pmatrix} = \alpha^{(2)} \begin{pmatrix} 1 & w^2 \\ w^2 & w^2 \end{pmatrix} \\
(v) & I_s : \alpha \mapsto \alpha \begin{pmatrix} a(s) & b(s) \\ c(s) & a(s) \end{pmatrix}, \ s \in F \\
(vi) & I_\infty I_s : \alpha \mapsto \alpha \begin{pmatrix} w^2c(s) & w^2a(s) \\ a(s) & b(s) \end{pmatrix}, \ s \in F \\
(vii) & \psi I_s : \alpha \mapsto \alpha^{(2)} \begin{pmatrix} w^2(a(s)+c(s)) & w^2(a(s)+b(s)) \\ w^2a(s)+wc(s) & w^2b(s)+wa(s) \end{pmatrix}, \ s \in F \\
(viii) & I_\infty\psi I_s : \alpha \mapsto \alpha^{(2)} \begin{pmatrix} a(s)+w^2c(s) & w^2a(s)+b(s) \\ w^2(a(s)+c(s)) & w^2(a(s)+b(s)) \end{pmatrix}, s \in F
\end{array}
\tag{63}
$$

To determine the orbits of $\mathcal{G}_0$ on $\mathrm{PG}(1,q)$ we need more information about the matrices $D_s^T = \begin{pmatrix} a(s) & b(s) \\ c(s) & a(s) \end{pmatrix}$, where $v(s) = s^2 + ws + 1$ and

$$
\begin{array}{ll}
(i) & a(s) = (w^2s^4 + w^2s)/v(s)^{5/2} \\
(ii) & b(s) = (ws^5 + s^4 + ws + w)/v(s)^{5/2} \\
(iii) & c(s) = (w^2s^5 + w^2s^4 + ws + w^2)/v(s)^{5/2}.
\end{array}
\tag{64}
$$

From condition (i) in theorem 5.2 there are two basic relationships for $a = a(s)$, $b = b(s)$, $c = c(s)$:

$$
\begin{array}{ll}
(i) & a^2 + 1 = bc \\
(ii) & a + b + w^2c = 0
\end{array}
\tag{65}
$$

In the computations that follow, there are several times when we need to know that some 5th degree polynominal with no root in $\mathrm{GF}(4)$ is actually irreducible over $\mathrm{GF}(4)$. To do this we show that the remainder after division by a general irreducible quadratic polynominal over GF(4) cannot be zero. The general (monic) irreducible quadratic polynominal over GF(4) has the form

$$d(x) = x^2 + px + p^2q, \text{ where } p = 1, w \text{ or } w^2 \text{ and } q = w \text{ or } w^2. \tag{66}$$

So $p^3 = 1 = q^3$ and $q^2+q+1 = 0$. In what follows $d(x)$ is always such a polynominal and the dividend always has no root in $GF(4)$.

$$\begin{array}{ll}
(i) & a(s) = 0 \iff w^2s^4 + ws = 0 \iff s \in \mathrm{GF}(4) \\
(ii) & b(s) = 0 \iff s^5 + w^2s^4 + s + 1 = d(s)[s^3 + (p + w^2)s^2 + \\
& (p^2q^2 + pw^2)s + 1 + p^2q^2w^2] + ws + p^2q + pw^2 = 0 \\
(iii) & c(s) = 0 \iff a(s) = b(s) \iff 0 = s^5 + s^4 + w^2s + 1 = \\
& d(s)[s^3 + (1 + p)s^2 + (p + p^2q^2)s + 1 + p^2q^2] + ws + 1 + p + p^2q \\
(iv) & a(s) = c(s) \iff 0 = s^5 + ws + 1 = \\
& d(s)[s^3 + ps^2 + p^2q^2s + 1] + ws + 1 + p^2q.
\end{array} \tag{67}$$

It is a curious fact that in each division the coefficient on s in the remainder is w, which of course is not zero.

Immediately from equations (65) and (67) we have

$$\begin{array}{ll}
(i) & a(s) = 0 \iff s \in \mathrm{GF}(4), \\
& \text{giving 4 values of } s \text{ for which } (a(s), b(s), c(s)) = (0, w, w^2) \\
(ii) & b(s) = 0 \text{ has no solution if } 5 \not| r. \\
& \text{If } 5|r, \text{ there are 5 values of } s \text{ for which } (a(s), b(s), c(s)) = (1, 0, w) \\
(iii) & c(s) = 0 \text{ has no solution if } 5 \not| r. \\
& \text{If } 5|r, \text{ there are 5 values of } s \text{ for which } (a(s), b(s), c(s)) = (1, 1, 0).
\end{array} \tag{68}$$

From equation (65) we note that $a = c$ iff $b = wa$ and $b = c$ iff $a = wc$. It is also easy to check that:

$$\begin{array}{ll}
(i) & a(s + w) = a(s) \\
(ii) & b(s + w) = b(s) + a(s) \\
(iii) & c(s + w) = c(s) + wa(s).
\end{array} \tag{69}$$

Then using equations (65), (67) and (69) it is easy to verify that

$$a(s) = c(s) \iff b(s + w) = c(s + w). \tag{70}$$

And this pair of equations has no solution if $5 \not| r$. If $5|r$, there are five values of s for which both equations hold.

Theorem 7.5 *The stabilizer of $\alpha = (1, w)$ is $\mathcal{H} \cup (\cup\{\mathcal{H}I_s : s \in \mathrm{GF}(4)\})$, with order $20r(q - 1)$. So the $\mathcal{G}_0$-orbit of $(1, w)$ has size $4r(q^2 - 1)/20r(q - 1) = (q + 1)/5$, and it consists of all α of the form $(a(s) + wc(s), b(s) + wa(s))$, $s \in F$.*

Proof. Using equation (63) (i),(ii),(iii) and (iv) it is easy to check that the stabilizer of $\alpha = (1, w)$ includes $\mathcal{H}$. Then $I_s : (1, w) \mapsto (a(s) + wc(s),\ b(s) + wa(s)) \equiv (1, w)$ iff $b(s)+wa(s) = w(a(s)+wc(s))$ iff $b(s) = w^2c(s)$. But $a(s)+b(s)+w^2c(s) = 0$ by equation 67(ii), so $\mathcal{H}I_s$ fixes $(1, w)$ iff $a(s) = 0$ iff $s \in \mathrm{GF}(4)$ by equation (69)(i). □

Theorem 7.6 *If* $5 \not| \ e$, $\alpha = (0, 1)$ *belongs to an orbit of length* $4(q + 1)/5$. *Hence* $\mathcal{G}_0$ *has exactly two orbits on* $PG(1, q)$, *and therefore on ovals.*

Proof. Using equation (63) we see that $\mathcal{M}$ stabilizes $\alpha = (0, 1)$, but I_∞, ψ and $I_\infty\psi$ do not. I_s stabilizes $(0, 1)$ iff $c(s) = 0$. $I_\infty I_s$ stabilizes $(0, 1)$ iff $a(s) = 0$ iff $s \in \mathrm{GF}(4)$. ψI_s stabilizes $(0, 1)$ iff $c(s) = wa(s)$ iff $b(s) = 0$. And $I_\infty\psi I_s$ stabilizes $(0, 1)$ iff $a(s) = c(s)$. So using equation (69) we have the following: if $5 \not| \ e$, the stabilizer of $(0, 1)$ consists of $\mathcal{M} \cup \{\mathcal{M}I_\infty I_s : s \in \mathrm{GF}(4)\}$ and has order $5r(q - 1)$. So its orbit has length $4r(q^2 - 1)/5r(q - 1) = 4(q + 1)/5$. □

We conjecture that for $e \equiv 2 \pmod 4$ even when $5|r$ there will be a long orbit of length $4(q + 1)/5$.

8 The Subiaco translation planes

Let $PG(3, q)$ be the projective space associated with the 4−dimensional vector space $F^2 \times F^2$. Define $l_{t,u} = \{(\alpha, \alpha \begin{pmatrix} x_t & y_t + u \\ u & -z_t \end{pmatrix}) \mid \alpha \in F^2\}$ $(t, u \in F)$ and $l_\infty = \{(0, \alpha) \mid \alpha \in F^2\}$. Let

$$\mathcal{S}(\mathbf{C}) = \{l_{t,u} \mid t, u \in F\} \cup \{l_\infty\}. \tag{71}$$

As $\mathbf{C}$ is a q-clan, we can prove with a direct calculation that $\mathcal{S}(\mathbf{C})$ is a spread of PG$(3, q)$. If $\mathcal{R}_t = \{l_{t,u} \mid u \in F\} \cup \{l_\infty\}(t \in F)$, then $\mathcal{R}_t$ is a regulus and $\mathcal{S}(\mathbf{C})= \cup_{t\in F}\mathcal{R}_t$ is the union of q reguli, each two meeting in the line l_∞.

We point out that $\mathcal{S}(\mathbf{C})$ is not the spread $\mathcal{S}$ associated with the flock $\mathcal{F}(\mathbf{C})$ using the Klein correspondence, whose lines are $l_\infty = \{(0, \alpha) \mid \alpha \in F^2\}$ and $m_{t,u} = \{(\alpha, \alpha \begin{pmatrix} y_t + u & -z_t \\ x_t & u \end{pmatrix}) \mid \alpha \in F^2\}$ $(t, u \in F)$ (see [3] for more details). But the collineation $\tau : (\alpha, (b_1, b_2)) \mapsto (\alpha, (b_2, b_1))$ transforms $\mathcal{S}(\mathbf{C})$ into $\mathcal{S}$. We will say that $\mathcal{S}(\mathbf{C})$ is the spread associated with $\mathbf{C}$.

Proposition 8.1 *If* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *are two q-clans, then* $\mathcal{S}(\mathbf{C}_1)$ *and* $\mathcal{S}(\mathbf{C}_2)$ *are isomorphic if and only if* $\mathcal{F}(\mathbf{C}_1)$ *and* $\mathcal{F}(\mathbf{C}_2)$ *are projectively equivalent.*

Proof. For $i = 1, 2$ let $\mathcal{S}_i$ be the spread associated with the flock $\mathcal{F}(\mathbf{C}_i)$ using the Klein correspondence. As $\mathcal{S}(\mathbf{C}_i)\tau = \mathcal{S}_i$, the spreads $\mathcal{S}(\mathbf{C}_1)$ and $\mathcal{S}(\mathbf{C}_2)$ are isomorphic if and only if $\mathcal{S}_1$ and $\mathcal{S}_2$ are isomorphic. By [3] $\mathcal{S}_1 \simeq \mathcal{S}_2$ if and only if $\mathcal{F}(\mathbf{C}_1)$ and $\mathcal{F}(\mathbf{C}_2)$ are projectively equivalent. □

We will denote by $T(\mathbf{C})$ the affine translation plane constructed using the spread $\mathcal{S}(\mathbf{C})$, whose points are the vectors of $F^2 \times F^2$ and whose lines are the cosets $A+v$ ($v \in F^2 \times F^2$), where A is any vector subspace of $F^2 \times F^2$ which defines a line of $\mathcal{S}(\mathbf{C})$. Denote by C the translation complement of $T(\mathbf{C})$. If τ_X denotes the collineation of $\mathrm{PG}(3,q)$ defined by the non-singular semilinear map $X : F^2 \times F^2 \to F^2 \times F^2$, then $C = \{X \in \Gamma L(4,q) \mid \mathcal{S}(\mathbf{C})\tau_X = \mathcal{S}(\mathbf{C})\}$. The subgroup H of $\mathrm{P\Gamma L}(4,q)$ defined by C is the stabilizer of $\mathcal{S}(\mathbf{C})$ in $\mathrm{P\Gamma L}(4,q)$.

For $u \in F$, let $g_u : F^2 \times F^2 \to F^2 \times F^2$ be the linear map defined by $g_u : (\alpha,\beta) \mapsto (\alpha, \alpha \begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix} + \beta)$ and let $E = \{g_u \mid u \in F\}$. Then each element of E fixes $\mathcal{S}(\mathbf{C})$ and each of the reguli $\mathcal{R}_t$ for any $t \in F$. Moreover E is an elation group of $T(\mathbf{C})$ with axis the line $\{(0,\alpha) \mid \alpha \in F^2\}$.

If $\mathcal{S}(\mathbf{C})$ is not a regular spread (i.e., $\mathrm{GQ}(\mathbf{C})$ is not isomorphic to the classical generalized quadrangle $H(3,q^2)$) and $\bar{H}$ is the stabilizer of $\mathcal{F}(\mathbf{C})$ in $\mathrm{P\Gamma L}(4,q)$, then there is a surjective homomorphism $^- : H \to \bar{H}$ whose kernel is E such that:

(1) if $g \in H \cap \mathrm{PGL}(4,q)$, then $\bar{g} \in \bar{H} \cap \mathrm{PGL}(4,q)$;

(2) an element g of H fixes all the reguli $\mathcal{R}_t$ if and only if $\bar{g}$ acts as the identity over $\mathcal{F}(\mathbf{C})$ (see [1] corollary 3, we point out that the proof of theorem 2 and corollary 3 of [1] does not depend on the characteristic.).

Theorem 8.2 *If $\mathbf{C}$ is a Subiaco q-clan, then the plane $T(\mathbf{C})$ has an elation group of order $2q$ and no Baer-involution.*

Proof. If $\mathbf{C}$ is a Subiaco q-clan obtained with the construction III, then using lemma 6.2 and theorem 6.3 one may compute the unique involution I_∞ fixing $[A(\infty)]$ to be $I_\infty = \theta(id, \begin{pmatrix} a & c \\ b & d \end{pmatrix}, 1)$ with $\bar{0} = \delta$ and $a = d = 1 + K^{-1}$, $b = c = K^{-1}$. (In checking that this collineation really exists, we find that $f(\bar{0}) = g(\bar{0}) = \delta^3 + \delta^4 + \delta^6 + \delta^7$ and leave the remaining details to the reader.)

Therefore there is always a non-identity involution θ of $\mathrm{GQ}(\mathbf{C})$ fixing $[A(\infty)]$.

By theorem 6.3 θ is uniquely defined and $\theta = \theta(\sigma, D, \lambda)$ with $\sigma = id$, $\lambda = 1$ and $\theta : A(0) \mapsto A(\sqrt{c_2})$. By equation (7)) there is a unique involution $\bar{g}$ of $\mathrm{PGL}(4,q)$ fixing $\mathcal{F}(\mathbf{C})$. Moreover $\bar{g} : \pi_0 \mapsto \pi_{\sqrt{c_2}}$ (we refer to notations of section 1).

Therefore, there is a non-identity element g of $\mathrm{PGL}(4,q)$ fixing $\mathcal{S}(\mathbf{C})$ such that $g^2 \in E$. Then either $g^2 = 1$ or $g^4 = 1$. Moreover g does not fix all the reguli $\mathcal{R}_t$.

As $\mathcal{S}(\mathbf{C})$ is not a regular spread, by [4] each collineation of H must fix the line l_∞. As the 2-elements of $\mathrm{PGL}(2,q)$ ($q = 2^e$) are all involutions, g induces an involution over l_∞ and over the set of planes through l_∞. Then g fixes a point of l_∞ and a plane through l_∞.

If g is the identity over l_∞, then there is an elation X of $T(\mathbf{C})$ such that $\tau_X = g$.

If g does not induce the identity over l_∞, we can choose a basis e_1, e_2, e_3, e_4 of $F^2 \times F^2$ such that $< e_1, e_2 >= l_\infty$ and $g = \tau_X$ where X is represented with respect to the fixed basis by a matrix of type $\begin{pmatrix} 1 & 1 & x & y \\ 0 & 1 & z & t \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. As q is a power of 2, $X^2 = 1$

and, therefore, $g^2 = 1$.

Then either X is an elation with axis $\{(0, \alpha) \mid \alpha \in F^2\}$ or X is a Baer-involution of $T(\mathbf{C})$.

Suppose that X is a Baer-involution. Then g fixes exactly $q + 1$ lines of $\mathcal{S}(\mathbf{C})$ one of which is l_∞. By [4] there are no reguli in $\mathcal{S}(\mathbf{C})$ different from $\mathcal{R}_t$ $(t \in F)$. Therefore if $l_{t,u}$ is another line of $\mathcal{S}(\mathbf{C})$ fixed by g, then g maps the regulus $\mathcal{R}_t$ into itself. As g fixes a point p of l_∞, g fixes the transversal line m in $\mathcal{R}_t$ incident with the point p. As $q - 1$ is odd, there is a third line $l_{t,v}$ of $\mathcal{R}_t$ fixed by g. Then g acts as the identity over m because $g \in \mathrm{PGL}(4, q)$. This implies that g induces over the plane $< l_\infty, m >$ an elation with axis m and center p. As m is a transversal line of $\mathcal{R}_t$, each line through p different from l_∞ is a transversal of one of the reguli of $\mathcal{S}(\mathbf{C})$. Therefore g fixes all the reguli of $\mathcal{S}(\mathbf{C})$: a contradiction. Hence X is an elation. □

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