The use of operators for the construction of normal bases for the space of continuous functions on V_q

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Abstract

Let a and q be two units of $\mathbb{Z}_p$, q not a root of unity, and let V_q be the closure of the set $\{aq^n \mid n = 0, 1, 2, ..\}$. K is a non-archimedean valued field, K contains $\mathbb{Q}_p$, and K is complete for the valuation $|.|$, which extends the p-adic valuation. $C(V_q \to K)$ is the Banach space of continuous functions from V_q to K, equipped with the supremum norm.

Let $\mathcal{E}$ and D_q be the operators on $C(V_q \to K)$ defined by $(\mathcal{E}f)(x) = f(qx)$ and $(D_q f)(x) = (f(qx) - f(x))/(x(q-1))$. We will find all linear and continuous operators that commute with $\mathcal{E}$ (resp. with D_q), and we use these operators to find normal bases for $C(V_q \to K)$.

1 Introduction

Let p be a prime, $\mathbb{Z}_p$ the ring of the p-adic integers, $\mathbb{Q}_p$ the field of the p-adic numbers. K is a non-archimedean valued field, $K \supset \mathbb{Q}_p$, and we suppose that K is complete for the valuation $|.|$, which extends the p-adic valuation. Let a and q be two units of $\mathbb{Z}_p$ (i. e. $|a| = |q| = 1$), q not a root of unity. Let V_q be the closure of the set $\{aq^n \mid n = 0, 1, 2, ..\}$. The set V_q has been described in [5].

$C(V_q \to K)$ (resp. $C(\mathbb{Z}_p \to K)$ will denote the set of all continuous functions $f : V_q \to K$ (resp. $f : \mathbb{Z}_p \to K$) equipped with the supremum norm.

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If f is an element of $C(V_q \to K)$ then we define the operators $\mathcal{E}$ and D_q as follows : $(\mathcal{E}f)(x) = f(qx), (D_qf)(x) = (f(qx) - f(x))/(x(q-1))$. The translation operator E on $C(\mathbb{Z}_p \to K)$ is the operator defined by $Ef(x) = f(x+1)$ (f an element of $C(\mathbb{Z}_p \to K)$).

We will call a sequence of polynomials $(p_n(x))$ a polynomial sequence if p_n is exactly of degree n for all natural numbers n.

In [4], L. Van Hamme finds all linear, continuous operators Q on $C(\mathbb{Z}_p \to K)$ wich commute with E. Such operators have the form $Q = \sum_{i=0}^{\infty} b_i \Delta^i$ where the sequence (b_n) is bounded, and where $(\Delta f)(x) = f(x+1) - f(x)$. If $|b_n| \leq |b_1| = 1$ if $n > 1$, and if b_0 is zero , he associates a (unique) polynomial sequence $(q_n(x))$ with such an operator Q and he concludes that these sequences $(q_n(x))$ form normal bases for $C(\mathbb{Z}_p \to K)$. Let f be an element of $C(\mathbb{Z}_p \to K)$, then there exists a uniformly convergent expansion such that $f(x) = \sum_{n=0}^{\infty} c_n q_n(x)$ and it is possible to give an expression for the coefficients c_n.

In sections 3 and 4, we give results analogous to the results of L. Van Hamme in [4], but with the space $C(\mathbb{Z}_p \to K)$ replaced by $C(V_q \to K)$, and the operator E replaced by the operators $\mathcal{E}$ or D_q.

In theorem 1, section 3 we find all linear, continuous operators that commute with $\mathcal{E}$. Such operators can be written in the form $Q = \sum_{i=0}^{\infty} b_i D^{(i)}$ where the sequence (b_n) is bounded, and where $(D^{(i)}f)(x) = ((\mathcal{E}-1)\cdots(\mathcal{E}-q^{i-1})f)x)$.

If Q is an operator such that $|b_n| < |b_N| = 1$ if $n > N$, and $b_n = 0$ if $n < N$ ($N \geq 1$), then we can associate polynomial sequences $(p_n(x))$ with Q. These sequences form normal bases for the space $C(V_q \to K)$. If f is an element of $C(V_q \to K)$, then f can be written as a uniformly convergent series $f(x) = \sum_{n=0}^{\infty} c_n p_n(x)$ and we are able to give an expression for the coefficients c_n. These results can be found in theorem 3, section 4.

If we replace the operator $\mathcal{E}$ by the operator D_q we have the following results :

We can find all linear, continuous operators that commute with D_q (theorem 2, section 3). Such operators can be written in the form $Q = \sum_{i=0}^{\infty} b_i D_q^i$ where the sequence $(b_n/(q-1)^n)$ is bounded.

If Q is an operator such that $|b_N| = |(q-1)^N|$, $|b_n| \leq |(q-1)^n|$ if $n > N$, and $b_n = 0$ if $n < N$ ($N \geq 1$), then we can associate polynomial sequences $(p_n(x))$ with Q. These sequences form normal bases for the space $C(V_q \to K)$. If $f(x) = \sum_{n=0}^{\infty} c_n p_n(x)$ we can give an expression for the coefficients c_n. This can be found in theorem 4, section 4.

Theorems 3 and 4 are more extensive than the theorem in [4].

We remark that the operator $\mathcal{E}$ does not commute with D_q. Furthermore, the operator D_q lowers the degree of a polynomial with one, whereas the operator $\mathcal{E}$ does not. Now let R and Q be operators on $C(V_q \to K)$, such that $R = \sum_{i=1}^{\infty} d_i D^{(i)}$ (i.e. R commutes with $\mathcal{E}$), and $Q = \sum_{i=1}^{\infty} b_i D_q^i$ (i.e. R commutes with D_q). The main difference between the operators Q and R is that Q lowers the degree of each polynomial with at least one, where R does not necessarily lowers the degree of a polynomial.

In section 5 our aim is to find other normal bases for $C(V_q \to K)$ (theorems 5 and 6). Therefore we will use linear, continuous operators wich commute with D_q

or with $\mathcal{E}$.

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2 Preliminary Lemmas

Before we can prove the theorems, we need some lemmas and some notations.

Let V_q, K and $C(V_q \to K)$ be as in the introduction. The supremum norm on $C(V_q \to K)$ will be denoted by $\|.\|$.

We introduce the following :

$[n]! = [n][n-1]..[1]$, $[0]! = 1$, where $[n] = \frac{q^n-1}{q-1}$ if $n \geq 1$.

$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$ if $n \geq k$, $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ if $n < k$

The polynomials $\begin{bmatrix} n \\ k \end{bmatrix}$ are the Gauss-polynomials.

$(x-\alpha)^{(k)} = (x-\alpha)(x-\alpha q)\cdots(x-\alpha q^{k-1})$ if $k \geq 1$, $(x-\alpha)^{(0)} = 1$,

$\left\{ \begin{matrix} x \\ k \end{matrix} \right\}_\alpha = \dfrac{(x-\alpha)(x-\alpha q)\cdots(x-\alpha q^{k-1})}{(\alpha q^k - \alpha)(\alpha q^k - \alpha q)\cdots(\alpha q^k - \alpha q^{k-1})}$ if $k \geq 1$, $\left\{ \begin{matrix} x \\ 0 \end{matrix} \right\}_\alpha = 1$ where α is an element of V_q. If α equals a, then we will use the notation $\left\{ \begin{matrix} x \\ k \end{matrix} \right\}$.

We will need the following properties of these symbols :

$\| \left\{ \begin{matrix} x \\ k \end{matrix} \right\} \| = 1$, since $\begin{bmatrix} n \\ k \end{bmatrix} = \left\{ \begin{matrix} x \\ k \end{matrix} \right\}$ if $x = aq^n$, $| \begin{bmatrix} n \\ k \end{bmatrix} | \leq 1$ for all n, k in $\mathbb{N}$ (this follows from [3], p. 120, (3)), $\left\{ \begin{matrix} aq^k \\ k \end{matrix} \right\} = \begin{bmatrix} k \\ k \end{bmatrix} = 1$ and since $\left\{ \begin{matrix} x \\ k \end{matrix} \right\}$ is continuous. If we replace a by α, we find that $\| \left\{ \begin{matrix} x \\ k \end{matrix} \right\}_\alpha \| = 1$.

Further, $\frac{(x-\alpha)^{(n)}}{[n]!} = \left\{ \begin{matrix} x \\ n \end{matrix} \right\}_\alpha (q-1)^n q^{n(n-1)/2} \alpha^n$, so $\|\frac{(x-a)^{(n)}}{[n]!}\| = |(q-1)^n|$.

Definition

Let f be a function from V_q to K. We define the following operators :

$$(D_q f)(x) = \frac{f(qx) - f(x)}{x(q-1)}$$

$(D_q^n f)(x) = (D_q(D_q^{n-1} f))(x)$

$(\mathcal{E} f)(x) = f(qx), (\mathcal{E}^n f)(x) = f(q^n x)$

$D f(x) = D^{(1)} f(x) = f(qx) - f(x) = ((\mathcal{E} - 1)f)(x)$

$D^{(n)} f(x) = ((\mathcal{E} - 1)..(\mathcal{E} - q^{n-1})f)(x), \quad D^{(0)} f(x) = f(x)$

The following properties are easily verified :

$D_q^j x^k = [k][k-1]\cdots[k-j+1]x^{k-j}$ if $k \geq j \geq 1$, $D_q^j x^k = 0$ if $k < j$

$D_q^j (x-\alpha)^{(k)} = [k][k-1]\cdots[k-j+1](x-\alpha)^{(k-j)}$ if $k \geq j \geq 1$, $D_q^j (x-\alpha)^{(k)} = 0$ if $j > k$

$$D_q^j \left\{ \begin{matrix} x \\ k \end{matrix} \right\}_\alpha = \left\{ \begin{matrix} x \\ k-j \end{matrix} \right\}_\alpha \frac{1}{\alpha^j (q-1)^j q^{jk - j(j+1)/2}} \quad \text{if } j \leq k, \ D_q^j \left\{ \begin{matrix} x \\ k \end{matrix} \right\}_\alpha = 0 \quad \text{if } j > k.$$

In particular $D_q^n \left\{ {x \atop n} \right\}_\alpha = \dfrac{1}{\alpha^n (q-1)^n q^{n(n-1)/2}}$

$D^{(j)} x^k = (q-1)^j q^{j(j-1)/2} [k][k-1] \cdots [k-j+1] x^k \quad$ if $k \geq j \geq 1$,
$D^{(j)} x^k = 0$ if $k < j$.
$D^{(j)} \left\{ {x \atop k} \right\}_\alpha = (x/\alpha)^j q^{j(j-k)} \left\{ {x \atop k-j} \right\}_\alpha \quad$ if $j \leq k$, $D^{(j)} \left\{ {x \atop k} \right\}_\alpha = 0 \quad$ if $j > k$.

In particular $D^{(n)} \left\{ {x \atop n} \right\}_\alpha = (x/\alpha)^n$

Lemma 1

i) $\quad x^n q^{n(n-1)/2} (q-1)^n (D_q^n f)(x) = \sum_{k=0}^n (-1)^{n-k} \left[{n \atop k} \right] q^{(n-k)(n-k-1)/2} f(q^k x)$

ii) $\quad (D^{(n)} f)(x) = \sum_{k=0}^n (-1)^{n-k} \left[{n \atop k} \right] q^{(n-k)(n-k-1)/2} f(q^k x)$

iii) $\quad (D^{(n)} f)(x) = x^n q^{n(n-1)/2} (q-1)^n (D_q^n f)(x)$ *where f is a function from V_q to K.*

Proof

By induction.
i) can also be found in [3], p. 121, and iii) can be found in [1], p 60.

Lemma 2

Let f be an element of $C(V_q \to K)$, then
i) $\|(q-1)^n D_q^n f\| \leq \|f\|$
ii) $\|D^{(n)} f\| \leq \|f\|$ *and this for all n in $\mathbb{N}$.*

Proof

i) can be shown by induction, and ii) follows from i), using lemma 1, iii).

Lemma 3

Let f be an element of $C(V_q \to K)$. Then
i) $(q-1)^n D_q^n f(x) \to 0$ *uniformly.*
ii) $D^{(n)} f(x) \to 0$ *uniformly.*

Proof

The proof of i) can be found in [3] p. 124-125,
ii) follows from i) by using lemma 1, iii).

Lemma 4

Let f be an element of $C(V_q \to K)$. Then
i) $D_q^{n+1} f = 0 \Leftrightarrow f$ *polynomial of degree* $\leq n$
ii) $D^{(n+1)} f = 0 \Leftrightarrow f$ *polynomial of degree* $\leq n$.

Proof

The proof of i) is analogous to the proof of lemma 1 in [4],
ii) follows from lemma 1, iii).

Lemma 5

Let p be a polynomial of degree n in $K[x]$ and let Q and R be linear, continuous operators such that $\mathcal{E}Q = Q\mathcal{E}$ and $D_q R = R D_q$. Then Qp and Rp are polynomial of degree less or equal to n.

Proof

Analogous to the proof of [4], lemma 2.

Lemma 6

If Q is a linear, continuous operator such that $QD_q = D_qQ$, then the integer $m = \deg p - \deg Qp$ is the same for all p in $K[x]$ which are not in the kernel of Q.

Proof

Analogous to the proof of lemma 3 in [4].

Corollary

Let (p_n) be a polynomial sequence. If Ker Q contains p_{n-1}, then Q lowers the degree of each polynomial with at least n.

Lemma 7

Suppose $\mathcal{E}Q = Q\mathcal{E}$. Let N be the smallest natural number n such that x^n does not belong to Ker Q (if this N exists). Then x^N divides $(Qp)(x)$ for all p in $K[x]$.

Proof

Take $x^n, n \geq N$. Let $Qx^n = \sum_{k=0}^{n} a_k x^k$. Then $QD^{(N)}x^n = cQx^n = c\sum_{k=0}^{n} a_k x^k$ with $c \neq 0$ by lemma 1, iii), and $D^{(N)}Qx^n = D^{(N)}\sum_{k=0}^{n} a_k x^k = \sum_{k=N}^{n} d_k x^k$. Since $D^{(N)}Qx^n = QD^{(N)}x^n$, we have $Qx^n = \sum_{k=N}^{n} a_k x^k$.

Corollary

If (p_n) is a polynomial sequence, and Ker Q contains $p_0, p_1, \cdots, p_{n-1}$ and p_n, then Ker Q contains every polynomial p of degree less or equal to n and x^{n+1} divides Qp for every polynomial p.

Now we are ready to prove the first theorems.

3 Linear Continuous Operators which Commute with $\mathcal{E}$ or with D_q

With the aid of the lemmas in section 2, we can prove the following theorems :

Theorem 1

An operator Q on $C(V_q \to K)$ is continuous, linear and commutes with $\mathcal{E}$ if and only if the sequence (b_n) is bounded, where $b_n = (QB_n)(a)$, $B_n = \left\{ {x \atop n} \right\}$.

Proof

This proof is analogous to the proof of the analogous proposition in [4], except for the construction of the bounded sequence (b_n). The construction works as follows :

Suppose Q is a linear continuous operator on $C(V_q \to K)$ and $Q\mathcal{E} = \mathcal{E}Q$. Define $b_0 = QB_0$.

Then Ker $(Q - b_0 I)$ contains B_0 since $QB_0 - b_0 = 0$ (I is the identity-operator). $(Q - b_0 I)B_1$ is a K-multiple of x (lemma 5 and corollary to lemma 7), and so we put $(Q - b_0 I)B_1 = (x/a)b_1$.

Ker $(Q - b_0 I - b_1 D^{(1)})$ contains B_1 since $(Q - b_0 I)B_1 - b_1 D^{(1)} B_1 = (x/a)b_1 - (x/a)b_1 = 0$.

So Ker $(Q - b_0 I - b_1 D^{(1)})$ contains B_0 and B_1 etc...

If $b_0, b_1, \cdots, b_{n-1}$ are already defined, then we have that Ker $(Q - b_0 I - \sum_{i=1}^{n-1} b_i D^{(i)})$ contains $B_0, B_1, \cdots, B_{n-1}$. The polynomial $(Q - b_0 I - \sum_{i=1}^{n-1} b_i D^{(i)})B_n$ is divisible by x^n and its degree is at most n (lemma 5 and corollary to lemma 7). Hence it is a K-multiple of x^n and we can put $(Q - b_0 I - \sum_{i=1}^{n-1} b_i D^{(i)})B_n = (x/a)^n b_n$.

From now on the proof is analogous to the proof of the analogous proposition in [4].

Just as in [4], it follows that Q can be written in the form $Q = \sum_{i=0}^{\infty} b_i D^{(i)}$. If f is an element of $C(V_q \to K)$, then $(Qf)(x) = \sum_{i=0}^{\infty} b_i (D^{(i)} f)(x)$ and the series on the right-hand-side is uniformly convergent (lemma 3). Clearly we have $b_n = (QB_n)(a)$.

Using lemma 1, iii) we have the following :

Corollary

Qx^n is a K-multiple of x^n.

If $b_0 = \cdots = b_{N-1} = 0, b_N \neq 0$, and if $p(x)$ is a polynomial, then x^N divides $(Qp)(x)$.

Analogous to theorem 1 we have :

Theorem 2

An operator Q on $C(V_q \to K)$ is continuous, linear and commutes with D_q if and only if the sequence $(b_n/(q-1)^n)$ is bounded, where $b_n = (QC_n)(a)$, $C_n(x) = \frac{(x-a)^{(n)}}{[n]!}$.

The proof is analogous to the proof of the proposition in [4]. Theorem 2 can also be found in [6]. Such an operator Q can be written in the form $Q = \sum_{i=0}^{\infty} b_i D_q^i$, and if f is an element of $C(V_q \to K)$ it follows that $(Qf)(x) = \sum_{i=0}^{\infty} b_i (D_q^i f)(x)$, where the series on the right-hand-side converges uniformly (lemma 3). Furthermore, we have $b_n = (QC_n)(a)$.

4 Normal bases for $C(V_q \to K)$

We use the operators of theorems 1 and 2 to make polynomials sequences $(p_n(x))$ which form normal bases for $C(V_q \to K)$.

The operator $R = \sum_{i=0}^{\infty} b_i D^{(i)}$ does not necessarily lowers the degree of a polynomial.

This leads us to the following lemma :

Lemma 8

Let $Q = \sum_{i=N}^{\infty} b_i D^{(i)}$ $(N \geq 0)$, with $|b_N| > |b_k|$ if $k > N$, $b_k = 0$ if $k < N$.
If $p(x)$ is a polynomial of degree $n \geq N$, then the degree of $(Qp)(x)$ is also n.

Proof

We prove the lemma for $p(x) = \frac{x^n}{[n]!}$ with $n \geq N$. The lemma then follows by linearity. Then $(Qp)(x) = \sum_{i=N}^{n} b_i D^{(i)} \frac{x^n}{[n]!} = \sum_{i=N}^{n} b_i (q-1)^i q^{i(i-1)/2} \frac{x^n}{[n-i]!}$.

The coefficient of x^n in this expansion is $\sum_{i=N}^{n} b_i (q-1)^i q^{i(i-1)/2} \frac{1}{[n-i]!}$.

Multiply this coefficient with $[n-N]!$: this does not change the fact that the coefficient is zero or not : $[n-N]! \sum_{i=N}^{n} b_i (q-1)^i q^{i(i-1)/2} \frac{1}{[n-i]!}$.

If $i > N$: $|b_i (q-1)^i q^{i(i-1)/2} \frac{[n-N]!}{[n-i]!}| \leq |b_i (q-1)^i| \leq |b_i (q-1)^N| < |b_N (q-1)^N|$

If $i = N$: $|b_N (q-1)^N q^{N(N-1)/2}| = |b_N (q-1)^N|$.

So $|[n-N]! \sum_{i=N}^{n} b_i (q-1)^i q^{i(i-1)/2} \frac{1}{[n-i]!}| = |b_N (q-1)^N| \neq 0$, and we conclude that the coefficient of x^n in $(Qp)(x)$ is different from zero.

If Q is an operator as found in theorem 1, with b_0 equal to zero, we associate a (unique) polynomial sequence $(p_n(x))$ with Q :

Proposition 1

Let $Q = \sum_{i=N}^{\infty} b_i D^{(i)}$ $(N \geq 1)$ with $|b_N| > |b_n|$ if $n > N$ and let α be a fixed element of V_q.

There exists a unique polynomial sequence $(p_n(x))$ such that

$(Qp_n)(x) = x^N p_{n-N}(x)$ if $n \geq N$, $p_n(\alpha q^i) = 0$ if $n \geq N$, $0 \leq i < N$ and $p_n(x) = \left\{ {x \atop n} \right\}_\alpha$ if $n < N$.

Proof

The series $(p_n(x))$ is constructed by induction.

Suppose that $p_0, p_1, \cdots, p_{n-1} (n \geq N)$ have already been constructed.

Since $p_n(x)$ is a polynomial of degree $n \geq N$, $(Qp_n)(x)$ is also a polynomial of degree n (lemma 8), and x^N divides $(Qp_n)(x)$ (corollary to theorem 1).

We can write $p_n(x)$ in the following way : $p_n(x) = \sum_{j=0}^{n} c_{n;j} x^j$.

Since Qx^k is a K-multiple of x^k (corollary to theorem 1), we have $Qx^k = \beta_k x^k$ where $\beta_k \neq 0$, if $k \geq N$ (lemma 8), and $\beta_0 = \beta_1 = \cdots = \beta_{N-1} = 0$.

So $(Qp_n)(x) = \sum_{j=0}^{n} c_{n;j} \beta_j x^j$ and this must equal $x^N p_{n-N}(x)$.

This gives us the coefficients $c_{n;n}, c_{n;n-1}, \cdots, c_{n;N}$.

The fact that $p_n(\alpha q^i)$ must equal zero gives us the equations

$$c_{n;0} + c_{n;1} \alpha q^i + \cdots + c_{n;N-1} (\alpha q^i)^{N-1} = - \sum_{k=N}^{n} c_{n;k} (\alpha q^i) k \qquad (0 \leq i \leq N-1).$$

To find a unique solution for the coefficients, the determinant must be different from zero.

We have :

$$\det \begin{pmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{N-1} \\ 1 & \alpha q & (\alpha q)^2 & \cdots & (\alpha q)^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha q^{N-1} & (\alpha q^{N-1})^2 & \cdots & (\alpha q^{N-1})^{N-1} \end{pmatrix} = \alpha^{N(N-1)/2} \prod_{i<j} q^{i-1} (q^{j-i} - 1)$$

and this is different from zero since q is different from zero and since q is not a root of unity.

This gives us the coefficients $c_{n;0,n;1}, \cdots, c_{n;N-1}$. From this it follows that the polynomial sequence $(p_n(x))$ exists and is unique.

In the same way as in proposition 1 we have

Proposition 2

Let $Q = \sum_{i=N}^{\infty} b_i D_q^i (N \geq 1)$, $b_N \neq 0, (b_n/(q-1)^n)$ bounded, and let α be a fixed element of V_q. Then there exists a unique polynomial sequence $(p_n(x))$ such that

$(Qp_n)(x) = p_{n-N}(x)$ if $n \geq N$, $p_n(\alpha q^i) = 0$ if $n \geq N, 0 \leq i < N$

and $p_n(x) = \left\{ {x \atop n} \right\}_\alpha$ if $n < N$.

Proof

The series $(p_n(x))$ is constructed by induction, analogous as in the proof of proposition 1, by writing p_n in the following way : $p_n(x) = c_n x^n + \sum_{i=0}^{n-1} c_i p_i(x)$.

Lemma 9

Let N be a natural number different from zero, let α be a fixed element of V_q and let $p(x)$ be a polynomial in $K[x]$ such that $p(\alpha q^i) = 0$ if $0 \leq i < N$.

Then $(D^{(k)}p)(\alpha) = (D_q^k p)(\alpha) = 0$ if $0 \leq k < N$.

Proof

If deg $p < N$, there is nothing to prove. Now suppose deg $p = n \geq N$. We can write p in the following way : $p(x) = \sum_{j=N}^{n} d_j \frac{(x-\alpha)^{(j)}}{[j]!}$ since $p(\alpha q^i) = 0$ if $0 \leq i < N$.

Then $(D_q^k p)(x) = \sum_{j=N}^{n} d_j \frac{(x-\alpha)^{(j-k)}}{[j-k]!}$ and so $(D_q^k p)(\alpha) = 0$ if $0 \leq k < N$.

$(D^{(k)}p)(\alpha) = 0$ if $0 \leq k < N$ follows from lemma 1, iii).

We use the operators of theorems 1 and 2 to make polynomials sequences $(p_n(x))$ which form normal bases for $C(V_q \to K)$. If f is an element of $C(V_q \to K)$, there exist coefficients c_n such that $f(x) = \sum_{n=0}^{\infty} c_n p_n(x)$ where the series on the right-hand-side is uniformly convergent. In some cases, it is also possible to give an expression for the coefficients c_n.

Before we prove the next theorem, we remark that the sequence $(\left\{ {x \atop n} \right\}_\alpha)$ (where α is a fixed element of V_q) forms a normal base for $C(V_q \to K)$ ([5], theorem 4, iii), applied on the sequence $p_n(x) = \left\{ {x \atop n} \right\}$)

Theorem 3

Let $Q = \sum_{i=N}^{\infty} b_i D^{(i)} \quad (N \geq 1)$ with $|b_n| < |b_N| = 1$ if $n > N$ and let α be a fixed element of V_q.

1) There exists a unique polynomial sequence $(p_n(x))$ such that

$$(Qp_n)(x) = x^N p_{n-N}(x) \text{ if } n \geq N, p_n(\alpha q^i) = 0 \text{ if } n \geq N, 0 \leq i < N$$

$$\text{and } p_n(x) = \left\{ {x \atop n} \right\}_\alpha \text{ if } n < N$$

This sequence forms a normal base for $C(V_q \to K)$ and the norm of Q equals one.

2) If f is an element of $C(V_q \to K)$, then f can be written as a uniformly convergent series $f(x) = \sum_{n=0}^{\infty} c_n p_n(x)$, $c_n = ((D^{(i)}(x^{-N}Q)^k)f)(\alpha)$ if $n = i + kN (0 \le i < N)$, with $\|f\| = max_{0\le k; 0\le i<N}\{|((D^{(i)}(x^{-N}Q)^k)f)(\alpha)|\}$, where $x^{-N}Q$ is a linear continuous operator with norm equal to one.

Proof

The existence and the uniqueness of the sequence follows from proposition 1.

Now we prove that the sequence $(p_n(x))$ forms a normal base.

We can write p_n in the following way : $p_n(x) = \sum_{j=0}^{n} c_{n;j} \left\{ {x \atop j} \right\}_\alpha$.

If we can prove that $|c_{n;j}| \le 1, |c_{n;n}| = 1$, then the sequence $(p_n(x))$ forms a normal base of $C(V_q \to K)$ ([5], theorem 4, iii)).

We prove the inequality for $|c_{n;j}|$ by induction on n. For $n = 0, 1, \cdots, N-1$ the assertion holds. Take $n \ge N$. Suppose the assertion holds for $i = 0, \cdots, n-1$.

We can write $p_n(x)$ in the following way :

$p_n(x) = \sum_{j=N}^{n} c_{n;j} \left\{ {x \atop j} \right\}_\alpha$ since $p_n(\alpha q^i) = 0 (0 \le i < N)$ if $n \ge N$. So $|c_{n;0}| = |c_{n;1}| = \cdots = |c_{n;N-1}| = 0 \le 1$. Now

$$\begin{aligned}(Qp_n)(x) &= \sum_{i=N}^{n} b_i D^{(i)} \sum_{j=N}^{n} c_{n;j} \left\{ {x \atop j} \right\}_\alpha \\ &= (x/\alpha)^N \sum_{j=N}^{n} c_{n;j} \sum_{i=N}^{j} b_i (x/\alpha)^{i-N} q^{i(i-j)} \left\{ {x \atop j-i} \right\}_\alpha = x^N p_{n-N}(x).\end{aligned}$$

We divide both sides by $(x/\alpha)^N$ and we find

$$\sum_{j=N}^{n} c_{n;j} \sum_{i=N}^{j} b_i (x/\alpha)^{i-N} q^{i(i-j)} \left\{ {x \atop j-i} \right\}_\alpha = \alpha^N p_{n-N}(x). \tag{1}$$

Comparing the norm of the coefficient of x^{n-N} on both sides we find, using the fact that $\frac{(x-a)^{(k)}}{[k]!} = \left\{ {x \atop k} \right\}_\alpha (q-1)^k q^{k(k-1)/2} \alpha^k$ (2)

$$|c_{n;n} \sum_{i=N}^{n} \frac{b_i (1/\alpha)^{i-N} q^{i(i-n)}}{[n-i]!(q-1)^{n-i} q^{(n-i)(n-i-1)/2} \alpha^{n-i}}| = |\frac{1}{[n-N]!(q-1)^{n-N}}|,$$

since $|c_{n-N;n-N}| = |\alpha| = 1$. Multiply both sides with $[n-N]!(q-1)^{n-N}$:

$$|c_{n;n} \sum_{i=N}^{n} \frac{b_i (1/\alpha)^{i-N} q^{i(i-n)} [n-N]!(q-1)^{n-N}}{[n-i]!(q-1)^{n-i} q^{(n-i)(n-i-1)/2} \alpha^{n-i}}| = 1.$$

If i is different from N we find

$$|\frac{b_i (1/\alpha)^{i-N} q^{i(i-n)} [n-N]!(q-1)^{n-N}}{[n-i]!(q-1)^{n-i} q^{(n-i)(n-i-1)/2} \alpha^{n-i}}| = |\frac{b_i [n-N]!(q-1)^{n-N}}{[n-i]!(q-1)^{n-i}}| \le |b_i| < |b_N|$$

and if i equals N we find

$$|\frac{b_N q^{N(N-n)} [n-N]!(q-1)^{n-N}}{[n-N]!(q-1)^{n-N} q^{(n-N)(n-N-1)/2} \alpha^{n-N}}| = |b_N| = 1.$$

So we conclude $|c_{n;n}| = 1$.

We proceed by subinduction :
suppose $|c_{n;i}| \leq 1$ for $N+1 \leq k+1 \leq i \leq n \quad (k \geq N)$.

We want to prove $|c_{n;k}| \leq 1$.

In (1) we move all terms of the L.H.S. with $j > k$ to the R.H.S. This gives something of the form

$$\sum_{j=N}^{k} c_{n;j} \sum_{i=N}^{j} b_i (x/\alpha)^{i-N} q^{i(i-j)} \left\{ \begin{matrix} x \\ j-i \end{matrix} \right\}_\alpha = \sum_{i=0}^{k-N} d_{k-N;i}^{(n)} \left\{ \begin{matrix} x \\ i \end{matrix} \right\}_\alpha \tag{3}$$

By the induction-hypothesis, the supremum norm of the R.H.S. is less or equal than one, so $|d_{k-N;i}^{(n)}| \leq 1$, since $(\left\{ \begin{smallmatrix} x \\ k \end{smallmatrix} \right\}_\alpha)$ forms a normal base for $C(V_q \to K)$.

Using (2), the coefficient of x^{k-N} on the L.H.S. in (3) equals

$$c_{n;k} \sum_{i=N}^{k} \frac{b_i (1/\alpha)^{i-N} q^{i(i-k)}}{[k-i]!(q-1)^{k-i} q^{(k-i)(k-i-1)/2} \alpha^{k-i}}.$$

Again by using (2), the coefficient of $\left\{ \begin{smallmatrix} x \\ k-N \end{smallmatrix} \right\}_\alpha$ on the L.H.S. in (3) equals

$$c_{n;k} [k-N]!(q-1)^{k-N} q^{(k-N)(k-N-1)/2} \alpha^{k-N} \sum_{i=N}^{k} \frac{b_i (1/\alpha)^{i-N} q^{i(i-k)}}{[k-i]!(q-1)^{k-i} q^{(k-i)(k-i-1)/2} \alpha^{k-i}}.$$

If i is different from N we find

$$| \, [k-N]!(q-1)^{k-N} q^{(k-N)(k-N-1)/2} \alpha^{k-N} \frac{b_i (1/\alpha)^{i-N} q^{i(i-k)}}{[k-i]!(q-1)^{k-i} q^{(k-i)(k-i-1)/2} \alpha^{k-i}} \, |$$

$$= | \, [k-N]!(q-1)^{k-N} \frac{b_i}{[k-i]!(q-1)^{k-i}} \, | \leq |b_i| < |b_N|$$

and if i equals N we find

$$| \, [k-N]!(q-1)^{k-N} q^{(k-N)(k-N-1)/2} \alpha^{k-N} \frac{b_N q^{N(N-k)}}{[k-N]!(q-1)^{k-N} q^{(k-N)(k-N-1)/2} \alpha^{k-N}} \, |$$

$$= |b_N| = 1$$

But this means that

$$|c_{n;k} [k-N]!(q-1)^{k-N} q^{(k-N)(k-N-1)/2} \alpha^{k-N} \sum_{i=N}^{k} \frac{b_i (1/\alpha)^{i-N} q^{i(i-k)}}{[k-i]!(q-1)^{k-i} q^{(k-i)(k-i-1)/2} \alpha^{k-i}}|$$

$$= |c_{n;k}|$$

So if we compare the norms of the coefficients of $\left\{ \begin{smallmatrix} x \\ k-N \end{smallmatrix} \right\}_\alpha$ on the L.H.S. and on the R.H.S in (3) , we find $|c_{n;k}| = |d_{k-N;k-N}^{(n)}| \leq 1$. So the sequence forms a normal base.

Now we prove the norm of Q equals one.

Therefore, let f be an element of $C(V_q \to K)$. Then $(Qf)(x) = \sum_{i=0}^{\infty} b_i(D^{(i)}f)(x)$. So $\|Qf\| \leq \max_{0\leq i}\{|b_i|\}\|f\| \leq \|f\|$ (lemma 2). So $\|Q\| \leq 1$.

If $n \geq N$, then $\|Qp_n\| = \|x^N p_{n-N}\| = \|p_{n-N}\| = 1 = \|p_n\|$, since $(p_n(x))$ forms a normal base for $C(V_q \to K)$. So $\|Q\| = 1$.

It is clear that $x^{-N}Q$ is linear. Further, $\|x^{-N}Q\| \leq \|x^{-N}\|\|Q\| \leq \|Q\| \leq 1$. So $x^{-N}Q$ is continuous. If $n \geq N$, then $\|x^{-N}Qp_n\| = \|x^{-N}x^N p_{n-N}\| = \|p_{n-N}\| = 1 = \|p_n\|$, since $(p_n(x))$ forms a normal base for $C(V_q \to K)$. So $\|x^{-N}Q\| = 1$.

Let f be an element of $C(V_q \to K)$. Since the sequence $(p_n(x))$ forms a normal base for $C(V_q \to K)$, there exists coefficients (c_n) such that $f(x) = \sum_{n=0}^{\infty} c_n p_n(x)$. We prove that c_n equals $((D^{(i)}(x^{-N}Q)^k)f)(\alpha)$ if $n = i + kN \quad (0 \leq i < N)$.

Since $f(x) = \sum_{n=0}^{\infty} c_n p_n(x)$, we have $((x^{-N}Q)f)(x) = \sum_{n=0}^{\infty} c_{n+N} p_n(x)$.

If we continue this way, we have

$$\begin{aligned} (((x^{-N}Q)^k)f)(x) &= \sum_{n=0}^{\infty} c_{n+kN} p_n(x) \\ &= \sum_{n=0}^{N-1} c_{n+kN} p_n(x) + \sum_{n=N}^{\infty} c_{n+kN} p_n(x) \\ &= \sum_{n=0}^{N-1} c_{n+kN} \left\{ \begin{matrix} x \\ n \end{matrix} \right\}_\alpha + \sum_{n=N}^{\infty} c_{n+kN} p_n(x). \end{aligned}$$

Using lemma 9 , we conclude that $((D^{(i)}(x^{-N}Q)^k)f)(\alpha) = c_{i+kN}$.

$\|f\| = \max_{0\leq k;0\leq i<N}\{|((D^{(i)}(x^{-N}Q)^k)f)(\alpha)|\}$ follows from the fact that $(p_n(x))$ forms a normal base for $C(V_q \to K)$. This finishes the proof.

Remark

If $Qx^k = \beta_k x^k$, there is a connection between the constants β_k and the constants b_k, namely $\beta_k = b_k = 0$ if $k < N$, and $\beta_k = \sum_{i=N}^{k} b_i (q-1)^i q^{i(i-1)/2}[k]\cdots[k-i+1]$ if $k \geq N$. In particular, $\beta_N = b_N(q-1)^N q^{N(N-1)/2}[N]!$.

An example

Let us consider the operator D. Then $b_1 = 1$ and $b_n = 0$ if $n \neq 1$. If α equals a, the polynomials $p_n(x)$ are given by $p_n(x) = a^n q^{n(n-1)/2} \left\{ \begin{matrix} x \\ n \end{matrix} \right\}$. It can be shown that the expansion $f(x) = \sum_{n=0}^{\infty}(((x^{-1}D)^n)f)(a)a^n q^{n(n-1)/2} \left\{ \begin{matrix} x \\ n \end{matrix} \right\}$ is essentially the same as Jacksons's interpolation formula ([2], [3]).

Theorem 4

Let $Q = \sum_{i=N}^{\infty} b_i D_q^i \quad (N \geq 1)$ *with* $|b_N| = |(q-1)^N|$, $|b_n| \leq |(q-1)^n|$ *if* $n > N$, *and let* α *be a fixed element of* V_q.

1) There exists a unique polynomial sequence $(p_n(x))$ *such that* $(Qp_n)(x) = p_{n-N}(x)$ *if* $n \geq N$, $p_n(\alpha q^i) = 0$ *if* $n \geq N, 0 \leq i < N$ *and* $p_n(x) = \left\{ \begin{matrix} x \\ n \end{matrix} \right\}_\alpha$ *if* $n < N$.

This sequence forms a normal base for $C(V_q \to K)$ *and the norm of* Q *equals one.*

2) If f *is an element of* $C(V_q \to K)$*, there exists a unique, uniformly convergent expansion of the form* $f(x) = \sum_{n=0}^{\infty} c_n p_n(x)$ *, where* $c_n = \alpha^i(q-1)^i q^{i(i-1)/2}(D_q^i Q^k f)(\alpha)$ *if* $n = i + kN(0 \leq i < N)$*, with* $\|f\| = \max_{0\leq k;0\leq i<N}\{|(q-1)^i(D_q^i Q^k f)(\alpha)|\}$.

Remark : Here we have $|b_n| \leq |b_N|$, in contrast with theorem 3, where we need $|b_n| < |b_N|(n > N)$.

Proof

The proof follows the same pattern as the proof of theorem 3.

We prove that the sequence forms a normal base by induction on n using [5], theorem 4, iii).

For $n = 0, 1, \cdots, N-1$ there is nothing to prove.

Now suppose $n \geq N$ and put $p_n(x) = \sum_{j=0}^{n} c_{n;j} \frac{(x-\alpha)^{(j)}}{[j]!}$, $\|\frac{(x-\alpha)^{(j)}}{[j]!}\| = |(q-1)^j|$.

If we apply [5], theorem 4, iii) on the sequence $(\left\{ {x \atop j} \right\}_\alpha)$ we find the following :

Let $(r_n(x))$ be a polynomial sequence such that $r_n(x) = \sum_{j=0}^{n} e_{n;j} \left\{ {x \atop j} \right\}_\alpha$ $(e_{n;n} \neq 0)$.
Then $(r_n(x))$ forms a normal base for $C(V_q \to K)$ if and only if $|e_{n;j}| \leq 1, |e_{n;n}| = 1$.
Using the fact that $\frac{(x-\alpha)^{(j)}}{[j]!}$ equals $\left\{ {x \atop j} \right\}_\alpha (q-1)^j q^{j(j-1)/2} \alpha^j$, this becomes :
Let $(r_n(x))$ be a polynomial sequence such that $r_n(x) = \sum_{j=0}^{n} d_{n;j} \frac{(x-\alpha)^{(j)}}{[j]!}$ $(d_{n;n} \neq 0)$.
Then $(r_n(x))$ forms a normal base for $C(V_q \to K)$ if and only if

$$|d_{n;j}| \leq |(q-1)^{-j}|, |d_{n;n}| = |(q-1)^{-n}|.$$

So, if we can prove that $|c_{n;j}| \leq |(q-1)^{-j}|, |c_{n;n}| = |(q-1)^{-n}|$, then the sequence $(p_n(x))$ forms a normal base of $C(V_q \to K)$.

We prove the inequality for $|c_{n;j}|$ by induction on n in an analogous way as in the proof of theorem 3.

The sequel of the proof follows the same pattern as the proof of theorem 3.

An example

Let us consider the following operator $Q = (q-1)D_q$. Then $b_1 = (q-1)$ and $b_n = 0$ if $n \neq 1$.

If α equals a, the polynomials $p_n(x)$ are given by $p_n(x) = \frac{(x-a)^{(n)}}{[n]!(q-1)^n}$, and they form a normal base for $C(V_q \to K)$. The expansion

$$f(x) = \sum_{n=0}^{\infty} ((q-1)^n D_q^n f)(a) p_n(x) = \sum_{n=0}^{\infty} (D_q^n f)(a) \frac{(x-a)^{(n)}}{[n]!}$$

is known as Jackson's interpolation formula ([2], [3]).

5 More Normal Bases

We want to make more normal bases, using the ones we found in theorems 3 and 4.

Proposition 3

Let $Q = \sum_{i=N}^{\infty} b_i D^{(i)}$ $(N \geq 0)$ with $1 = |b_N| > |b_k|$ if $k > N$, and let $p(x)$ be a polynomial of degree $n \geq N$, $p(x) = \sum_{j=0}^{n} c_j \left\{ {x \atop j} \right\}$ where $|c_j| \leq 1, |c_n| = 1$.

Then $(Qp)(x) = x^N r(x)$ where $r(x) = \sum_{j=0}^{n-N} d_j \left\{ {x \atop j} \right\}$ with $|d_j| \leq 1, |d_{n-N}| = 1$.

Proof

x^N divides $(Qp)(x)$ by the corollary to theorem 1, and $(Qp)(x)$ is a polynomial of degree n, by lemma 8, so $r(x)$ is a polynomial of degree $n-N$. Then

$$(Qp)(x) = \sum_{i=N}^{n} b_i D^{(i)} \sum_{j=0}^{n} c_j \left\{ \begin{matrix} x \\ j \end{matrix} \right\} = x^N \sum_{j=N}^{n} c_j \sum_{i=N}^{j} b_i \frac{x^{i-N}}{a^i} q^{i(i-j)} \left\{ \begin{matrix} x \\ j-i \end{matrix} \right\} = x^N r(x).$$

Now $\|Qp\| = \|x^N r\| = \|r\|$. Since $|c_j| \le 1$ and $|b_i| \le 1$ we have $\|Qp\| \le 1$ and so $\|r\| \le 1$. If $r(x) = \sum_{j=0}^{n-N} d_j \left\{ \begin{smallmatrix} x \\ j \end{smallmatrix} \right\}$ then $|d_j| \le 1$ (otherwise $\|r\| > 1$) . So it suffices to prove that $|d_{n-N}| = 1$. Since $r(x) = \sum_{j=N}^{n} c_j \sum_{i=N}^{j} b_i \frac{x^{i-N}}{a^i} q^{i(i-j)} \left\{ \begin{smallmatrix} x \\ j-i \end{smallmatrix} \right\}$, and since the coefficients of x^{n-N} on both sides must be equal we have

$$c_n \sum_{i=N}^{n} b_i \frac{1}{a^i} q^{i(i-n)} \frac{1}{[n-i]!(q-1)^{n-i} q^{(n-i)(n-i-1)/2} a^{n-i}} =$$

$$d_{n-N} \frac{1}{[n-N]!(q-1)^{n-N} q^{(n-N)(n-N-1)/2} a^{n-N}}.$$

If we multiply both sides with $[n-N]!(q-1)^{n-N}$, and if we use the same trick as in the proof of theorem 3, we find $|d_{n-N}| = 1$.

Now we have immediately the following theorem :

Theorem 5

Let $(p_n(x))$ be a polynomial sequence which forms a normal base for $C(V_q \to K)$, and let $Q = \sum_{i=N}^{\infty} b_i D^{(i)}$ $(N \ge 0)$ with $b_0 = \cdots = b_{N-1} = 0, 1 = |b_N| > |b_k|$ if $k > N$. If $Qp_n(x) = x^N r_{n-N}(x)$ $(n \ge N)$, then the polynomial sequence $(r_k(x))$ forms a normal base for $C(V_q \to K)$.

Proof

This follows immediately from proposition 3 and [5], theorem 4, iii) applied for $p_n(x) = \left\{ \begin{smallmatrix} x \\ n \end{smallmatrix} \right\}$.

And in the same way we have

Proposition 4

Let $Q = \sum_{i=N}^{\infty} b_i D_q^i$ $(N \ge 0)$ with $|b_N| = |(q-1)^N|, |b_n| \le |(q-1)^n|$ if $n > N$, and let $p(x)$ be a polynomial of degree $n \ge N$, $p(x) = \sum_{j=0}^{n} c_j \frac{(x-\alpha)^{(j)}}{[j]!}$ where $|c_j| \le |(q-1)^{-j}|, |c_n| = |(q-1)^{-n}|$.

Then $(Qp)(x) = r(x)$ where $r(x) = \sum_{j=0}^{n-N} d_j \frac{(x-\alpha)^{(j)}}{[j]!}$ with $|d_j| \le |(q-1)^{-j}|, |d_{n-N}| = |(q-1)^{-(n-N)}|$.

Proof

$r(x)$ is clearly a polynomial of degree $n-N$.

Then $(Qp)(x) = \sum_{i=N}^{n} b_i D_q^i \sum_{j=0}^{n} c_j \frac{(x-\alpha)^{(j)}}{[j]!} = \sum_{j=N}^{n} c_j \sum_{i=N}^{j} b_i \frac{(x-\alpha)^{(j-i)}}{[j-i]!} = r(x)$.

Now $\|Qp\| = \|r\|$. Since $|c_j| \leq |(q-1)^{-j}|$ and $|b_i| \leq |(q-1)^i|$ we have $\|Qp\| \leq 1$ and so $\|r\| \leq 1$. If $r(x) = \sum_{j=0}^{n-N} d_j \frac{(x-\alpha)^{(j)}}{[j]!}$, then we must have $|d_j| \leq |(q-1)^{-j}|$ (otherwise $\|r\| > 1$). So it suffices to prove that $|d_{n-N}| = |(q-1)^{-(n-N)}|$.

Since $r(x) = \sum_{j=N}^{n} c_j \sum_{i=N}^{j} b_i \frac{(x-\alpha)^{(j-i)}}{[j-i]!}$ and since the coefficients of $\frac{(x-\alpha)^{(n-N)}}{[n-N]!}$ on both sides must be equal we have $c_n b_N = d_{n-N}$ and so $|d_{n-N}| = |(q-1)^{-(n-N)}|$ since $|b_N| = |(q-1)^N|$ and $|c_n| = |(q-1)^{-n}|$.

Now we have immediately the following theorem :

Theorem 6

Let $(p_n(x))$ be a polynomial sequence which forms a normal base for $C(V_q \to K)$, and let $Q = \sum_{i=N}^{\infty} b_i D_q^i \quad (N \geq 0)$ with $|b_N| = |(q-1)^N|, |b_n| \leq |(q-1)^n|$ if $n > N$.

If $(Qp_n)(x) = r_{n-N}(x)(n \geq N)$, then the polynomial sequence $(r_k(x))$ forms a normal base for $C(V_q \to K)$.

Proof

This follows immediately from proposition 4, the fact that

$$\frac{(x-\alpha)^{(j)}}{[j]!} = \left\{ \begin{matrix} x \\ j \end{matrix} \right\}_{\alpha} (q-1)^j q^{j(j-1)/2} \alpha^j$$

and [5], theorem 4, iii) applied for $p_n(x) = \left\{ \begin{matrix} x \\ n \end{matrix} \right\}$.

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