LS-category of classifying spaces

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ABSTRACT. — Let X be a 1-connected topological space of finite type. Denote by $Baut X$ the base of the universal fibration of fibre X. We show in this paper that the Lusternik-Schnirelmann category of $B\,aut\,X$ is not finite when X is a wedge of spheres or a finite CW-complex with finite dimensional rational homotopy.

Introduction

In this paper X will denote a simply connected CW-complex of finite type. Recall that the Lusternik-Schnirelmann category of a topological space, $cat(S)$, is the least integer n such that S can be covered by $(n+1)$ open subsets contractible in S, and is ∞ if no such n exists.

Denote by X_0 the localization of X at zero, the rational Lusternik-Schnirelmann category, $cat_0(X)$, is defined by $cat_0(X) = cat(X_0)$. This invariant satisfies

$$nil\,H^*(X,\,\mathbb{Q})\ \leq e_0(X) \leq cat_0(X), \quad ([6],[12]), \tag{i}$$

$$cat_0(X) \leq cat(X)\ ([12]), \tag{ii}$$

where $nil\,H^*(X,\mathbb{Q})$ is the nilpotence of the cohomology ring with rational coefficients and $e_0(X)$ the Toomer invariant ([12]).

In this text we will use the theory of minimal models. The Sullivan minimal model of X is a free commutative cochain algebra $(\Lambda Z,\,d)$ such that $dZ \subset \Lambda^{\geq 2}Z$. Moreover $Z^n \cong Hom_{\mathbb{Q}}(\pi_*(X)\otimes\mathbb{Q},\,\mathbb{Q})$ ([10], [5]).

The Quillen minimal model of X is a free chain Lie algebra $(\mathbb{L}(V),\,\delta)$ satisfying $\delta V \subset \mathbb{L}^{\geq 2}V$ and the graded vector space V is related to the cohomology of X by

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$V_n \cong H^{n+1}(X, \mathbb{Q})$ ([10], [5]).

Let $(L,\ \delta)$ be a differential graded Lie algebra of finite type. The cochain algebra on $(L,\ \delta)$ is the commutative cochain algebra $C^*(L,\ \delta)$ defined as follows:

$$C^*(L,\ \delta) \cong (\Lambda s^{-1} L^\vee,\ d_1 + d_2);$$

$L^\vee$ denotes the dual of L, and d_1 and d_2 are defined by:

$$\begin{array}{l} <\ d_1 s^{-1} z,\ sx\ > = <\ z,\ \delta x\ > \\ <\ d_2 s^{-1} z;\ sx_1,\ sx_2\ > = (-1)^{|x_1|} <\ z,\ [\, x_1, x_2\,] >, \end{array}$$

where $z \in L^\vee,\ x_i,\ x \in L$.
Moreover, $(\mathbb{L}(V),\ \delta)$ is a Quillen model of X if and only if the commutative differential graded algebra $C^*(\mathbb{L}(V),\ \delta)$ is a Sullivan model of X.

We define from $(\Lambda Z,\ d)$ and $(L,\ \delta)$ two Lie algebras of derivations. First, the differential Lie algebra $(Der\ \Lambda Z,\ D)$ is defined by ([10]): in degree $k > 1$, take the derivations of ΛZ decreasing degree by k. In degree one, we only consider the derivations θ which decrease degree by one and verify $[\, d,\ \theta] = 0$. The differential D is defined by $D\theta = [\, d,\ \theta\,] = d\theta - (-1)^{|\theta|}\theta d$.

In the same way we define a differential Lie algebra $Der\mathbb{L}(V) = \underset{k\geq 1}{\oplus} Der_k(\mathbb{L}(V))$ where $Der_k(\mathbb{L}(V))$ is the vector space of derivations which increase the degree by k with the restriction that $Der_1(\mathbb{L}(V))$ is the vector space of derivations of degree one which commute with the differential δ.

Define the differential Lie algebra $(s\mathbb{L}(V) \underset{\sim}{\oplus} Der\mathbb{L}(V),\ \tilde{D})$ as follows:

- $s\mathbb{L}(V) \underset{\sim}{\oplus} Der\mathbb{L}(V)$ is isomorphic to $s\mathbb{L}(V) \oplus Der\mathbb{L}(V)$ as a graded vector space,
- If $\theta, \theta' \in Der\ \mathbb{L}(V); sx, sy \in s\mathbb{L}(V),\ [\theta, \theta'] = \theta\theta' - (-1)^{|\theta||\theta'|}\theta'\theta$, $[\theta, sx] = (-1)^{|\theta|}\ s\theta(x),\ [sx, sy] = 0$,
- $\tilde{D}(\theta) = [\delta, \theta], \tilde{D}(sx) = -s\delta x + ad\ x$, where $ad\ x$ is the derivation of $\mathbb{L}(V)$ defined by $(ad\ x)(y) = [x, y]$.

Theorem ([10], [9], [11]) *The graded differential Lie algebras* $(s\mathbb{L}(V) \underset{\sim}{\oplus} Der\mathbb{L}(V), \tilde{D})$ *and* $(Der\ \Lambda Z,\ D)$ *are models of the universal covering* $\widetilde{B}\,aut\ X$ *of* $B\,aut\ X$.

We shall use these models to compute the rational LS-category of $\widetilde{B}\,aut\,X$. Since $cat(B aut X) \geq cat(\widetilde{B} aut X) \geq cat_0(\widetilde{B} aut X)$, we shall conclude that the LS-category of $B\,aut\ X$ is not finite whenever $cat_0(\widetilde{B}\ aut\ X) = \infty$.

1 The theorem

Theorem *Let X be a simply connected finite CW-complex. The LS-category of $B\,aut\,X$ is not finite provided one of the following hypothesis is satisfied:*
(a) $dim\,\Pi_*(X)\otimes\mathbb{Q}$ is finite,
(b) X has the rational homotopy type of a wedge of spheres.

Remarks:

1. Observe that in case (b), $cat(X_0) = 1$ and $dim\,\Pi_*(X)\otimes\mathbb{Q}$ is not necessary finite.

2. Let $X = CP(\infty)$. Since $H^*(X,\ \mathbb{Q}) \cong \Lambda x,\ |x| = 2,\ cat_0(X) = \infty$. The graded Lie algebra $Der(\Lambda x)$ is reduced to the vector space of dimension one generated by the derivation θ of degree two defined by $\theta(x) = 1$. Then $\widetilde{B}\,aut\,X$ has the rational homotopy type of the sphere S^3. This shows that $cat_0(\widetilde{B}\,aut\,X)$ can be finite while $cat(X) = \infty$.

3. Let G be a connected Lie group acting on X. The Borel fibration $X \longrightarrow EG\times_G X \longrightarrow BG$ is classified by a map $f : BG \longrightarrow \widetilde{B}\,aut\,X$. Consider the map $H^*(f,\mathbb{Q}) : H^*(\widetilde{B}\,aut\,X,\ \mathbb{Q}) \longrightarrow H^*(BG,\mathbb{Q}) \cong \Lambda V$ where V is concentrated in even degrees. Suppose now that $cat_0(\widetilde{B}\,aut\,X)$ is finite. Then $\tilde{H}^*(f,\mathbb{Q})$ is trivial. (Otherwise, $nil(H^*(\widetilde{B}\,aut\,X,\mathbb{Q})) = \infty$.) Therefore $f : BG \longrightarrow \widetilde{B}\,aut\,X$ is rationally trivial, that is: the action of the group G on the space X is rationally trivial.

2 Proof of the theorem under hypothesis (a)

Let $(\Lambda V, d)$ be the Sullivan minimal of a simply connected space S. According to [3, corollary 6.12],

$$(cat_0(S),\ dim\,V < \infty) \Longrightarrow n\ is\ odd \tag{iii}$$

where n denotes the greatest integer such that $V^n \neq 0$.
Applying (iii) to X, we conclude that the greatest integer k such that $Z^k \neq 0$ is odd. Let $x \in Z^k,\ x \neq 0$. Define a derivation θ of ΛZ by $\theta(x) = 1,\ \theta(z) = 0$ if z belongs to a graded supplementary of $\mathbb{Q}.x$ in Z. Then θ is a cycle which is not a boundary since $(Der\Lambda Z)_p = 0$ if $p > k$. Denote by y the class of θ in $H_k(Der\ (\Lambda Z,\ d)) \cong \Pi_{k+1}(\widetilde{B}\,aut\,X)\otimes\mathbb{Q}$. Since $\Pi_*(\widetilde{B}\,aut\,X)\otimes\mathbb{Q} \cong H_*(Der\Lambda Z,\ D)$ is finite dimensional and $k+1$ is even, applying (iii) to the Sullivan minimal model

$$(\Lambda Y,\ \tilde{d}) \xrightarrow{\simeq} C^*(Der\Lambda Z,\ D)$$

of $(\widetilde{B}\,aut\,X)$ where $Y \cong H_*(Der\Lambda Z,\ D)$, we conclude that the rational LS-category of $\widetilde{B}\,aut\,X$ cannot be finite. ■

3 Proof of the theorem under hypothesis (b)

In this section, we shall suppose that X is a wedge of at least two spheres. The case when X is a sphere has been treated above. Therefore we will suppose that the number of spheres is ≥ 2.

The proof is divided in two parts, the first concerns the case where X is a wedge of odd dimensional spheres (Lemma 1) and the second case concerns the case there is at least one even dimensional sphere (Lemma 2).

Lemma 1 *Let X be a wedge of odd dimensional spheres, then the LS-category of $\widetilde{B}\,aut\,X$ is ∞.*

Proof: Let $L = \Pi_*(\Omega X) \otimes \mathbb{Q} \cong \mathbb{L}(a,\ b,\ c_i)$ be the Quillen minimal model of X and suppose that $|b| \geq |a| \geq |c_i|$. Denote $ad\,L$ the ideal of $DerL$ generated by the inner derivations of L. Define the projection

$$\phi : (sL \underset{\sim}{\oplus} DerL,\ D) \longrightarrow (Der\ L/(ad\,L),\ 0)$$

by $\phi(sx) = 0$ and $\phi(\theta)$ is the class of θ in $Der\ L/(ad\,L)$.
Clearly ϕ commutes with differentials. Since $im\tilde{D} \cong ad\ L$ and $ker\tilde{D} \cong Der\ L$, it is obvious that the map ϕ induces the identity in homology. Thus

$$\Pi_*(\Omega B\ aut\ X) \otimes \mathbb{Q} \cong Der\ L/(ad\ L).$$

For $p \geq 2$, define $\theta_p \in Der\ L/(ad\ L)$ by $\theta_p(a) = \theta_p(c_i) = 0$ and $\theta_p(b) = ad^p(a)(b)$. Denote, $L_1 = \overset{\infty}{\underset{p=2}{\oplus}} \mathbb{Q}.\theta_p$, the abelian Lie sub algebra of $\mathcal{L} = Der\ L/(ad\ L)$. Observe that $L_1 \subset \mathcal{L}/[\,\mathcal{L},\ \mathcal{L}\,]$.

Thus $H^*(C^*(L_1))$ is a sub algebra of $H^*(C^*(\mathcal{L})) \cong H^*(\widetilde{B}\ aut\ X) \otimes \mathbb{Q}$. Since $nilH^*(C^*(L_1)) = \infty,\ cat(\widetilde{B}\,aut\,X) = \infty$. ∎

Lemma 2 *Let $X = S^{\alpha_1} \vee \cdots \vee S^{\alpha_n}$ be a wedge of spheres such that α_1 is even, then the Lusternik-Schnirelmann category of $\widetilde{B}\,aut\,X$ is ∞.*

Proof: Let $L = \mathbb{L}(a_1, \cdots, a_p, b_1, \cdots, b_q)$ be the Quillen minimal model of X with $|a_i|$ odd, $|b_j|$ even, $|a_i| \leq |a_{i+1}|$, and $|b_j| \leq |b_{j+1}|$. There is an element $\theta \in \mathcal{L} = Der\ L/(ad\,L)$ such that $|\theta|$ is odd, $[\,\theta,\ \theta\,] = 0$ and $\theta \in \mathcal{L}/[\,\mathcal{L},\ \mathcal{L}\,]$.

In fact if $q \neq 0$, we can choose x and y in the sequence $\{\,a_1, \cdots, a_p,\ b_1, \ldots, b_q\,\}$ such that $|y| - |x|$ is the smallest odd positive integer. The derivation θ, defined by $\theta(x) = y$ and $\theta = 0$ on the other generators of L, satisfies the above properties. Moreover θ is not an inner derivation, therefore the class of θ is not zero in $\mathcal{L}$.

If $q = 0$, there is an element a_{i_0} of maximal degree such that $|a_{i_0}| < 2|a_1|$. The derivation θ, defined by $\theta(a_{i_0}) = [a_1,\ a_1]$ and null elsewhere, satisfies the desired properties. Note that there is no element $x \in L$ such that $\theta(a_{i_0}) = [a_{i_0},\ x]$, thus θ is not an inner derivation.

Denote $A = U\mathcal{L} \stackrel{\simeq}{\longrightarrow} C_*(\Omega\widetilde{B}\, aut\, X)$ the enveloping algebra of $\mathcal{L}$. Let x be the image of θ by the injection $\mathcal{L} \longrightarrow U\mathcal{L}$. Since $[\theta, \theta] = 0$, we have $x^2 = 0$.

Let $(\bar{B}(A),\, d)$ be the normalized bar construction on A ([7]). The bar filtration of $\bar{B}(A)$ induces a filtration of $H(\bar{B}(A),\, d)$ whose associated bigraded vector space is denoted by $E^{\infty}_{*,*}$. Recall that the Toomer invariant is defined by $e_0(X) = sup\ \{\ p\ |, E^{\infty}_{p,*} \neq 0\ \}$. The element $[x|\dots|x]$ is a cycle of $(\bar{B}(A),\, d)$ which is not a boundary since x is not decomposable in A. Therefore for each $p \geq 2$, the class of $[x|\dots|x]$ is a non zero element in $E^{\infty}_{p,*}$. Thus $e_0(\widetilde{B}autX) = \infty$, and $cat(\widetilde{B}autX) = \infty$. ∎

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