Solving some problems of advanced analytical nature posed in the SIAM-review

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In this paper, three SIAM-Review problems selected from Vol. 34 (1992) are reconsidered and treated using methods according to my own vision on them.

1 On alternating double sums[1]

Consider the functions $S(v)$ and $C(v)$ defined as the sums of two infinite double series :

$$S(v) = \sum_{m=0}^{+\infty}\sum_{n=1}^{+\infty}(-1)^{m+n}\frac{\sin(2v\sqrt{m^2+n^2})}{\sqrt{m^2+n^2}}, \tag{1.1}$$

$$C(v) = \sum_{m=0}^{+\infty}\sum_{n=1}^{+\infty}(-1)^{m+n}\frac{\cos(2v\sqrt{m^2+n^2})}{\sqrt{m^2+n^2}}, \tag{1.2}$$

whereby it is indifferent in which order of succession of m and n the summations are carried out on account of the symmetry of the summands with respect to m and n. Find closed expressions for $S(v)$ and $C(v)$ for arbitrary real v and try to deduce from them whether the conjectures

$$S(v) = -v/2 \text{ if } -\pi/\sqrt{2} < v < \pi/\sqrt{2}, \quad C(v) = 0 \text{ if } v = \pm 5/4, \tag{1.3}$$

based upon numerical calculations, hold or not.

[1]problem posed by Malte Henkel (University of Geneva, Geneva, Switzerland) and R.A. Weston (University of Durham, UK) (Problem 92-11*)

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These sums arose in finite-size scaling studies of the three-dimensional spherical model.

Solution

1a. First, one notices that

$$S(v) = \sum_{n=1}^{+\infty}(-1)^n \frac{\sin 2vn}{n} + \sum_{m=1}^{+\infty}(-1)^m \left(\sum_{n=1}^{+\infty}(-1)^n \frac{\sin(2v\sqrt{m^2+n^2})}{(m^2+n^2)^{1/2}}\right). \qquad (1.4)$$

The way in which (1.1) is rewritten indicates that the summations will be carried out horizontally and not diagonally or according to any other order of succession which could possibly influence ultimately the value of the sum of the double series. The same will hold for (1.2) in sect.1b.

It is known that

$$\sum_{n=1}^{+\infty}(-1)^n \frac{\sin nx}{n} = \begin{cases} -\dfrac{x}{2} + j\pi\,, & (2j-1)\pi < x < (2j+1)\pi\,,\ \forall j \in \mathbb{Z}\,, \\ 0\,, & x = \pm\pi, \pm 3\pi, \ldots \end{cases} \qquad (1.5)$$

being the "saw-tooth" curve extending from $-\infty$ to $+\infty$. By differentiation with respect to x, one finds in the Cesàro-sense (C1) :

$$\begin{aligned}\sum_{n=1}^{+\infty}(-1)^n \cos nx &= \begin{cases} -\dfrac{1}{2}\,, & -(2j-1)\pi < x < (2j+1)\pi\,,\ \forall j \in \mathbb{Z}\,, \\ \pi\delta[x-(2k+1)\pi]\,, & k = 0, \pm 1, \pm 2, \ldots \end{cases} \\ &= -\frac{1}{2} + \pi \sum_{k=-\infty}^{+\infty} \delta[x-(2k+1)\pi]\ \ (\mathrm{C1})\,, \qquad \forall x \in \mathbb{R}\,. \end{aligned} \qquad (1.6)$$

In [Chap. XIII, 13·47] of ref.[1], one reads on p. 415 :

$$\int_0^\infty J_\mu(bt) \frac{J_\nu(a\sqrt{t^2+z^2})}{(t^2+z^2)^{\nu/2}}\, t^{\mu+1}\, dt$$
$$= \begin{cases} 0\,, & (a<b)\,, \\ \dfrac{b^\mu}{a^\nu}\left\{\dfrac{\sqrt{a^2-b^2}}{z}\right\}^{\nu-\mu-1} J_{\nu-\mu-1}(z\sqrt{a^2-b^2})\,, & (a>b)\,, \\ & a, b \in \mathbb{R}^+\,. \end{cases}$$

For $\mu = 0, \nu = 1/2$, this simplifies to

$$\left(\frac{2}{\pi a}\right)^{1/2} \int_0^\infty J_0(bt)\, \frac{\sin(a\sqrt{t^2+z^2})}{(t^2+z^2)^{1/2}}\, t\, dt$$
$$= \begin{cases} 0\,, & (a<b)\,, \\ \dfrac{1}{\sqrt{a}}\left\{\dfrac{z}{\sqrt{a^2-b^2}}\right\}^{1/2}\left(\dfrac{2}{\pi z\sqrt{a^2-b^2}}\right)^{1/2} \cos z\sqrt{a^2-b^2}\,, & (a>b)\,, \end{cases}$$

or

$$\int_0^\infty tJ_0(bt)\, \frac{\sin(a\sqrt{t^2+z^2})}{(t^2+z^2)^{1/2}}\, dt = \begin{cases} 0\,, & (a<b)\,, \\ \dfrac{\cos z\sqrt{a^2-b^2}}{(a^2-b^2)^{1/2}}\,, & (a>b)\,. \end{cases}$$

Hankel inversion yields

$$\int_0^a b\, J_0(tb) \frac{\cos z\sqrt{a^2-b^2}}{(a^2-b^2)^{1/2}}\, db = \frac{\sin a\sqrt{t^2+z^2}}{(t^2+z^2)^{1/2}}\,, \qquad \forall a \in \mathbb{R}^+\,,$$

and substituting $(a^2-b^2)^{1/2} = ax$, one obtains

$$\frac{\sin a\sqrt{t^2+z^2}}{(t^2+z^2)^{1/2}} = a \int_0^1 J_0(at\sqrt{1-x^2}) \cos azx\, dx\,.$$

Setting $a = 2v, t = m$ and $z = n$, there comes :

$$\frac{\sin(2v\sqrt{m^2+n^2})}{(m^2+n^2)^{1/2}} = 2v \int_0^1 J_0(2vm\sqrt{1-x^2}) \cos 2vnx\, dx \tag{1.7}$$

valid for any real v and any integer values of m and n. This result can also be deduced from a more general formula given in ref.[2] (p.743, item 6.688(2)). Multiplying both sides by $(-1)^n$ and summing with respect to n from 1 to an arbitrarily chosen large positive integer N, one finds for any $m \geq 1$:

$$\begin{aligned}
&\sum_{n=1}^{N} (-1)^n \frac{\sin(2v\sqrt{m^2+n^2})}{(m^2+n^2)^{1/2}} \\
&\quad = 2v \int_0^1 J_0(2vm\sqrt{1-x^2}) \sum_{n=1}^{N} (-1)^n \cos 2vnx\, dx \\
&\quad = 2v \int_0^1 J_0(2vm\sqrt{1-x^2}) \left\{ -\frac{1}{2} + \frac{(-1)^N}{2} \frac{\cos(2N+1)vx}{\cos vx} \right\} dx \\
&\quad = -v \int_0^1 J_0(2vm\sqrt{1-x^2})\, dx \\
&\qquad\quad + v \int_0^1 J_0(2vm\sqrt{1-x^2}) \frac{\sin(2N+1)(vx-\pi/2)}{\sin(vx-\pi/2)}\, dx\,.
\end{aligned}$$

The function

$$\frac{\sin(2N+1)u}{\sin u}\,, \qquad \forall u \in \mathbb{R}$$

is an even π-periodic function of u. It is of oscillatory nature, with zeros at $u = j\pi/(2N+1)$, $j = 1, 2, \ldots, N$ in $0 < u \leq \pi/2$. It is descending from $2N+1$ at $u = 0$ to a negative local minimum situated between $\pi/(2N+1)$ and $2\pi/(2N+1)$, ascending from that minimum to a positive local maximum situated close to $5\pi/2(2N+1)$, again descending to a negative local minimum lying close to $7\pi/2(2N+1)$, etc., and it is equal to $(-1)^N$ at $u = \pi/2$. In the limit $N = +\infty$, it is positive infinite at $u = 0$. The union over N of the zero-sets of the functions comprised in the above expression for $N \geq N_0 > 1$ is dense in $[0, \pi/2]$. In every interval $[a, b] \subset]0, \pi/2]$, the sequence of distributions $[\sin(2N+1)u]/\sin u$, $N = N_0, N_0+1, \ldots$, converges to zero. Therefore, in the Cesàro-sense, it may be regarded as a periodically repeated Dirac δ-function,i.e.,

$$\lim_{N\to+\infty}\frac{\sin(2N+1)u}{\sin u}=C\sum_{k=-\infty}^{+\infty}\delta(u-k\pi)\,,\qquad C>0\,,$$

the whole of oscillations between 0 and π having zero measure when integrated. Since, for any $N>1$,

$$\int_{-\pi/2}^{\pi/2}\frac{\sin(2N+1)u}{\sin u}\,du$$
$$=\int_{-\pi/2}^{\pi/2}(1+2\cos 2u+2\cos 4u+\cdots+2\cos 2Nu)\,du=\pi\,,$$

independent of N, and

$$C\int_{-\pi/2}^{\pi/2}\delta(u)\,du=C\,,$$

it is clear that $C=\pi$ and so, in conclusion, for any $m\geq 1$:

$$\sum_{n=1}^{+\infty}(-1)^n\frac{\sin(2v\sqrt{m^2+n^2})}{(m^2+n^2)^{1/2}}=-v\int_0^1 J_0(2vm\sqrt{1-x^2})\,dx$$
$$+\pi v\int_0^1 J_0(2vm\sqrt{1-x^2})\sum_{k=-\infty}^{+\infty}\delta[vx-\frac{(2k+1)\pi}{2}]\,dx$$
$$=-v\int_0^1 J_0(2vm\sqrt{1-x^2})\,dx$$
$$+\pi\int_0^1 J_0(2vm\sqrt{1-x^2})\sum_{k=-\infty}^{+\infty}\delta\left[x-\frac{(2k+1)\pi}{2v}\right]dx\,. \tag{1.8}$$

Next,

$$\int_0^1 J_0(2vm\sqrt{1-x^2})\,dx=\int_0^1 J_0(2vmy)\frac{y}{\sqrt{1-y^2}}\,dy$$
$$=\sum_{j=0}^{+\infty}(-1)^j\frac{(vm)^{2j}}{j!\,j!}\int_0^1\frac{y^{2j+1}}{\sqrt{1-y^2}}\,dy=\sum_{j=0}^{+\infty}(-1)^j\frac{(vm)^{2j}}{j!\,j!}\frac{2.4\ldots(2j)}{3.5\ldots(2j+1)}$$
$$=\sum_{j=0}^{+\infty}(-1)^j\frac{(2vm)^{2j}}{(2j+1)!}=\frac{\sin 2vm}{2vm}\,,$$

in agreement with (1.7) when n is set equal to zero. Furthermore, a δ-function in (1.8) can only contribute to the integral when its singularity is located in $0<x\leq 1$.

Hence,

$$\sum_{n=1}^{+\infty}(-1)^n \frac{\sin(2v\sqrt{m^2+n^2})}{(m^2+n^2)^{1/2}}$$

$$= \begin{cases} -\dfrac{\sin 2vm}{2m}\,, & |v| < \dfrac{\pi}{2}\,, \\ \dfrac{\pi}{2}\mathrm{sgn}(v)\,, & v = \pm\dfrac{\pi}{2}\,, \\ -\dfrac{\sin 2vm}{2m} + \pi\mathrm{sgn}(v)\displaystyle\sum_{r=1}^{j} J_0(m\sqrt{4v^2-(2r-1)^2\pi^2})\,, & \\ \qquad (2j-1)\pi/2 < |v| < (2j+1)\pi/2\,,\ \ j=1,2,\dots\,, & \\ \pi\mathrm{sgn}(v)\left[\dfrac{1}{2} + \displaystyle\sum_{r=1}^{j} J_0(m\pi\sqrt{(2j+1)^2-(2r-1)^2})\right], & \\ \qquad v = \pm(2j+1)\pi/2\,,\ \ j=1,2,\dots\,. & \end{cases}$$

For the sake of clarity, let v be momentarily non-negative. Taking (1.4) and (1.5) into account, summation with respect to m yields :

$$S(v) = \begin{cases} -v + \displaystyle\sum_{m=1}^{+\infty}(-1)^m\left(-\frac{\sin 2vm}{2m}\right) = -v - \frac{1}{2}(-v) = -\frac{v}{2}, \\ \qquad 0 \le v < \dfrac{\pi}{2}\,, \\ 0 + \displaystyle\sum_{m=1}^{+\infty}(-1)^m\frac{\pi}{2} = \frac{\pi}{2}\sum_{m=1}^{+\infty}(-1)^m\,,\ \ v = \frac{\pi}{2}\,,\ \text{and similarly}\,, \\ -\dfrac{v}{2} + j\dfrac{\pi}{2} + \pi\displaystyle\sum_{r=1}^{j}\left[\sum_{m=1}^{+\infty}(-1)^m J_0(m\sqrt{4v^2-(2r-1)^2\pi^2})\right], \\ \qquad (2j-1)\pi/2 < v < (2j+1)\pi/2\,,\ \ j=1,2,\dots\,, \\ \dfrac{\pi}{2}\displaystyle\sum_{m=1}^{+\infty}(-1)^m + \pi\sum_{r=1}^{j}\left[\sum_{m=1}^{+\infty}(-1)^m J_0(m\pi\sqrt{(2j+1)^2-(2r-1)^2})\right], \\ \qquad v = (2j+1)\pi/2\,,\ \ j=1,2,\dots\,, \end{cases}$$

in which $\sum_{m=1}^{+\infty}(-1)^m$ has to be summed in the Cesàro-sense which entails continuity between the first two and the last two right-hand sides. Therefore, by virtue of

$$\sum_{m=1}^{+\infty}(-1)^m = -\frac{1}{2} \quad \text{(C1)}\,,$$

there comes :

$$S(v) = \begin{cases} -\dfrac{v}{2}\,, & 0 \le v \le \dfrac{\pi}{2}\,, \\ -\dfrac{v}{2} + j\dfrac{\pi}{2} + \pi\displaystyle\sum_{r=1}^{j}\left[\sum_{m=1}^{+\infty}(-1)^m J_0(m\sqrt{4v^2-(2r-1)^2\pi^2})\right]\,, & \\ \qquad (2j-1)\pi/2 < v \le (2j+1)\pi/2\,,\ \ j=1,2,\dots\,. & \end{cases}$$

By means of

$$\sum_{m=1}^{+\infty}(-1)^m J_0(mx) = \begin{cases} -\dfrac{1}{2}\,, & -\pi < x < \pi\,, \\ -\dfrac{1}{2} + 2\displaystyle\sum_{s=1}^{l}\frac{1}{\sqrt{x^2-(2s-1)^2\pi^2}}\,, \\ \qquad (2l-1)\pi \le |x| < (2l+1)\pi\,, \\ \qquad l = 1, 2, \ldots\,, \end{cases} \tag{1.9}$$

one obtains as final result for any $v \in \mathbb{R}$, taking into account that $S(-v) = -S(v)$,

$$S(v) = \begin{cases} -\dfrac{v}{2}\,, & -\dfrac{\pi}{\sqrt{2}} < v < \dfrac{\pi}{\sqrt{2}}\,, \\ -\dfrac{v}{2} + \operatorname{sgn}(v)\displaystyle\sum_{r=1}^{j}\sum_{s=1}^{l_v(r)}\frac{1}{\left[\dfrac{v^2}{\pi^2}-(r-\dfrac{1}{2})^2-(s-\dfrac{1}{2})^2\right]^{1/2}}\,, \\ \qquad |v| \ge \dfrac{\pi}{\sqrt{2}}\,, \end{cases} \tag{1.10}$$

in which j is the largest integer smaller than or equal to

$$\frac{1}{2} + \left(\frac{v^2}{\pi^2} - \frac{1}{4}\right)^{1/2} \tag{1.10$'$}$$

and $l_v(r)$ is the largest integer smaller than or equal to

$$\frac{1}{2} + \left[\frac{v^2}{\pi^2} - \left(r - \frac{1}{2}\right)^2\right]^{1/2} . \tag{1.10$''$}$$

The first right-hand side in the final result for $S(v)$ proves the first conjecture in (1.3). It is astonishing that in $v \ge \pi\sqrt{2} \;\wedge\; v \le -\pi/\sqrt{2}$, $S(v)$ has infinitely many vertical asymptotes, i.e., at the abscissae

$$v = \pm\pi\left[\left(r-\frac{1}{2}\right)^2 + \left(s-\frac{1}{2}\right)^2\right]^{1/2}, \qquad \forall r \in \mathbb{N}_0\,, \forall s \in \{1, 2, \ldots, r\}\,, \\ (\mathbb{N}_0 = \{1, 2, \ldots\})\,. \tag{1.11}$$

These are the v-values for which the double series in (1.1) suddenly becomes divergent.

Let the positive v-values comprised in (1.11) be classified in ascending order. To every half-open interval between two consecutive of these v-values, closed at the left-hand side and open at the right, there belongs a positive integer j determined by (1.10$'$), for instance, $j = 1$ for $\pi\sqrt{2}/2 \le v < \pi\sqrt{10}/2$, $j = 2$ for $\pi\sqrt{10}/2 \le v < \pi\sqrt{18}/2$ and $\pi\sqrt{18}/2 \le v < \pi\sqrt{26}/2$, $j = 3$ for $\pi\sqrt{26}/2 \le v < \pi\sqrt{34}/2$ and $\pi\sqrt{34}/2 \le v < \pi\sqrt{50}/2$, etc. Actually, the same j belongs to the set of above consecutive half-open intervals which are contained in the wider interval

$$\pi\left(j^2 - j + \frac{1}{2}\right)^{1/2} \le v < \pi\left(j^2 + j + \frac{1}{2}\right)^{1/2} .$$

What distinguishes the half-open intervals between two consecutive positive v-values of the ordered set (1.11) from one another in (1.10) are the upper bounds of s for $r = 1, 2, \ldots, j$ in the double sum. For instance,

– to $\pi\sqrt{2}/2 \le v < \pi\sqrt{10}/2$ belongs $j = 1, r = 1, s = 1$, hence $l_v(1) = 1$;

– to $\pi\sqrt{10}/2 \le v < \pi\sqrt{18}/2$ belongs $j = 2$,

$$\begin{aligned} r &= 1, \quad s = 1, 2, \text{ hence } l_v(1) = 2, \\ r &= 2, \quad s = 1, \text{ hence } l_v(2) = 1; \end{aligned}$$

– to $\pi\sqrt{18}/2 \le v < \pi\sqrt{26}/2$ belongs $j = 2$,

$$\begin{aligned} r &= 1, \quad s = 1, 2, \text{ hence } l_v(1) = 2, \\ r &= 2, \quad s = 1, 2, \text{ hence } l_v(2) = 2; \end{aligned}$$

– to $\pi\sqrt{26}/2 \le v < \pi\sqrt{34}/2$ belongs $j = 3$,

$$\begin{aligned} r &= 1, \quad s = 1, 2, 3, \text{ hence } l_v(1) = 3, \\ r &= 2, \quad s = 1, 2, \text{ hence } l_v(2) = 2, \\ r &= 3, \quad s = 1, \text{ hence } l_v(3) = 1; \end{aligned}$$

– to $\pi\sqrt{34}/2 \le v < \pi\sqrt{50}/2$ belongs $j = 3$,

$$\begin{aligned} r &= 1, \quad s = 1, 2, 3, \text{ hence } l_v(1) = 3, \\ r &= 2, \quad s = 1, 2, 3, \text{ hence } l_v(2) = 3, \\ r &= 3, \quad s = 1, 2, \text{ hence } l_v(3) = 2; \quad \text{etc.} \end{aligned}$$

Note that $l_v(1) = j$ because $(1.10')$ and $(1.10'')$ are identical conditions for $r = 1$. Furthermore, $l_v(2) \le j$ and $l_v(r) < j$ for $r = 3, 4, \ldots, j$. When v varies over $[\pi\sqrt{50}/2, \pi\sqrt{58}/2[$, the expression $(1.10')$ varies over $[4, (1/2)+(1/2)\sqrt{57}[$ and clearly $j = 4$. The values of r are $1, 2, 3, 4$ and $l_v(1) = 4$. With $r = 2$ inserted into $(1.10'')$, the expression varies over $[(1/2) + (1/2)\sqrt{41}, 4[$ and therefore $l_v(2) = 3$. With $r = 3$ inserted into $(1.10'')$, the expression varies over $[3, (1/2) + (1/2)\sqrt{33}[$ so that $l_v(3) = 3$. Finally, with $r = 4$ inserted into $(1.10'')$, the expression varies over $[1, 2[$ so that $l_v(4) = 1$. Consequently,

$$\begin{aligned} r = 1 \quad &, \quad s = 1, 2, 3, 4 \\ r = 2 \quad &, \quad s = 1, 2, 3 \\ r = 3 \quad &, \quad s = 1, 2, 3 \\ r = 4 \quad &, \quad s = 1 \end{aligned} \tag{1.12}$$

and the double sum in (1.10) contains eleven terms.

An equivalent way of writing the double sum in (1.10) is

$$\operatorname{sgn}(v) \sum_{r=1}^{+\infty} \sum_{s=1}^{+\infty} \frac{1}{\left[\dfrac{v^2}{\pi^2} - \left(r - \dfrac{1}{2}\right)^2 - \left(s - \dfrac{1}{2}\right)^2\right]^{1/2}},$$

$$\forall (r, s) : \frac{v^2}{\pi^2} - \left(r - \frac{1}{2}\right)^2 - \left(s - \frac{1}{2}\right)^2 \ge 0 \,. \tag{1.13}$$

For finite v, this double series never involves an infinite number of terms by virtue of the condition on the (r,s)-couples. For any finite v,

$$\left(r-\frac{1}{2}\right)^2+\left(s-\frac{1}{2}\right)^2$$

being an expression which increases with growing r,s and both, does not remain smaller than v^2/π^2. At a certain moment, it starts getting larger than v^2/π^2. (1.13) is called locally finite. For instance, when $\pi\sqrt{50}/2 \le v < \pi\sqrt{58}/2$, the (r,s)-couples in (1.13) are those listed in (1.12).

1b. In their solution, J. Boersma and P.J. de Doelder [3] combine $C(v)$ and $S(v)$ by forming $C(v)+iS(v)$ and subject $f(v;x,y)$ defined as

$$f(v;x,y) := \begin{cases} \dfrac{\exp(2iv\sqrt{x^2+y^2})-1}{\sqrt{x^2+y^2}}\,, & (x,y)\neq(0,0)\,, \\ 2iv\,, & (x,y)=(0,0)\,, \end{cases}$$

with $v \ge 0$ to two-dimensional complex Fourier transformation with respect to x and y. This offers the advantage that they simultaneously attain results for $S(v)$ and $C(v)$ both of the form (1.10). In what follows, several representations of $C(v)$, some more interesting from the numerical point of view and some mainly of theoretical importance, will be deduced.

In contrast to $S(0)=0$, one notices that $C(0)\not\equiv 0$ and I shall first concentrate on the constant $C(0)$ given by

$$\begin{aligned} \sum_{m=0}^{+\infty}\sum_{n=1}^{+\infty}\frac{(-1)^{m+n}}{(m^2+n^2)^{1/2}} &= \sum_{n=1}^{+\infty}\frac{(-1)^n}{n}+\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}\frac{(-1)^{m+n}}{(m^2+n^2)^{1/2}} \\ &= -\ln 2+\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}\frac{(-1)^{m+n}}{(n^2+m^2)^{1/2}}\,. \end{aligned}$$

Making use of form. (1) in [Chap. XIII,13·2] of ref. [1], i.e.,

$$\int_0^\infty \mathrm{e}^{-at}J_0(bt)\,dt=\frac{1}{(a^2+b^2)^{1/2}}\,, \qquad \mathrm{Re}(a)>0\,,$$

one can write

$$\begin{aligned} C(0) &= -\ln 2+\sum_{m=1}^{+\infty}(-1)^m\sum_{n=1}^{+\infty}(-1)^n\int_0^\infty \mathrm{e}^{-nt}J_0(mt)\,dt \\ &= -\ln 2-\int_0^\infty\frac{dt}{\mathrm{e}^t+1}\sum_{m=1}^{+\infty}(-1)^m J_0(mt)\,. \end{aligned}$$

Applying (1.9), one finds :

$$\begin{aligned}
C(0) &= -\ln 2 + \frac{1}{2}\int_0^\infty \frac{dt}{\mathrm{e}^t+1} \\
&\quad -2\sum_{s=1}^{+\infty}\int_{(2s-1)\pi}^\infty \frac{dt}{(\mathrm{e}^t+1)[t^2-(2s-1)^2\pi^2]^{1/2}} \\
&= -\frac{1}{2}\ln 2 - 2\sum_{s=1}^{+\infty}\int_1^\infty \frac{du}{\sqrt{u^2-1}\{1+\exp[(2s-1)\pi u]\}} \\
&= -\frac{1}{2}\ln 2 - 2\sum_{s=1}^{+\infty}\int_1^\infty \frac{\exp[-(2s-1)\pi u]}{\sqrt{u^2-1}\{1+\exp[-(2s-1)\pi u]\}}\, du \\
&= -\frac{1}{2}\ln 2 + 2\sum_{s=1}^{+\infty}\int_1^\infty \frac{du}{\sqrt{u^2-1}}\sum_{r=1}^{+\infty}(-1)^r\exp[-r(2s-1)\pi u] \\
&= -\frac{1}{2}\ln 2 + 2\sum_{s=1}^{+\infty}\left\{\sum_{r=1}^{+\infty}(-1)^r\int_1^\infty \frac{\exp[-r(2s-1)\pi u]}{\sqrt{u^2-1}}\, du\right\},
\end{aligned}$$

which permits the use of the modified Bessel function of the second kind and order zero (cfr. [1], VI,6.3 p.185) :

$$K_0(z) = \int_1^\infty \frac{\mathrm{e}^{-zu}}{\sqrt{u^2-1}}\, du\,.$$

Hence,

$$\begin{aligned}
C(0) &= -\frac{1}{2}\ln 2 + 2\sum_{s=1}^{+\infty}\left\{\sum_{r=1}^{+\infty}(-1)^r K_0[r(2s-1)\pi]\right\} \\
&= -0.403\,885\,656\,\ldots
\end{aligned}$$

where the numerical value of $C(0)$ was obtained with the help of a computer program for the numerical integration of

$$\mathcal{J}(a) := \int_1^\infty \frac{du}{\sqrt{u^2-1}(1+\mathrm{e}^{au})} = \int_0^\infty \frac{dx}{\sqrt{1+x^2}[1+\exp(a\sqrt{1+x^2})]}$$

applicable for any real $a \geq 1$, say. A high accuracy is easily attained on account of the rapid decrease of the integrand with growing x. Similarly, due to

$$0 < \mathcal{J}(a) < K_0(a) \simeq \left(\frac{\pi}{2a}\right)^{1/2}\mathrm{e}^{-a}[1+\mathcal{O}(1/a)]\,,$$

the convergence of the series in

$$C(0) = -\frac{1}{2}\ln 2 - 2\sum_{s=1}^{+\infty}\mathcal{J}[(2s-1)\pi]$$

is also fast enough to obtain easily the mentioned numerical value with all the decimals being significant.

As far as an analytical representation of $C(0)$ in terms of known functions is concerned, one can appeal to Table I in [4] in which one finds :

$$\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}\frac{(-1)^{m+n}}{(m^2+n^2)^s}=(1-2^{1-2s})\zeta(2s)-(1-2^{1-2s})\beta(s)\zeta(s)\,, \qquad (1.14^{false})$$

where $\zeta(s)$ is the well-known Riemann zeta function and

$$\beta(s):=\sum_{j=0}^{+\infty}\frac{(-1)^j}{(2j+1)^s}\,, \qquad s>0\,.$$

There is a misprint in (1.14^{false}) since the formula leads to $C(0)=0$. Indeed, if it were correct, one would have

$$\begin{aligned} C(0) &= \sum_{n=1}^{+\infty}\frac{(-1)^n}{n}+\lim_{s\to 1/2}[(1-2^{1-2s})\zeta(2s)-(1-2^{1-2s})\beta(s)\zeta(s)] \\ &= -\ln 2+\lim_{s\to 1/2}[(1-2^{1-2s})\zeta(2s)]-0.\beta(1/2)\zeta(1/2) \end{aligned}$$

whereby $\beta(1/2)$ and $\zeta(1/2)$ are finite numbers :

$$1/2<\beta(1/2)<\pi/4\,, \qquad -(\sqrt{2}+1)\ln 2<\zeta(1/2)<-(\sqrt{2}+1)/2\,.$$

Hence,

$$\begin{aligned} C(0) &= -\ln 2+\lim_{s\to 1/2}\left(1-\mathrm{e}^{(1-2s)\ln 2}\right)\left(\frac{1}{2s-1}+\mathcal{O}(s^0)\right) \\ &= -\ln 2+\lim_{s\to 1/2}\left[-(1-2s)\ln 2+\mathcal{O}((1-2s)^2)\right]\left(\frac{1}{2s-1}+\mathcal{O}(s^0)\right) \\ &= -\ln 2+\ln 2=0\,. \end{aligned}$$

In [5], the correct formula replacing the erroneous one appears to be

$$\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}\frac{(-1)^{m+n}}{(m^2+n^2)^s}=(1-2^{1-2s})\zeta(2s)-(1-2^{1-s})\beta(s)\zeta(s)\,, \qquad (1.14)$$

so that

$$\begin{aligned} C(0) &= (\sqrt{2}-1)\beta(1/2)\zeta(1/2) \\ &= -\left(1-\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{5}}-\cdots\right)\left(1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\cdots\right) \end{aligned}$$

from which one can deduce

$$-\frac{\pi}{4}\ln 2<C(0)<-\frac{1}{4}$$

or

$$-0.544\,40<C(0)<-0.25\,,$$

indeed satisfied by $C(0) = -0.403\,885\,656\ldots$, very close to the middle of the interval.

The cosine analogue of (1.5) is

$$\sum_{n=1}^{+\infty}(-1)^n\frac{\cos nx}{n} = -\ln\left(2\left|\cos\frac{x}{2}\right|\right)\,, \qquad \forall x\in\mathbb{R}\,.$$

By differentiation with respect to x, one finds in the Cesàro-sense (C1) :

$$\sum_{n=1}^{+\infty}(-1)^n\sin nx = \begin{cases} -\frac{1}{2}\tan\frac{x}{2}\,, & (2j-1)\pi < x < (2j+1)\pi\,,\ \forall j\in\mathbb{Z}\,,\\ 0\,, & x=\pm\pi,\pm3\pi,\ldots \end{cases} \tag{1.15}$$

One could consider treating the problem with $C(v)$ along the same lines as that with $S(v)$. For that purpose, one has to dispose of the analogue of (1.7) which is

$$\begin{aligned}\frac{\cos(2v\sqrt{m^2+n^2})}{(m^2+n^2)^{1/2}} &= -2v\int_0^1 J_0(2vm\sqrt{1-x^2})\sin 2vnx\,dx\\ &\quad+\frac{2n}{\pi}\int_0^1\frac{\cos 2vmu}{[m^2(1-u^2)+n^2]\sqrt{1-u^2}}\,du\,,\end{aligned} \tag{1.16}$$

but in contrast to (1.7), this result is valid for $0\le m<n$ only. This restriction, as well as the appearance of two integrals in the right-hand side, make (1.16) less attractive. Instead, one can make use of form. (4) in [Chap. XIII, 13·47] of ref. [1], i.e.,

$$\int_0^\infty J_0(bt)\frac{\exp[-a\sqrt{t^2-y^2}]}{(t^2-y^2)^{1/2}}t\,dt = \frac{\exp[\mp iy\sqrt{a^2+b^2}]}{(a^2+b^2)^{1/2}}$$

where the upper or lower sign is taken according as the indentation passes above or below the y-axis. This leads to

$$\begin{aligned}C(v) &= \sum_{n=1}^{+\infty}(-1)^n\frac{\cos(2vn)}{n}+\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}(-1)^{n+m}\frac{\cos(2v\sqrt{m^2+n^2})}{(m^2+n^2)^{1/2}}\\ &= -\ln(2|\cos v|)+\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}(-1)^{n+m}\left\{-\int_0^{2|v|}tJ_0(nt)\frac{\sin(m\sqrt{4v^2-t^2})}{(4v^2-t^2)^{1/2}}\,dt\right.\\ &\quad\left.+\int_{2|v|}^\infty tJ_0(nt)\frac{\exp[-m\sqrt{t^2-4v^2}]}{(t^2-4v^2)^{1/2}}\,dt\right\}.\end{aligned} \tag{1.17}$$

Taking (1.15) into account, (1.17) becomes :

$$\begin{aligned}
C(v) &= -\ln(2|\cos v|) \\
&\quad + \sum_{n=1}^{+\infty}(-1)^n \left\{ \frac{1}{2}\mathcal{P}\int_0^{2|v|} tJ_0(nt)\frac{\tan(\sqrt{4v^2-t^2}/2)}{(4v^2-t^2)^{1/2}}\,dt \right. \\
&\qquad \left. -\int_{2|v|}^{\infty}\frac{tJ_0(nt)}{(t^2-4v^2)^{1/2}[1+\exp(\sqrt{t^2-4v^2})]}\,dt \right\} \\
&= -\ln(2|\cos v|) \\
&\quad + \sum_{n=1}^{+\infty}(-1)^n \left\{ \frac{1}{2}\mathcal{P}\int_0^{2|v|} J_0(n\sqrt{4v^2-x^2})\tan(x/2)\,dx \right. \\
&\qquad \left. -\int_0^{\infty}\frac{J_0(n\sqrt{4v^2+u^2})}{e^u+1}\,du \right\} .
\end{aligned} \tag{1.18}$$

The appearance of the Cauchy principal value operator $\mathcal{P}$ stems from the fact that the right-hand side of (1.15) involves infinitely many discontinuities which may be removed by means of cut-offs which are symmetric with respect to the vertical asymptotes. In connection herewith, it may be preferable to incorporate the same prescription as for taking the Cauchy principal value of a divergent integral into (1.15) by writing :

$$\sum_{n=1}^{+\infty}(-1)^n \sin nx = \lim_{\epsilon\to+0} \begin{cases} -\dfrac{1}{2}\tan\dfrac{x}{2}\,, \quad (2j-1)\pi+\epsilon < x < (2j+1)\pi-\epsilon\,, \\ \hfill \forall j\in\mathbb{Z}\,, \qquad (1.15') \\ 0\,, \quad (2j+1)\pi-\epsilon \le x \le (2j+1)\pi+\epsilon\,,\ \forall j\in\mathbb{Z}\,. \end{cases}$$

In order to remove the summation with respect to n in (1.18), one can apply (1.9) and obtain :

– for $-\pi/2 < v < \pi/2$:

$$\begin{aligned}
C(v) = &-\frac{1}{2}\ln(2\cos v) \\
&-2\sum_{s=1}^{+\infty}\int_{[(2s-1)^2\pi^2-4v^2]^{1/2}}^{\infty}\frac{du}{(e^u+1)[u^2-(2s-1)^2\pi^2+4v^2]^{1/2}}\,;
\end{aligned} \tag{1.19}$$

– for $(2j-1)\pi/2 < |v| < (2j+1)\pi/2\,, \qquad \forall j \in \mathbb{N}_0$:

$$\begin{aligned} C(v) = \ & -\frac{1}{2}\ln(2|\cos v|) \\ & +\sum_{s=1}^{j}\mathcal{P}\int_0^{[4v^2-(2s-1)^2\pi^2]^{1/2}} \frac{\tan(x/2)}{[4v^2-(2s-1)^2\pi^2-x^2]^{1/2}}\,dx \\ & -2\sum_{s=1}^{j}\int_0^{\infty}\frac{du}{(\mathrm{e}^u+1)[u^2+4v^2-(2s-1)^2\pi^2]^{1/2}} \\ & -2\sum_{s=j+1}^{+\infty}\int_{[(2s-1)^2\pi^2-4v^2]^{1/2}}^{\infty}\frac{du}{(\mathrm{e}^u+1)[u^2-(2s-1)^2\pi^2+4v^2]^{1/2}}\,. \end{aligned} \tag{1.20}$$

When $|v|$ converges towards $\pi/2$, the first term in (1.19) tends to $+\infty$, but at the same time, the first integral in the series also tends to $+\infty$ and therefore an indeterminacy of the type $(+\infty-\infty)$ occurs. It can be eliminated making use of some formulae belonging to the theory of the Gamma-function and so,

$$\begin{aligned} C(\pm\pi/2) \ &= \ \gamma-\frac{1}{2}\ln(\pi/2) \\ & \qquad -2\sum_{s=2}^{+\infty}\int_{2\pi\sqrt{s(s-1)}}^{\infty}\frac{du}{(\mathrm{e}^u+1)[u^2-4\pi^2 s(s-1)]^{1/2}}. \end{aligned} \tag{1.21}$$

In a similar way, one gets :

$$\begin{aligned} & C(\pm(2j+1)\pi/2) = \gamma-\frac{1}{2}\ln[\pi/2(2j+1)] \\ & +\sum_{s=1}^{j}\mathcal{P}\int_0^{2\pi\sqrt{(j+s)(j-s+1)}}\frac{\tan(x/2)}{[4\pi^2(j+s)(j-s+1)-x^2]^{1/2}}\,dx \\ & -2\sum_{s=1}^{j}\int_0^{\infty}\frac{du}{(\mathrm{e}^u+1)[u^2+4\pi^2(j+s)(j-s+1)]^{1/2}} \\ & -2\sum_{s=j+2}^{+\infty}\int_{2\pi\sqrt{(s+j)(s-j-1)}}^{\infty}\frac{du}{(\mathrm{e}^u+1)[u^2-4\pi^2(s+j)(s-j-1)]^{1/2}}\,, \\ & \qquad\qquad\qquad \forall j\in\mathbb{N}_0\,. \end{aligned} \tag{1.22}$$

Taken together, the formulae (1.19)–(1.22) constitute a representation of $C(v)$, $\forall v \in \mathbb{R}$, clearly generalizing

$$C(0) = -\frac{1}{2}\ln 2 - 2\sum_{s=1}^{+\infty}\int_{(2s-1)\pi}^{\infty}\frac{du}{(\mathrm{e}^u+1)[u^2-(2s-1)^2\pi^2]^{1/2}}\,. \tag{1.19$'$}$$

They involve only rapidly converging series.

From (1.19) and (1.21), it follows that $C(v)$ is a continuous function of v in $[-\pi/2, \pi/2]$. In (1.20), there is continuity for $|v| \in [(2j-1)\pi/2, (2j+1)\pi/2]$ as far as the first term, the third part and the fourth part are concerned, but the second part generates infinite discontinuities at the same abscissae as $S(v)$, namely, at

$$v = \pm\pi\left[\left(r-\frac{1}{2}\right)^2+\left(s-\frac{1}{2}\right)^2\right]^{1/2},\ \forall r\in\mathbb{N}_0\,,\ \forall s\in\{1,2,\ldots,r\}. \tag{1.23}$$

Indeed, each time the upper bound of integration $[4v^2-(2s-1)^2\pi^2]^{1/2}$ attains a positive odd integer multiple of π, the tangent function causes divergence of the integral involved, as it gives rise to a (single) pole at that bound. The infinite jumps closest to the origin being located at $v=\pm\pi/\sqrt{2}, C(v)$ is a continuous function of v in $]-\pi/\sqrt{2},\pi/\sqrt{2}[$ just as $S(v)$, but it is by far not as simple.

From (1.19), it follows that $C(v)<0$ for any $v\in[0,\pi/3]$. For $v\in]\pi/3,\pi/2]$, the first term in (1.19) is positive whereas the second part continues to furnish a negative contribution. Computer tabulation reveals that $C(v)$ increases monotonically from $-0.140\,972\ldots$ at $v=\pi/3$ to $0.351\,309\ldots$ at $v=\pi/2$. Consequently, $C(v)$ has a single zero inside $[\pi/3,\pi/2]$. That zero is $v_0=1.252\,129\ldots$ and not $5/4$ as is conjectured by the proposers in (1.3). Their numerical work clearly was not sufficiently accurate, but $5/4$ differs from v_0 only by approximately 0.002. It could therefore have been used in practice when a precision of two decimals would be sufficient.

A second representation of $C(v)$ no longer containing integrals but at the expense of involving a double series, can be obtained as follows for $-\pi/2<v<\pi/2$, making use of (1.19) and (1.19$'$) :

$$\begin{aligned}C(v)-C(0) = & -\frac{1}{2}\ln\cos v\\ & -2\sum_{s=1}^{+\infty}\int_1^\infty\frac{dt}{\sqrt{t^2-1}\{1+\exp[\sqrt{(2s-1)^2\pi^2-4v^2}t]\}}\\ & +2\sum_{s=1}^{+\infty}\int_1^\infty\frac{dt}{\sqrt{t^2-1}\{1+\exp[(2s-1)\pi t]\}}\\ = & -\frac{1}{2}\ln\cos v-2\sum_{s=1}^{+\infty}\int_1^\infty\frac{\exp[-\sqrt{(2s-1)^2\pi^2-4v^2}t]}{\sqrt{t^2-1}\{1+\exp[-\sqrt{(2s-1)^2\pi^2-4v^2}t]\}}]\,dt\\ & +2\sum_{s=1}^{+\infty}\int_1^\infty\frac{\exp[-(2s-1)\pi t]}{\sqrt{t^2-1}\{1+\exp[-(2s-1)\pi t]\}}\,dt\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\ln\cos v + 2\sum_{s=1}^{+\infty}\sum_{r=1}^{+\infty}(-1)^r\Big\{K_0(r[(2s-1)^2\pi^2-4v^2]^{1/2}) \\
&\qquad\qquad -K_0[r(2s-1)\pi]\Big\} \\
&= -\frac{1}{2}\ln\cos v + 2\sum_{s=1}^{+\infty}\sum_{r=1}^{+\infty}(-1)^r\Big\{\int_0^\infty \frac{\cos ru}{[(2s-1)^2\pi^2-4v^2+u^2]^{1/2}}\,du \\
&\qquad\qquad -\int_0^\infty \frac{\cos ru}{[(2s-1)^2\pi^2+u^2]^{1/2}}\,du\Big\} \\
&= -\frac{1}{2}\ln\cos v + 2\sum_{s=1}^{+\infty}\int_0^\infty\Bigg(\frac{1}{[(2s-1)^2\pi^2-4v^2+u^2]^{1/2}} \\
&\qquad -\frac{1}{[(2s-1)^2\pi^2+u^2]^{1/2}}\Bigg)\Bigg(-\frac{1}{2}+\pi\sum_{r=-\infty}^{+\infty}\delta[u-(2r-1)\pi]\Bigg)\,du \\
&= -\frac{1}{2}\ln\cos v + \frac{1}{2}\sum_{s=1}^{+\infty}\ln\left(1-\frac{4v^2}{(2s-1)^2\pi^2}\right) \\
&\qquad +\sum_{s=1}^{+\infty}\sum_{r=1}^{+\infty}\Bigg(\frac{1}{\left[\left(s-\frac{1}{2}\right)^2-\frac{v^2}{\pi^2}+\left(r-\frac{1}{2}\right)^2\right]^{1/2}} \\
&\qquad\qquad -\frac{1}{\left[\left(s-\frac{1}{2}\right)^2+\left(r-\frac{1}{2}\right)^2\right]^{1/2}}\Bigg).
\end{aligned}
\tag{1.24}
$$

In these calculations, use was made of two integral representations of $K_0(z)$:

$$
K_0(z)=\int_1^\infty \frac{\mathrm{e}^{-zt}}{\sqrt{t^2-1}}\,dt \qquad\text{and}\qquad K_0(xz)=\int_0^\infty \frac{\cos xu}{\sqrt{z^2+u^2}}\,du\,.
$$

A known infinite product representation of the cosine function has as consequence the cancellation of the above first two parts against one another. Hence,

$$
\begin{aligned}
C(v) &= C(0)+\sum_{r=1}^{+\infty}\sum_{s=1}^{+\infty}\Bigg\{\frac{1}{\left[\left(r-\frac{1}{2}\right)^2+\left(s-\frac{1}{2}\right)^2-\frac{v^2}{\pi^2}\right]^{1/2}} \\
&\qquad -\frac{1}{\left[\left(r-\frac{1}{2}\right)^2+\left(s-\frac{1}{2}\right)^2\right]^{1/2}}\Bigg\}\,, \qquad -\pi/2<v<\pi/2\,,
\end{aligned}
\tag{1.25}
$$

showing a greater similarity to (1.10) than previous representations such as (1.19)–(1.22) or the first right-hand side in (1.24). This expression is less suited for computational evaluation of $C(v)$ than (1.19) although it converges faster than the definition of $C(v)$ since its general term is of the form

$$\frac{v^2/2\pi^2}{\left[\left(r-\frac{1}{2}\right)^2+\left(s-\frac{1}{2}\right)^2\right]^{3/2}}+\mathcal{O}\left(\frac{v^4}{\left[\left(r-\frac{1}{2}\right)^2+\left(s-\frac{1}{2}\right)^2\right]^{5/2}}\right).$$

The double series in (1.25) is absolutely convergent by virtue of its general term being positive for all r and $s \in \mathbb{N}_0$.

(1.25) may be extended to any real v. When $(2k-1)\pi/2 < v < (2k+1)\pi/2$, $\forall k \in \mathbb{N}_0$, (1.24) changes into

$$\begin{aligned}
C(v)-C(0) &= -\frac{1}{2}\ln\cos(v-k\pi)\\
&\quad +2\sum_{s=k+1}^{+\infty}\sum_{r=1}^{+\infty}(-1)^r\Big\{K_0(r[(2s-1)^2\pi^2-4v^2]^{1/2})-K_0[r(2s-1)\pi]\Big\}\\
&\quad +\sum_{s=1}^{k}\sum_{r=1}^{+\infty}(-1)^r\Big\{K_0(ir[4v^2-(2s-1)^2\pi^2]^{1/2})\\
&\quad +K_0(-ir[4v^2-(2s-1)^2\pi^2]^{1/2})-2K_0[r(2s-1)\pi]\Big\}\\
&= -\frac{1}{2}\ln[(-1)^k\cos v]+2\sum_{s=k+1}^{+\infty}\sum_{r=1}^{+\infty}(-1)^r\left\{\int_0^\infty\frac{\cos ru\,du}{[(2s-1)^2\pi^2-4v^2+u^2]^{1/2}}\right.\\
&\quad \left.-\int_0^\infty\frac{\cos ru}{[(2s-1)^2\pi^2+u^2]^{1/2}}\,du\right\}\\
&\quad +2\sum_{s=1}^{k}\sum_{r=1}^{+\infty}(-1)^r\left\{\int_{[4v^2-(2s-1)^2\pi^2]^{1/2}}^\infty\frac{\cos ru\,du}{[u^2-(4v^2-(2s-1)^2\pi^2)]^{1/2}}\right.\\
&\quad \left.-\int_0^\infty\frac{\cos ru}{[u^2+(2s-1)^2\pi^2]^{1/2}}\,du\right\}\\
&= -\frac{1}{2}\sum_{s=1}^{k}\ln\left(\frac{4v^2}{(2s-1)^2\pi^2}-1\right)-\frac{1}{2}\sum_{s=k+1}^{+\infty}\ln\left(1-\frac{4v^2}{(2s-1)^2\pi^2}\right)\\
&\quad +2\sum_{s=k+1}^{+\infty}\int_0^\infty\left(\frac{1}{[u^2+(2s-1)^2\pi^2-4v^2]^{1/2}}-\frac{1}{[u^2+(2s-1)^2\pi^2]^{1/2}}\right)\\
&\qquad \times\left(-\frac{1}{2}+\pi\sum_{r=-\infty}^{+\infty}\delta[u-(2r-1)\pi]\right)du
\end{aligned}$$

$$
\begin{aligned}
&+2\sum_{s=1}^{k}\int_{[4v^2-(2s-1)^2\pi^2]^{1/2}}^{\infty}\left(\frac{1}{[u^2+(2s-1)^2\pi^2-4v^2]^{1/2}}\right.\\
&\left.-\frac{1}{[u^2+(2s-1)^2\pi^2]^{1/2}}\right)\left(-\frac{1}{2}+\pi\sum_{r=-\infty}^{+\infty}\delta[u-(2r-1)\pi]\right)du\\
&-2\sum_{s=1}^{k}\int_{0}^{[4v^2-(2s-1)^2\pi^2]^{1/2}}\frac{1}{[u^2+(2s-1)^2\pi^2]^{1/2}}\\
&\qquad\times\left(-\frac{1}{2}+\pi\sum_{r=-\infty}^{+\infty}\delta[u-(2r-1)\pi]\right)du\,.
\end{aligned}
$$

On account of

$$
\begin{aligned}
&\int_0^{\infty}\left(\frac{1}{\sqrt{u^2+A^2}}-\frac{1}{\sqrt{u^2+B^2}}\right)du\\
&=\lim_{\lambda\to+\infty}\int_0^{\lambda}\left(\frac{1}{\sqrt{u^2+A^2}}-\frac{1}{\sqrt{u^2+B^2}}\right)du\\
&=\lim_{\lambda\to+\infty}\ln\left(\frac{u+\sqrt{u^2+A^2}}{u+\sqrt{u^2+B^2}}\right)\Bigg|_{u=0}^{u=\lambda}=\ln\frac{B}{A}\,,\qquad A>0,B>0\,,
\end{aligned}
\tag{1.26}
$$

there comes :

$$
\begin{aligned}
C(v)-C(0)&=-\frac{1}{2}\sum_{s=1}^{k}\ln\left(\frac{4v^2}{(2s-1)^2\pi^2}-1\right)\\
&+\sum_{s=k+1}^{+\infty}\sum_{r=1}^{+\infty}\left(\frac{1}{\left[\left(r-\frac{1}{2}\right)^2+\left(s-\frac{1}{2}\right)^2-\frac{v^2}{\pi^2}\right]^{1/2}}\right.\\
&\qquad\left.-\frac{1}{\left[\left(r-\frac{1}{2}\right)^2+\left(s-\frac{1}{2}\right)^2\right]^{1/2}}\right)\\
&+2\pi\sum_{s=1}^{k}\sum_{r=1}^{+\infty}\int_{[4v^2-(2s-1)^2]^{1/2}}^{\infty}\left(\frac{1}{[u^2+(2s-1)^2\pi^2-4v^2]^{1/2}}\right.\\
&\qquad\left.-\frac{1}{[u^2+(2s-1)^2\pi^2]^{1/2}}\right)\delta[u-(2r-1)\pi]\,du
\end{aligned}
$$

$$-\sum_{s=1}^{k}\int_0^\infty\left(\frac{1}{[u^2+4v^2-(2s-1)^2\pi^2]^{1/2}}\right.$$
$$\left.-\frac{1}{[u^2+(2s-1)^2\pi^2]^{1/2}}\right)du$$
$$-2\pi\sum_{s=1}^{k}\sum_{r=1}^{+\infty}\int_0^{[4v^2-(2s-1)^2\pi^2]^{1/2}}\frac{\delta[u-(2r-1)\pi]}{[u^2+(2s-1)^2\pi^2]^{1/2}}\,du\,, \tag{1.27}$$

in which use was made of

$$\int_A^\lambda\frac{du}{(u^2-A^2)^{1/2}}=\int_0^{\sqrt{\lambda^2-A^2}}\frac{dt}{(t^2+A^2)^{1/2}}=\int_0^{\sqrt{\lambda^2-A^2}}\frac{du}{(u^2+A^2)^{1/2}}\,.$$

The right-hand side of (1.27) is even in v and therefore holds in both regions $v<-\pi/2$ and $v>\pi/2$. By virtue of (1.26), its first part is cancelled by its fourth part. Now, for $s=1$ in the third part, the δ-function has its peak inside the integration interval when

$$(2r-1)\pi>(4v^2-\pi^2)^{1/2}$$

which gives

$$r>\frac{1}{2}+\left(\frac{v^2}{\pi^2}-\frac{1}{4}\right)^{1/2}.$$

In accordance with (1.10″) this means

$$r>l_v(1)\,.$$

Similarly for $s=2,3,\ldots,k$, the δ-function has its peak inside the integration interval when

$$r>\frac{1}{2}+\left[\frac{v^2}{\pi^2}-\left(s-\frac{1}{2}\right)^2\right]^{1/2}$$

which means

$$r>l_v(s)\,.$$

Consequently,

$$C(v)-C(0)=\left\{\sum_{s=k+1}^{+\infty}\sum_{r=1}^{+\infty}+\sum_{s=1}^{k}\sum_{r=l_v(s)+1}^{+\infty}\right\}\left(\frac{1}{\left[\left(r-\frac{1}{2}\right)^2+\left(s-\frac{1}{2}\right)^2-\frac{v^2}{\pi^2}\right]^{1/2}}\right.$$
$$\left.-\frac{1}{\left[\left(r-\frac{1}{2}\right)^2+\left(s-\frac{1}{2}\right)^2\right]^{1/2}}\right)-\sum_{s=1}^{k}\sum_{r=1}^{l_v(s)}\frac{1}{\left[\left(r-\frac{1}{2}\right)^2+\left(s-\frac{1}{2}\right)^2\right]^{1/2}}\,.$$

Finally, to make this result better comparable with (1.10), let us rename k as j, s as r and vice-versa. There comes :

$$C(v)=\begin{cases} C(0)+\displaystyle\sum_{r=1}^{+\infty}\sum_{s=1}^{+\infty}\left\{\frac{1}{\left[\left(r-\frac{1}{2}\right)^2+\left(s-\frac{1}{2}\right)^2-\frac{v^2}{\pi^2}\right]^{1/2}}\right. \\ \qquad\left.-\dfrac{1}{\left[\left(r-\frac{1}{2}\right)^2+\left(s-\frac{1}{2}\right)^2\right]^{1/2}}\right\}, \quad -\pi/\sqrt{2}<v<\pi/\sqrt{2}, \\ C(0)-\displaystyle\sum_{r=1}^{j}\sum_{s=1}^{l_v(r)}\frac{1}{\left[\left(r-\frac{1}{2}\right)^2+\left(s-\frac{1}{2}\right)^2\right]^{1/2}} \\ \qquad+\displaystyle\sum_{r=1}^{+\infty}\sum_{s=l_v(r)+1}^{+\infty}\left(\frac{1}{\left[\left(r-\frac{1}{2}\right)^2+\left(s-\frac{1}{2}\right)^2-\frac{v^2}{\pi^2}\right]^{1/2}}\right. \\ \qquad\left.-\dfrac{1}{\left[\left(r-\frac{1}{2}\right)^2+\left(s-\frac{1}{2}\right)^2\right]^{1/2}}\right), \quad |v|\geq\pi/\sqrt{2}, \end{cases} \tag{1.28}$$

whereby j is still defined as in $(1.10')$ and $l_v(r)$ as in $(1.10'')$ when $r=1,2,\ldots,j$ whereas $l_v(r)=0$ for $r=j+1,j+2,\ldots$. This result confirms the occurrence of infinite discontinuities at the abscissae (1.23). It is worth noticing that (1.10) and (1.28) are related in such a way that their combination yields :

$$\begin{aligned} &\sum_{m=0}^{+\infty}\sum_{n=1}^{+\infty}(-1)^{m+n}\frac{\exp(2iv\sqrt{m^2+n^2})}{\sqrt{m^2+n^2}} \\ &\quad= C(0)+\sum_{r=1}^{+\infty}\sum_{s=1}^{+\infty}\left\{\frac{1}{\left[\left(r-\frac{1}{2}\right)^2+\left(s-\frac{1}{2}\right)^2-\frac{v^2}{\pi^2}\right]^{1/2}}\right. \\ &\qquad\left.-\frac{1}{\left[\left(r-\frac{1}{2}\right)^2+\left(s-\frac{1}{2}\right)^2\right]^{1/2}}\right\}-\frac{v}{2}i, \qquad \forall v\in\mathbb{R}, \end{aligned}$$

or

$$\begin{aligned} &\sum_{m=0}^{+\infty}\sum_{n=1}^{+\infty}(-1)^{m+n}\frac{\exp(2iv\sqrt{m^2+n^2})-1}{\sqrt{m^2+n^2}} \\ &\quad= \sum_{r=1}^{+\infty}\sum_{s=1}^{+\infty}\left\{\frac{1}{\left[\left(r-\frac{1}{2}\right)^2+\left(s-\frac{1}{2}\right)^2-\frac{v^2}{\pi^2}\right]^{1/2}}\right. \\ &\qquad\left.-\frac{1}{\left[\left(r-\frac{1}{2}\right)^2+\left(s-\frac{1}{2}\right)^2\right]^{1/2}}\right\}-\frac{v}{2}i, \qquad \forall v\in\mathbb{R}, \end{aligned} \tag{1.29}$$

if one agrees upon setting

$$\left[\left(r-\frac{1}{2}\right)^2+\left(s-\frac{1}{2}\right)^2-\frac{v^2}{\pi^2}\right]^{1/2}$$
$$=-i\,\mathrm{sgn}(v)\left[\frac{v^2}{\pi^2}-\left(r-\frac{1}{2}\right)^2-\left(s-\frac{1}{2}\right)^2\right]^{1/2},$$

when v exceeds

$$\pi\left[\left(r-\frac{1}{2}\right)^2+\left(s-\frac{1}{2}\right)^2\right]^{1/2}.$$

This is entirely like in the definition of the branch of $(z^2-1)^{1/2}$ in $\mathbb{C}$ which is determined by the arithmetic square root of x^2-1 when $z(=x+yi)\in]1,+\infty[$. Indeed, in that case,

$$\left[(z^2-1)^{1/2}\right]_{z=x\pm 0.i}=\pm\, i\sqrt{1-x^2}\,,\qquad \forall x\in]-1,1[\,.$$

The fraction in the left-hand side of (1.29) explains why Boersma and de Doelder have started their calculation of $S(v)$ and $C(v)$ by introducing $f(v;x,y)$ (cfr. the beginning of subsection 1b) (see also [3]). At the end of their paper, Boersma and de Doelder derive a convergent single series representing $C(v)$ in $-\pi/\sqrt{2}<v<\pi/\sqrt{2}$:

$$C(v)=\left(\frac{2}{\pi}\right)^{1/2}\sum_{n=0}^{+\infty}\frac{\Gamma(n+\frac{1}{2})}{n!}(1-2^{-n-(1/2)})\beta(n+\frac{1}{2})\zeta(n+\frac{1}{2})\left(\frac{2v^2}{\pi^2}\right)^n. \tag{1.30}$$

Independently, for the purpose of computing $C(v)$-values in the same interval, I established :

$$\begin{aligned}C(v) &= C(0)+\sum_{n=1}^{+\infty}\frac{1.3\ldots(2n-1)}{2.4.\ldots 2n}\\ &\quad\times\left(\sum_{r=1}^{+\infty}\sum_{s=1}^{+\infty}\frac{1}{\left[\left(r-\frac{1}{2}\right)^2+\left(s-\frac{1}{2}\right)^2\right]^{n+(1/2)}}\right)\frac{v^{2n}}{\pi^{2n}}\\ &= C(0)+\sum_{n=1}^{+\infty}A_n v^{2n}\cong -0.403\,885\,656+\sum_{n=1}^{10}A_n v^{2n}\end{aligned}$$

whereby

$$\begin{aligned}A_1 &= 2.091\,942\,879\,4\times 10^{-1}, & A_2 &= 2.281\,414\,756\,3\times 10^{-2},\\ A_3 &= 3.707\,098\,721\,0\times 10^{-3}, & A_4 &= 6.530\,541\,970\,7\times 10^{-4},\\ A_5 &= 1.189\,582\,057\,8\times 10^{-4}, & A_6 &= 2.209\,193\,287\,4\times 10^{-5},\\ A_7 &= 4.156\,799\,684\,0\times 10^{-6}, & A_8 &= 7.896\,899\,562\,4\times 10^{-7},\\ A_9 &= 1.511\,341\,062\,6\times 10^{-7}, & A_{10} &= 2.909\,485\,373\,4\times 10^{-8}.\end{aligned} \tag{1.31}$$

For v not too close to $\pm\pi/\sqrt{2}$, this permits the calculation of $C(v)$ with high accuracy. The formula (1.30) presents the advantage that it enables one to express the A-coefficients in terms of the functions β and ζ :

$$A_n = \frac{\sqrt{2}}{\pi^{2n}} \frac{1.3.\ldots(2n-1)}{n!}\left(1 - \frac{1}{2^{n+(1/2)}}\right)\beta(n+\frac{1}{2})\zeta(n+\frac{1}{2}), \quad n = 1, 2, \ldots$$

The asymptotic approximation of A_n is

$$A_n \simeq \sqrt{\frac{2}{\pi n}}\left(\frac{2}{\pi^2}\right)^n \left(1 + \frac{2}{5^{n+(1/2)}} + \mathcal{O}\left(\frac{1}{9^{n+(1/2)}}\right)\right), \quad n \gg 1\,.$$

By means of d'Alembert's test, this formula shows that the series $\sum_{n=1}^{\infty} A_n v^{2n}$ with solely positive coefficients is convergent for $|v| < \pi/\sqrt{2}$.

Another practical formula for the same purpose is obtained on the basis of

$$\begin{aligned} C(v) &= C(0) + \sqrt{2}\left[\left(1 - \frac{2}{\pi^2}v^2\right)^{-1/2} - 1\right] \\ &\quad + \sum_{n=1}^{+\infty}\left(A_n - \frac{1.3.\ldots(2n-1)}{n!}\frac{\sqrt{2}}{\pi^{2n}}\right)v^{2n} \\ &\cong -0.403\,885\,656 + \sqrt{2}\left[\left(1 - \frac{2}{\pi^2}v^2\right)^{-1/2} - 1\right] + \sum_{n=1}^{5} a_n v^{2n} \end{aligned}$$

with

$$\begin{aligned} &a_1 = 0.065\,904\,495\,885, && a_2 = 0.001\,036\,710\,560, \\ &a_3 = 0.000\,029\,572\,564\,5, && a_4 = 0.000\,000\,984\,416\,73, \\ &a_5 = 0.000\,000\,034\,938\,255, && (a_6 = 0.000\,000\,001\,281\,455, \\ & && a_n < 0.5 \times 10^{-10}\,, \quad n \geq 7). \end{aligned} \tag{1.32}$$

The exact expression and the asymptotic approximation of a_n are given by :

$$\begin{aligned} a_n &= \frac{\sqrt{2}}{\pi^{2n}}\frac{1.3.\ldots(2n-1)}{n!}\left[\left(1 - \frac{1}{2^{n+(1/2)}}\right)\beta(n+\frac{1}{2})\zeta(n+\frac{1}{2}) - 1\right] \\ &\simeq 2\sqrt{\frac{2}{5\pi n}}\left(\frac{2}{5\pi^2}\right)^n + \cdots, \qquad n \gg 1\,. \end{aligned}$$

The series $\sum_{n=1}^{\infty} a_n v^{2n}$ is convergent for $|v| < \pi\sqrt{10}/2$, the new upper bound being the one following $\pi/\sqrt{2}$ in (1.23) (for $r = 2, s = 1$). The formula (1.32) allows a much closer approach of v towards $\pm\pi/\sqrt{2}$ if need be, because it comprises the exact way in which $C(v)$ tends to infinity at $\pm\pi/\sqrt{2}$. At the same time, it enables one to attain comparable precisions as obtained by means of (1.31), but with roughly one half of the terms by virtue of the faster convergence of the a-coefficients towards zero.

Acknowledgment

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References

[1] **G.N. Watson**, *A treatise on the theory of Bessel functions*, sec. ed., Cambridge University Press, 1948.

[2] **I.S. Gradshteyn** and **I.M. Ryzhik**, *Table of Integrals, Series and Products*, Acad. Press, New York, London, 4th ed., 1965.

[3] **J. Boersma** and **P.J. de Doelder**, *SIAM Review*, Vol.35, No.3, Sept. 1993, pp.498–500.

[4] **M.L. Glasser**, *The evaluation of lattice sums.* I. *Analytic procedures*, J. Math. Phys., Vol.14, No.3, March 1973, pp.409–413.

[5] **M.L. Glasser** and **I.J. Zucker**, in *Theoretical Chemistry : Advances and Perspectives*, Vol.5, D. Henderson and H. Eyring, Acad. Press, New York, 1980.

2 Summing an alternating double series involving a Bessel function[2]

Find a closed expression representing

$$S_\nu(v) := \sum_{m=0}^{+\infty}\sum_{n=1}^{+\infty}(-1)^{m+n}\frac{J_\nu(2v\sqrt{m^2+n^2})}{(m^2+n^2)^{\nu/2}}\,, \qquad \nu \geq 0, \tag{2.33}$$

for any real $v \geq 0$.

<u>Solution</u>

Setting $\nu = 1/2$, one finds :

$$S_{1/2}(v) = \frac{1}{\sqrt{\pi v}}\sum_{m=0}^{+\infty}\sum_{n=1}^{+\infty}(-1)^{m+n}\,\frac{\sin(2v\sqrt{m^2+n^2})}{\sqrt{m^2+n^2}} = \frac{S(v)}{\sqrt{\pi v}} \tag{2.34}$$

in which $S(v)$ is given by (1.1). This relation will be useful for verification purposes. It also shows that (2.1) is a logical generalization of (1.1) which may have applications in higher dimensional extensions of Henkel and Weston's work.

Hankel inversion of form. (1) on p. 415 of [1] yields, after replacement of a by $2v$, t by n and z by m,

$$\frac{J_\nu(2v\sqrt{n^2+m^2})}{(n^2+m^2)^{\nu/2}} = \frac{1}{(2v)^\nu}\int_0^{2v} b^{\mu+1}\left\{\frac{\sqrt{4v^2-b^2}}{m}\right\}^{\nu-\mu-1}$$

$$\times J_{\nu-\mu-1}(m\sqrt{4v^2-b^2})\frac{J_\mu(nb)}{n^\mu}\,db\,, \qquad v>0\,, \tag{2.35}$$

where n and m can take on any positive integer value and $\nu > \mu > -1$. Although the integrand is in general not invariant for permutation of m and n, the integral also represents

$$\frac{J_\nu(2v\sqrt{m^2+n^2})}{(m^2+n^2)^{\nu/2}}$$

[2] a generalization of Problem 1 proposed and solved by C.C. Grosjean

for m and n independently belonging to $\{1,2,\ldots\}$. Eq.(2.3) may therefore be replaced by

$$\frac{J_\nu(2v\sqrt{m^2+n^2})}{(m^2+n^2)^{\nu/2}} = \frac{1}{(2v)^\nu n^\mu}\int_0^{2v} b^{\mu+1}\left\{\frac{\sqrt{4v^2-b^2}}{m}\right\}^{\nu-\mu-1} \times J_{\nu-\mu-1}(m\sqrt{4v^2-b^2})J_\mu(nb)\,db\,, \quad v>0\,, \quad (2.3')$$

valid for $-1<\mu<\nu$, any positive integer n, any positive integer m and even for $m=0$ as can be directly verified, on the condition, however, that for $m=0$

$$\frac{J_{\nu-\mu-1}(m\sqrt{4v^2-b^2})}{m^{\nu-\mu-1}}$$

be replaced by

$$\frac{(\sqrt{4v^2-b^2})^{\nu-\mu-1}}{2^{\nu-\mu-1}\Gamma(\nu-\mu)}\,.$$

This is justified by the fact that before z was set equal to the integer m in the Hankel inverse of form.(1) on p.415 of [1], letting z tend to zero in that inverse would have lead to the above limit. Thus, with $m=0$ in the right-hand side of (2.3'), there comes still for $v>0$:

$$\begin{aligned}
&\frac{1}{2^{2\nu-\mu-1}n^\mu v^\nu\Gamma(\nu-\mu)}\int_0^{2v} b^{\mu+1}(4v^2-b^2)^{\nu-\mu-1}J_\mu(nb)\,db\\
&= \frac{2v^{\nu-\mu}}{n^\mu\Gamma(\nu-\mu)}\int_0^1 t^{\mu+1}(1-t^2)^{\nu-\mu-1}J_\mu(2nvt)\,dt\\
&= \frac{2v^{\nu-\mu}}{n^\mu\Gamma(\nu-\mu)}\sum_{k=0}^{+\infty}(-1)^k\frac{(nv)^{\mu+2k}}{k!\,\Gamma(\mu+k+1)}\int_0^1 t^{2\mu+2k+1}(1-t^2)^{\nu-\mu-1}\,dt\\
&= \frac{v^{\nu-\mu}}{n^\mu\Gamma(\nu-\mu)}\sum_{k=0}^{+\infty}(-1)^k\frac{(nv)^{\mu+2k}}{k!\,\Gamma(\mu+k+1)}\int_0^1 u^{\mu+k}(1-u)^{\nu-\mu-1}\,du\\
&= \frac{v^{\nu-\mu}}{n^\mu\Gamma(\nu-\mu)}\sum_{k=0}^{+\infty}(-1)^k\frac{(nv)^{\mu+2k}}{k!\,\Gamma(\mu+k+1)}\frac{\Gamma(\mu+k+1)\Gamma(\nu-\mu)}{\Gamma(\nu+k+1)}\\
&= \sum_{k=0}^{+\infty}(-1)^k\frac{n^{2k}v^{\nu+2k}}{k!\,\Gamma(\nu+k+1)} = \frac{J_\nu(2vn)}{n^\nu}\,.
\end{aligned}$$

By virtue of what preceded, we may write for $v>0$:

$$\begin{aligned}
S_\nu(v) &= \sum_{m=0}^{+\infty}\sum_{n=1}^{+\infty}\frac{(-1)^{m+n}}{(2v)^\nu n^\mu}\int_0^{2v} b^{\mu+1}\left\{\frac{\sqrt{4v^2-b^2}}{m}\right\}^{\nu-\mu-1}\\
&\qquad\times J_{\nu-\mu-1}(m\sqrt{4v^2-b^2})J_\mu(nb)\,db\\
&= 2v\sum_{m=0}^{+\infty}\sum_{n=1}^{+\infty}\frac{(-1)^{m+n}}{m^{\nu-\mu-1}n^\mu}\int_0^1 t^{\mu+1}(\sqrt{1-t^2})^{\nu-\mu-1}\\
&\qquad J_{\nu-\mu-1}(2mv\sqrt{1-t^2})J_\mu(2nvt)\,dt\,.
\end{aligned}$$

Summation with respect to n in (2.1), using the integral representation (2.3), necessitates the calculation of

$$\sum_{n=1}^{+\infty}(-1)^n \frac{J_\mu(nb)}{n^\mu}\,.$$

One way to do this makes use of the integral representation

$$J_\nu(z) = \frac{2(z/2)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^1 (1-t^2)^{\nu-(1/2)}\cos zt\,dt$$

being form. (2) on p. 48 in [1], valid for $\mathrm{Re}(\nu) > -1/2$. Renaming ν as μ and setting $z = nb$, one obtains :

$$\begin{aligned}\sum_{n=1}^{+\infty}(-1)^n \frac{J_\mu(nb)}{n^\mu} &= \frac{b^\mu}{2^{\mu-1}\sqrt{\pi}\Gamma(\mu+\frac{1}{2})}\int_0^1 (1-t^2)^{\mu-(1/2)} \\ &\quad \times \left(\sum_{n=1}^{+\infty}(-1)^n \cos nbt\right) dt\,, \qquad \mu > -1/2\,, \end{aligned} \tag{2.36}$$

and by virtue of (1.6),

$$\begin{aligned}&\sum_{n=1}^{+\infty}(-1)^n \frac{J_\mu(nb)}{n^\mu} \\ &= \frac{b^\mu}{2^{\mu-1}\sqrt{\pi}\Gamma(\mu+\frac{1}{2})}\int_0^1 (1-t^2)^{\mu-(1/2)}\Big\{-\frac{1}{2}+\pi\sum_{k=-\infty}^{+\infty}\delta[bt-(2k+1)\pi]\Big\}\,dt \\ &= \begin{cases} -\dfrac{b^\mu}{2^{\mu+1}\Gamma(\mu+1)}\,, \qquad 0 \le b < \pi\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad \mu > -1/2\,, \\ -\dfrac{b^\mu}{2^{\mu+1}\Gamma(\mu+1)} + \dfrac{\sqrt{\pi}}{2^{\mu-1}\Gamma(\mu+\frac{1}{2})b^\mu} \\ \qquad \times \displaystyle\sum_{r=1}^{j}[b^2-(2r-1)^2\pi^2]^{\mu-(1/2)} \\ \qquad (2j-1)\pi \le b < (2j+1)\pi\,, \quad j = 1, 2, \ldots\,, \end{cases}\end{aligned} \tag{2.5}$$

in which the last term of the sum should be multiplied by 1/2 when $b = (2j-1)\pi, j \geq 1$, because

$$\left\{\int_0^1 \delta\left(t - \frac{2k+1}{2j-1}\right)\,dt\right\}_{k=j-1} = \frac{1}{2}\,.$$

As in the case of (1.8), for any positive b, one has to examine which ones of the infinitely many δ-functions have their singularity located in $]0,1]$. For those whereby $(2k+1)\pi/b$ belongs to $]0,1[$, the integral takes on the value

$$\frac{1}{b^{2\mu}}[b^2-(2k+1)^2\pi^2]^{\mu-(1/2)}\,.$$

When the singularity is located at the upper bound $t = 1$ of the integration interval, this expression must be multiplied by $1/2$, as was just indicated. All other δ-functions with singularity outside the integration interval do not contribute to the value of the integral.

In what precedes, the dummy integer index k was replaced by $r - 1$ to facilitate comparison with results obtained in subsection 1a. The multiplication by $1/2$ in the last term of the sum in (2.5) when $b = (2j-1)\pi, j \geq 1$, is in fact solely of importance when $\mu = 1/2$ because otherwise the last term with $b = (2j-1)\pi$ is either 0 (for $\mu > 1/2$) or infinite (for $-1/2 < \mu < 1/2$). The left-hand side of (2.3) being independent of μ (although this parameter is present in the right-hand side), one may choose μ arbitrarily as long as $-1 < \mu < \nu$ is fulfilled. Since the expression for $S_\nu(v)$ must be obtained only for $\nu > 0$, momentarily at least, the case $\mu = 1/2$ can be avoided by assuming $-1/2 < \mu < \nu$ when $0 < \nu \leq 1/2$ and $-1/2 < \mu < 1/2$ when $\nu > 1/2$ in (2.3). The right-hand side of (2.5) is the limit of summing with respect to n at first from 1 to some finite N exceeding 1 and afterwards letting N tend to $+\infty$. The condition of validity $\mu > -1/2$ which stems from the requirement of convergence of the integral representation of $J_\mu(nb)$ used in (2.4) is in agreement with the requirement of convergence of the infinite series in the left-hand side of (2.4). Indeed, in the main part of the asymptotic form of its general term, the denominator comprises $n^{\mu+(1/2)}$ and since there is piecewise alternation of sign, convergence requires $\mu + (1/2) > 0$. When summation with respect to n is carried out, one obtains :
a) for $0 < v < \pi/2$:

$$\begin{aligned}
\sum_{n=1}^{+\infty}(-1)^n \frac{J_\nu(2v\sqrt{m^2+n^2})}{(m^2+n^2)^{\nu/2}} &= -\frac{1}{2^{\mu+\nu+1}\Gamma(\mu+1)v^\nu} \\
&\quad \times \int_0^{2v} b^{2\mu+1}\left\{\frac{\sqrt{4v^2-b^2}}{m}\right\}^{\nu-\mu-1} J_{\nu-\mu-1}(m\sqrt{4v^2-b^2})\,db \\
&= -\frac{v^{\mu+1}}{\Gamma(\mu+1)m^{\nu-\mu-1}}\int_0^1 x^{2\mu+1}(1-x^2)^{(\nu-\mu-1)/2} J_{\nu-\mu-1}(2vm\sqrt{1-x^2})\,dx\,.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\int_0^1 x^{2\mu+1}(1-x^2)^{(\nu-\mu-1)/2} J_{\nu-\mu-1}(2vm\sqrt{1-x^2})\,dx \\
&= \int_0^1 (1-u^2)^\mu u^{\nu-\mu} J_{\nu-\mu-1}(2vmu)\,du \\
&= \sum_{k=0}^{+\infty}(-1)^k \frac{(vm)^{\nu-\mu+2k-1}}{k!\,\Gamma(\nu-\mu+k)}\int_0^1 u^{2\nu-2\mu+2k-1}(1-u^2)^\mu\,du \\
&= \frac{1}{2}\sum_{k=0}^{+\infty}(-1)^k \frac{(vm)^{\nu-\mu+2k-1}}{k!\,\Gamma(\nu-\mu+k)}\int_0^1 t^{\nu-\mu+k-1}(1-t)^\mu\,dt \\
&= \frac{1}{2}\sum_{k=0}^{+\infty}(-1)^k \frac{(vm)^{\nu-\mu+2k-1}\Gamma(\mu+1)}{k!\,\Gamma(\nu+k+1)} = \frac{1}{2}\frac{\Gamma(\mu+1)}{(vm)^{\mu+1}}J_\nu(2vm)
\end{aligned}$$

confirmed by item 6.683(6) on p.740 of [2]. Therefore,

$$\sum_{n=1}^{+\infty}(-1)^n \frac{J_\nu(2v\sqrt{m^2+n^2})}{(m^2+n^2)^{\nu/2}} = -\frac{J_\nu(2vm)}{2m^\nu}\,, \qquad 0 < v < \pi/2\,. \tag{2.6}$$

Note that the parameter μ, present in the right-hand side of (2.3), vanishes automatically in the course of the calculations;

b) for $(2j-1)\pi/2 \leq v < (2j+1)\pi/2\,,\ j = 1, 2, \dots$:

$$\begin{aligned}
\sum_{n=1}^{+\infty}(-1)^n \frac{J_\nu(2v\sqrt{m^2+n^2})}{(m^2+n^2)^{\nu/2}} &= -\frac{J_\nu(2vm)}{2m^\nu} + \frac{\sqrt{\pi}}{2^{\mu+\nu-1}\Gamma(\mu+\frac{1}{2})v^\nu} \\
&\quad \times \sum_{r=1}^{j}\int_{(2r-1)\pi}^{2v} b\left\{\frac{\sqrt{4v^2-b^2}}{m}\right\}^{\nu-\mu-1} J_{\nu-\mu-1}(m\sqrt{4v^2-b^2}) \\
&\quad \times [b^2-(2r-1)^2\pi^2]^{\mu-(1/2)}\,db \\
= \quad & -\frac{J_\nu(2vm)}{2m^\nu} + \frac{\sqrt{\pi}}{2^{2\mu-2}\Gamma(\mu+\frac{1}{2})v^{\mu-1}m^{\nu-\mu-1}} \\
&\quad \times \sum_{r=1}^{j}\int_{(2r-1)\pi/2v}^{1} x(1-x^2)^{(\nu-\mu-1)/2} J_{\nu-\mu-1}(2vm\sqrt{1-x^2}) \\
&\quad \times [4v^2x^2-(2r-1)^2\pi^2]^{\mu-(1/2)}\,dx\,.
\end{aligned}$$

The integral may be rewritten as

$$\int_0^\alpha u^{\nu-\mu} J_{\nu-\mu-1}(2vmu)[4v^2-(2r-1)^2\pi^2-4v^2u^2]^{\mu-(1/2)}\,du$$

with $\alpha = [4v^2-(2r-1)^2\pi^2]^{1/2}/2v$, and also as

$$\begin{aligned}
&\frac{[4v^2-(2r-1)^2\pi^2]^{(\nu+\mu)/2}}{(2v)^{\nu-\mu+1}} \int_0^1 t^{\nu-\mu}(1-t^2)^{\mu-(1/2)} \\
&\qquad \times J_{\nu-\mu-1}(mt\sqrt{4v^2-(2r-1)^2\pi^2})\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{[4v^2-(2r-1)^2\pi^2]^{(\nu+\mu)/2}}{(2v)^{\nu-\mu+1}} \sum_{k=0}^{+\infty}(-1)^k \frac{(\frac{m}{2}\sqrt{4v^2-(2r-1)^2\pi^2})^{\nu-\mu+2k-1}}{k!\,\Gamma(\nu-\mu+k)} \\
&\quad \times \int_0^1 t^{2\nu-2\mu+2k-1}(1-t^2)^{\mu-(1/2)}\,dt \\
&= \frac{1}{2}\frac{[4v^2-(2r-1)^2\pi^2]^{(\nu+\mu)/2}\Gamma(\mu+\frac{1}{2})}{(2v)^{\nu-\mu+1}} \\
&\quad \times \sum_{k=0}^{+\infty}(-1)^k \frac{(\frac{m}{2}\sqrt{4v^2-(2r-1)^2\pi^2})^{\nu-\mu+2k-1}}{k!\,\Gamma(\nu+\frac{1}{2}+k)} \\
&= \frac{1}{2}\frac{[4v^2-(2r-1)^2\pi^2]^{(\nu/2)-(1/4)}}{(2v)^{\nu-\mu+1}(m/2)^{\mu+(1/2)}} \\
&\quad \times\Gamma(\mu+\frac{1}{2})J_{\nu-(1/2)}\left\{m[4v^2-(2r-1)^2\pi^2]^{1/2}\right\}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\sum_{n=1}^{+\infty}(-1)^n\frac{J_\nu(2v\sqrt{m^2+n^2})}{(m^2+n^2)^{\nu/2}} \\
&= -\frac{J_\nu(2vm)}{2\,m^\nu} + \frac{\sqrt{\pi}}{(2m)^{\nu-(1/2)}v^\nu}\sum_{r=1}^{j}[4v^2-(2r-1)^2\pi^2]^{(\nu/2)-(1/4)} \\
&\qquad \times J_{\nu-(1/2)}\left\{m[4v^2-(2r-1)^2\pi^2]^{1/2}\right\}, \\
&\qquad (2j-1)\pi/2 \le v < (2j+1)\pi/2\,, \quad j=1,2,\ldots, \quad \nu > -1/2\,. \qquad (2.6')
\end{aligned}
$$

Since $\nu > \mu$ should hold in (2.3) and use was made of (2.5), the condition of validity is $\nu > -1/2$ which is again confirmed by the requirement of convergence of the series in the left-hand side of (2.6)–(2.6′). The main part of the asymptotic form of the general term has $(m^2+n^2)^{(\nu/2)+(1/4)}$ in its denominator and therefore, convergence is ensured for $\nu > -1/2$.

Finally, summation with respect to m gives, according to (2.1),

$$
S_\nu(v) = \sum_{n=1}^{+\infty}(-1)^n\frac{J_\nu(2vn)}{n^\nu} + \sum_{m=1}^{+\infty}(-1)^m\left(\sum_{n=1}^{+\infty}(-1)^n\frac{J_\nu(2v\sqrt{m^2+n^2})}{(m^2+n^2)^{\nu/2}}\right).
$$

For $0 < v < \pi/2$, there comes on account of (2.6) :

$$
\begin{aligned}
S_\nu(v) &= \sum_{n=1}^{+\infty}(-1)^n\frac{J_\nu(2vn)}{n^\nu} - \sum_{m=1}^{+\infty}(-1)^m\frac{J_\nu(2vm)}{2\,m^\nu} \\
&= \frac{1}{2}\sum_{n=1}^{+\infty}(-1)^n\frac{J_\nu(2vn)}{n^\nu}
\end{aligned}
$$

and by virtue of (2.5) in which b is set equal to $2v$ and μ is replaced by ν, the result is

$$S_\nu(v) = -\frac{v^\nu}{4\Gamma(\nu+1)}\,, \qquad 0 < v < \pi/2\,, \quad \nu > -1/2\,.$$

For $(2j-1)\pi/2 \le v < (2j+1)\pi/2$, $j = 1, 2, \ldots$, one finds in the same way :

$$\begin{aligned} S_\nu(v) &= \sum_{n=1}^{+\infty}(-1)^n \frac{J_\nu(2vn)}{n^\nu} - \sum_{m=1}^{+\infty}(-1)^m \frac{J_\nu(2vm)}{2\,m^\nu} \\ &\quad + \frac{\sqrt{\pi}}{2^{\nu-(1/2)}v^\nu}\sum_{r=1}^{j}[4v^2-(2r-1)^2\pi^2]^{(\nu/2)-(1/4)} \\ &\qquad \times \sum_{m=1}^{+\infty}(-1)^m \frac{J_{\nu-(1/2)}\left\{m[4v^2-(2r-1)^2\pi^2]^{1/2}\right\}}{m^{\nu-(1/2)}} \\ &= \frac{1}{2}\sum_{n=1}^{+\infty}(-1)^n\frac{J_\nu(2vn)}{n^\nu} + \frac{\sqrt{\pi}}{2^{\nu-(1/2)}v^\nu}\sum_{r=1}^{j}\ldots\sum_{m=1}^{+\infty}\ldots, \qquad \nu > -1/2\,. \end{aligned}$$

The final simplification is carried out by applying twice form. (2.5), firstly to the first series, setting b equal to $2v$ and replacing μ by ν, and secondly to the infinite series contained in the second part setting $b = [4v^2-(2r-1)^2\pi^2]^{1/2}$ and $\mu = \nu - (1/2)$. The result is :

$$\begin{aligned} S_\nu(v) &= -\frac{v^\nu}{4\Gamma(\nu+1)} + \frac{\sqrt{\pi}}{2^{2\nu}\Gamma(\nu+\frac{1}{2})v^\nu}\sum_{q=1}^{j}[4v^2-(2q-1)^2\pi^2]^{\nu-(1/2)} \\ &+ \frac{\sqrt{\pi}}{2^{\nu-(1/2)}v^\nu}\sum_{r=1}^{j}[4v^2-(2r-1)^2\pi^2]^{(\nu/2)-(1/4)} \\ &\times\left\{-\frac{[4v^2-(2r-1)^2\pi^2]^{(\nu/2)-(1/4)}}{2^{\nu+(1/2)}\Gamma(\nu+\frac{1}{2})}\right. \\ &\quad + \frac{\sqrt{\pi}}{2^{\nu-(3/2)}\Gamma(\nu)[4v^2-(2r-1)^2\pi^2]^{(\nu/2)-(1/4)}} \\ &\quad \left.\times\sum_{s=1}^{l_v(r)}[4v^2-(2r-1)^2\pi^2-(2s-1)^2\pi^2]^{\nu-1}\right\}. \end{aligned}$$

It is fairly surprising that the second part in this right-hand side which stems from the first application of (2.5), cancels exactly the first contribution to the third part which stems from the second application of (2.5). Ultimately, there comes :

$$S_\nu(v) = -\frac{v^\nu}{4\Gamma(\nu+1)} + \frac{\pi}{v^\nu\Gamma(\nu)}\sum_{r=1}^{j}\sum_{s=1}^{l_v(r)}[v^2-(r-\frac{1}{2})^2\pi^2-(s-\frac{1}{2})^2\pi^2]^{\nu-1},$$

$$\nu > 0\,, \quad (2j-1)\pi/2 \le v < (2j+1)\pi/2\,, \quad j = 1, 2, \ldots,$$

with

$$-\frac{1}{2}+\left[\frac{v^2}{\pi^2}-\left(r-\frac{1}{2}\right)^2\right]^{1/2}<l_v(r)\leq\frac{1}{2}+\left[\frac{v^2}{\pi^2}-\left(r-\frac{1}{2}\right)^2\right]^{1/2},$$

$$r=1,2,\ldots,j. \tag{2.7}$$

The reason that the validity of this result is provisionally guaranteed only for $\nu>0$ lies in the fact that (2.5) was proved only for $\mu>-1/2$ and that in its second application, μ was set equal to $\nu-(1/2)$. Combined, this yields $\nu>0$. For $\pi/2\leq v<3\pi/2$, $j=1$ and $r=1$ and the above formula becomes :

$$S_\nu(v)=-\frac{v^\nu}{4\Gamma(\nu+1)}+\frac{\pi}{v^\nu\Gamma(\nu)}\sum_{s=1}^{l_v(1)}\left[v^2-\frac{\pi^2}{4}-\left(s-\frac{1}{2}\right)^2\pi^2\right]^{\nu-1}$$

with

$$-\frac{1}{2}+\left(\frac{v^2}{\pi^2}-\frac{1}{4}\right)^{1/2}<l_v(1)\leq\frac{1}{2}+\left(\frac{v^2}{\pi^2}-\frac{1}{4}\right)^{1/2}.$$

Being an integer, $l_v(1)$ is equal to zero when

$$\frac{1}{2}+\left(\frac{v^2}{\pi^2}-\frac{1}{4}\right)^{1/2}<1$$

which entails $v<\pi/\sqrt{2}$. Therefore,

$$S_\nu(v)=-\frac{v^\nu}{4\Gamma(\nu+1)}\quad\text{for}\quad\pi/2\leq v<\pi/\sqrt{2}\,.$$

When $\pi/\sqrt{2}\leq v<3\pi/2$, it appears that $l_v(1)=1$ and so,

$$S_\nu(v)=-\frac{v^\nu}{4\Gamma(\nu+1)}+\frac{\pi}{v^\nu\Gamma(\nu)}\left(v^2-\frac{\pi^2}{2}\right)^{\nu-1}.$$

For $(2j-1)\pi/2\leq v<(2j+1)\pi/2$, $\;j=2,3,\ldots$, whereby $r=1,2,\ldots,j$, no case can be found in which the double sum in the last right-hand side of (2.7) is empty. Hence, the final result which cannot undergo any further simplification reads :

$$S_\nu(v)=\begin{cases}-\dfrac{v^\nu}{4\Gamma(\nu+1)}, & 0\leq v<\pi/\sqrt{2}\,,\\ -\dfrac{v^\nu}{4\Gamma(\nu+1)}+\dfrac{\pi}{v^\nu\Gamma(\nu)}\displaystyle\sum_{r=1}^{j}\sum_{s=1}^{l_v(r)}\Big[v^2-(r-\frac{1}{2})^2\pi^2 & \\ \qquad -(s-\frac{1}{2})^2\pi^2\Big]^{\nu-1}, & v\geq\pi/\sqrt{2}\quad\nu>0,\end{cases} \tag{2.8}$$

in which j is the largest integer smaller than or equal to

$$\frac{1}{2}+\left(\frac{v^2}{\pi^2}-\frac{1}{4}\right)^{1/2}$$

and $l_v(r)$ is the largest integer smaller than or equal to

$$\frac{1}{2}+\left[\frac{v^2}{\pi^2}-\left(r-\frac{1}{2}\right)^2\right]^{1/2},$$

being the same conditions determining j and $l_v(r)$ as in (1.10′) and (1.10″). The same kind of comments as given on (1.10)–(1.10″) also apply to (2.8). In analogy to (1.13), the double sum in (2.8) may also be written as

$$\sum_{r=1}^{+\infty}\sum_{s=1}^{+\infty}\left[v^2-\left(r-\frac{1}{2}\right)^2\pi^2-\left(s-\frac{1}{2}\right)^2\pi^2\right]^{\nu-1},$$

$$\forall(r,s): v^2-\left(r-\frac{1}{2}\right)^2\pi^2-\left(s-\frac{1}{2}\right)^2\pi^2\geq 0\,, \tag{2.9}$$

being a locally finite series. Since $J_\nu(0)=0$ when $\nu>0$, $v=0$ may be included in the first condition on v. Also in that inequality $v=\pi/\sqrt{2}$ may be included when $\nu>1$.

For $\nu=1/2$, (2.8) yields

$$S_{1/2}(v)=\begin{cases}-\dfrac{\sqrt{v}}{2\sqrt{\pi}}\,, & 0\leq v<\pi/\sqrt{2}\,,\\[2mm] -\dfrac{\sqrt{v}}{2\sqrt{\pi}}+\dfrac{\sqrt{\pi}}{\sqrt{v}}\displaystyle\sum_{r=1}^{j}\sum_{s=1}^{l_v(r)}\frac{1}{\left[v^2-(r-\frac{1}{2})^2\pi^2-(s-\frac{1}{2})^2\pi^2\right]^{1/2}}\,, & \\ \qquad v\geq\pi/\sqrt{2} & \end{cases}$$

and comparison with (1.10) when $v\geq 0$ shows that

$$S_{1/2}(v)=\frac{S(v)}{\sqrt{\pi v}}\,,$$

confirming (2.2). $S_\nu(v)$ as defined by (2.1) and given by (2.8) really is a generalization of (1.1).

The terms in the double sum of (2.8) are by far the simplest when $\nu=1$. In that special case, there comes :

$$S_1(v)=-\frac{v}{4}+\frac{\pi}{v}\sum_{r=1}^{j}l_v(r)\,,\qquad \forall v\in\mathbb{R}\,,$$

with the above definitions of j and $l_v(r)$ holding good. Hence,

$$
\begin{aligned}
S_1(v) &= -\frac{v}{4}, && |v| < \frac{\pi}{\sqrt{2}}, \\
&= -\frac{v}{4} + \frac{\pi}{v}, && \frac{\pi\sqrt{2}}{2} \le |v| < \frac{\pi\sqrt{10}}{2}, \\
&= -\frac{v}{4} + \frac{3\pi}{v}, && \frac{\pi\sqrt{10}}{2} \le |v| < \frac{\pi\sqrt{18}}{2}, \\
&= -\frac{v}{4} + \frac{4\pi}{v}, && \frac{\pi\sqrt{18}}{2} \le |v| < \frac{\pi\sqrt{26}}{2}, \\
&= -\frac{v}{4} + \frac{6\pi}{v}, && \frac{\pi\sqrt{26}}{2} \le |v| < \frac{\pi\sqrt{34}}{2},
\end{aligned}
$$

etc. Clearly, $\sum_{r=1}^{j} l_v(r)$ represents the number of terms in the double sum of (2.8) for every half-open v-interval bounded by two consecutive positive abscissae contained in (1.11). For $\nu = 1$, the considered terms are equal to unity and the double sum in (2.8) gives rise to an infinite number of finite jumps in $\mathbb{R}^+$.

The result

$$
S_\nu(v) = -\frac{v^\nu}{4\Gamma(\nu+1)},
$$
$$
0 \le v < \frac{\pi}{\sqrt{2}} \text{ for } 0 < \nu \le 1 \quad \text{and} \quad 0 \le v \le \frac{\pi}{\sqrt{2}} \text{ for } \nu > 1,
$$

generalizes the first conjecture in (1.3) of Problem 92-11*. It is also worthwhile noticing that if one agrees upon

$$
\left(\frac{J_\nu(2v\sqrt{m^2+n^2})}{(m^2+n^2)^{\nu/2}} \right)_{m=n=0} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{J_\nu(2v\sqrt{x^2+y^2})}{(x^2+y^2)^{\nu/2}} = \frac{v^\nu}{\Gamma(\nu+1)},
$$

one can write

$$
\sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} (-1)^{m+n} \frac{J_\nu(2v\sqrt{m^2+n^2})}{(m^2+n^2)^{\nu/2}} = 0,
$$
$$
0 \le v < \frac{\pi}{\sqrt{2}} \text{ for } 0 < \nu \le 1 \quad \text{and} \quad 0 \le v \le \frac{\pi}{\sqrt{2}} \text{ for } \nu > 1.
$$

* * *

It is solely because, in order to obtain (2.7), we had to apply (2.5) proven for $\mu > -1/2$ with $\mu = \nu - 1/2$, that the final result (2.8) is shown by the foregoing calculations only for $\nu > 0$ and not for $\nu = 0$. In what follows, the special case $\nu = 0$ will be treated separately.

Firstly, for the purpose of using the final result which was attained for $S(v)$ (cfr. (1.1) and (1.10)–(1.10″)), one can start with

$$\begin{aligned} J_0(x) &= \frac{2}{\pi}\int_0^1 \frac{\cos xt}{(1-t^2)^{1/2}}\,dt = \frac{2}{\pi x}\int_0^1 \frac{1}{(1-t^2)^{1/2}}\,d(\sin xt - \sin x) \\ &= \frac{2}{\pi}\frac{\sin x}{x} - \frac{2}{\pi x}\int_0^1 t\,\frac{\sin xt - \sin x}{(1-t^2)^{3/2}}\,dt\,. \end{aligned}$$

Setting $x = 2v(m^2+n^2)^{1/2}$ and summing as in (2.1), one obtains :

$$\begin{aligned} S_0(v) &:= \sum_{m=0}^{+\infty}\sum_{n=1}^{+\infty}(-1)^{m+n}J_0(2v\sqrt{m^2+n^2}) \\ &= \frac{1}{\pi v}\left\{S(v) - \int_0^1 t\,\frac{S(vt)-S(v)}{(1-t^2)^{3/2}}\,dt\right\}. \end{aligned} \tag{2.10}$$

For $v = 0$, S_0 exists only in the Cesàro (C1) sense :

$$\sum_{n=1}^{+\infty}(-1)^n J_0(0\sqrt{m^2+n^2}) = \sum_{n=1}^{+\infty}(-1)^n = -\frac{1}{2} \quad (\mathrm{C1})\,,$$

$$S_0(0) = \sum_{m=0}^{+\infty}(-1)^m\left(-\frac{1}{2}\right) = -\frac{1}{2}\cdot\frac{1}{2} = -\frac{1}{4} \quad (\mathrm{C1})\,. \tag{2.11}$$

Since $S_0(-v) = S_0(v)$, v may be restricted to positive values without loss of generality in the following calculations. For $0 < v < \pi/\sqrt{2}$, one finds :

$$\begin{aligned} S_0(v) &= \frac{1}{\pi v}\left\{-\frac{v}{2} - \int_0^1 t\left[-\frac{vt}{2}+\frac{v}{2}\right]\frac{dt}{(1-t^2)^{3/2}}\right\} \\ &= -\frac{1}{2\pi}\left[1 + \int_0^1 \frac{t(1-t)}{(1-t^2)^{3/2}}\,dt\right] \\ &= -\frac{1}{2\pi}\left[1 + \int_0^1 \frac{t}{(1+t)(1-t^2)^{1/2}}\,dt\right] \\ &= -\frac{1}{2\pi}\left[1 + \int_0^{\pi/2} \frac{\cos\phi}{1+\cos\phi}\,d\phi\right] \\ &= -\frac{1}{2\pi}\left[1 + \frac{\pi}{2} - \int_0^{\pi/2}\frac{d(\phi/2)}{\cos^2(\phi/2)}\right] \\ &= -\frac{1}{2\pi}\left[1 + \frac{\pi}{2} - \tan\frac{\pi}{4}\right] = -\frac{1}{4}\,. \end{aligned} \tag{2.12}$$

Admitting the above Cesàro-sum for $S_0(0)$ comes down to regarding $S_0(v)$ as continuous in $-\pi/\sqrt{2} < v < \pi/\sqrt{2}$. The result obtained here is in fact the extrapolation to $\nu = 0+$ in the first equality of (2.8). Secondly, one gets in a similar manner for

$0 < a < 1$:

$$
\begin{aligned}
J_0(x) &= \frac{2}{\pi}\int_0^a \frac{\cos xt}{(1-t^2)^{1/2}}\,dt + \frac{2}{\pi}\int_a^1 \frac{\cos xt}{(1-t^2)^{1/2}}\,dt \\
&= \frac{2}{\pi x}\int_0^a \frac{d\,\sin xt}{(1-t^2)^{1/2}} + \frac{2}{\pi x}\int_a^1 \frac{1}{(1-t^2)^{1/2}} d(\sin xt - \sin x) \\
&= \frac{2}{\pi x}\frac{\sin xa}{(1-a^2)^{1/2}} - \frac{2}{\pi x}\int_0^a \frac{t\sin xt}{(1-t^2)^{3/2}}\,dt \\
&\quad -\frac{2}{\pi x}\frac{\sin xa - \sin x}{(1-a^2)^{1/2}} - \frac{2}{\pi x}\int_a^1 t\frac{\sin xt - \sin x}{(1-t^2)^{3/2}}\,dt \\
&= \frac{2}{\pi x}\left\{\frac{\sin x}{(1-a^2)^{1/2}} - \int_0^a t\frac{\sin xt}{(1-t^2)^{3/2}}\,dt - \int_a^1 t\frac{\sin xt - \sin x}{(1-t^2)^{3/2}}\,dt\right\}.
\end{aligned}
$$

The analogue of (2.10) is

$$
\begin{aligned}
S_0(v) = \frac{1}{\pi v}\bigg\{&\frac{S(v)}{(1-a^2)^{1/2}} - \int_0^a t\frac{S(vt)}{(1-t^2)^{3/2}}\,dt \\
&-\int_a^1 t\frac{S(vt)-S(v)}{(1-t^2)^{3/2}}\,dt\bigg\}, \quad 0<a<1\,. \qquad (2.13)
\end{aligned}
$$

For $\pi\sqrt{2}/2 \le v < \pi\sqrt{10}/2$, (1.10) yields

$$
S(v) = -\frac{v}{2} + \frac{\pi}{[v^2-(\pi^2/2)]^{1/2}}\,.
$$

Setting $a = \pi/v\sqrt{2}$ in (2.13), with $v > \pi/\sqrt{2}$ on account of $a < 1$, one finds :

$$
\begin{aligned}
S_0(v) = \frac{1}{\pi v}\bigg\{&\frac{1}{(1-\pi^2/2v^2)^{1/2}}\left(-\frac{v}{2} + \frac{\pi}{[v^2-(\pi^2/2)]^{1/2}}\right) \\
&-\int_0^{\pi/v\sqrt{2}} t\,\frac{(-vt/2)}{(1-t^2)^{3/2}}\,dt - \int_{\pi/v\sqrt{2}}^1 t\,\Big[-\frac{vt}{2} + \frac{\pi}{[v^2t^2-(\pi^2/2)]^{1/2}} \\
&+\frac{v}{2} - \frac{\pi}{[v^2-(\pi^2/2)]^{1/2}}\Big]\frac{dt}{(1-t^2)^{3/2}}\bigg\}, \quad \pi\sqrt{2}/2 < v < \pi\sqrt{10}/2\,. \qquad (2.14)
\end{aligned}
$$

Now,

$$
\begin{aligned}
&\int_0^{\pi/v\sqrt{2}} t\frac{(-vt/2)}{(1-t^2)^{3/2}}\,dt = -\frac{v}{2}\int_0^{\pi/v\sqrt{2}} \frac{t^2}{(1-t^2)^{3/2}}\,dt \\
&= \;-\frac{v}{2}\int_0^{\pi/v\sqrt{2}} t\,d(1-t^2)^{-1/2} = -\frac{\pi}{2\sqrt{2}}\frac{1}{\left(1-\frac{\pi^2}{2v^2}\right)^{1/2}} + \frac{v}{2}\arcsin\frac{\pi}{v\sqrt{2}}\,;
\end{aligned}
$$

$$\begin{aligned}
\int_{\pi/v\sqrt{2}}^{1} t\left(-\frac{vt}{2}+\frac{v}{2}\right)\frac{dt}{(1-t^2)^{3/2}} &= \frac{v}{2}\int_{\pi/v\sqrt{2}}^{1}\frac{t}{(1+t)(1-t^2)^{1/2}}\,dt \\
&= \frac{v}{2}\int_0^{\arccos(\pi/v\sqrt{2})}\frac{\cos\phi}{1+\cos\phi}\,d\phi = \frac{v}{2}\arccos\frac{\pi}{v\sqrt{2}} - \frac{v}{2}\tan\frac{\phi}{2}\Big|_{\phi=0}^{\phi=\arccos(\pi/v\sqrt{2})} \\
&= \frac{v}{2}\arccos\frac{\pi}{v\sqrt{2}} - \frac{v}{2}\frac{1-\pi/v\sqrt{2}}{\left(1-\frac{\pi^2}{2v^2}\right)^{1/2}}\,.
\end{aligned}$$

The last two results inserted into (2.14) give the provisional formula :

$$\begin{aligned}
S_0(v) = &-\frac{1}{4}+\frac{1}{v^2-(\pi^2/2)} \\
&-\frac{1}{v}\int_{\pi/v\sqrt{2}}^{1} t\left(\frac{1}{[v^2t^2-(\pi^2/2)]^{1/2}}-\frac{1}{[v^2-(\pi^2/2)]^{1/2}}\right)\frac{dt}{(1-t^2)^{3/2}} \\
= &-\frac{1}{4}+\frac{1}{v^2-(\pi^2/2)}-\frac{v}{2[v^2-(\pi^2/2)]^{1/2}} \\
&\times\int_{\pi^2/2v^2}^{1}\frac{1}{(1-u)^{1/2}[v^2u-(\pi^2/2)]^{1/2}} \\
&\qquad\times\frac{du}{\{[v^2-(\pi^2/2)]^{1/2}+[v^2u-(\pi^2/2)]^{1/2}\}}\,. \qquad (2.15)
\end{aligned}$$

The last integral can be calculated using the substitution $1-u=s^2$:

$$\begin{aligned}
&\int_{\pi^2/2v^2}^{1}\cdots \\
&= \int_0^{[1-\pi^2/2v^2]^{1/2}}\frac{1}{s[v^2-(\pi^2/2)-v^2s^2]^{1/2}} \\
&\qquad\times\frac{2s\,ds}{\{[v^2-(\pi^2/2)]^{1/2}+[v^2-(\pi^2/2)-v^2s^2]^{1/2}\}}\,,
\end{aligned}$$

followed by the substitutions $s=[1-(\pi^2/2v^2)]^{1/2}w$ and $(1-w^2)^{1/2}=y$:

$$\begin{aligned}
\int_{\pi^2/2v^2}^{1}\cdots &= \frac{2}{v[v^2-(\pi^2/2)]^{1/2}}\int_0^1\frac{dw}{(1-w^2)^{1/2}[1+(1-w^2)^{1/2}]} \\
&= \frac{2}{v[v^2-(\pi^2/2)]^{1/2}}\int_0^1\frac{dy}{(1+y)(1-y^2)^{1/2}} \\
&= \frac{2}{v[v^2-(\pi^2/2)]^{1/2}}\int_0^{\pi/2}\frac{d\phi}{(1+\cos\phi)} = \frac{2}{v[v^2-(\pi^2/2)]^{1/2}}\,, \\
&\qquad \pi\sqrt{2}/2 < v < \pi\sqrt{10}/2\,.
\end{aligned}$$

Inserted into (2.15), this last result simplifies the right-hand side in such measure that

$$S_0(v) = -\frac{1}{4} \quad \text{for} \quad \pi\sqrt{2}/2 < v < \pi\sqrt{10}/2\,. \tag{2.16}$$

In contrast to

$$S_\nu(v) = \begin{cases} -\dfrac{v^\nu}{4\Gamma(\nu+1)}\,, & 0 \le v < \dfrac{\pi}{\sqrt{2}}\,, \\ -\dfrac{v^\nu}{4\Gamma(\nu+1)} + \dfrac{\pi}{v^\nu\Gamma(\nu)}\left(v^2 - \dfrac{\pi^2}{2}\right)^{\nu-1}, & \\ \qquad\qquad \dfrac{\pi\sqrt{2}}{2} \le v < \dfrac{\pi\sqrt{10}}{2}\,, \quad \nu > 0 & \end{cases} \tag{2.17}$$

according to (2.8), we obtained so far

$$S_0(v) = -\frac{1}{4}\,, \qquad 0 \le v < \pi\sqrt{2}/2 \text{ and } \pi\sqrt{2}/2 < v < \pi\sqrt{10}/2\,, \tag{2.18}$$

by putting (2.11), (2.12) and (2.16) together. We notice immediately that this is in agreement with the extrapolation of (2.17) to $\nu = 0+$ by virtue of $\Gamma(0+) = +\infty$ exception made for $v = \pi\sqrt{2}/2$ which is excluded in (2.18) on account of the condition under which (2.14) was derived. (If $v = \pi/\sqrt{2}$ were tolerated in (2.14), infinities would appear in the right-hand side). Since $\Gamma(\nu)$ is also comprised in a denominator of the general result (2.8), something similar may be expected to occur for $v > \pi\sqrt{10}/2$, with exceptions at the v-values where a term in the double sum is infinite, i.e. at the positive v-values in (1.11). This has to be confirmed by direct calculations, taking the entire expression of $S(v)$ in (1.10) into account.

Both $S(v)$ and its representation (1.10) are odd functions of v. We may therefore provisionally restrict v to positive values in the discussion and calculations which will follow here. Each of the terms in the summation with respect to r and s contributes to $S(v)$ from some value of v onward, e.g.,

$$\begin{aligned} &\frac{1}{\left(\dfrac{v^2}{\pi^2} - \dfrac{1}{2}\right)^{1/2}} \quad \text{for} \quad \pi\sqrt{2}/2 \le v < +\infty\,, \\ &\frac{1}{\left(\dfrac{v^2}{\pi^2} - \dfrac{5}{2}\right)^{1/2}} \quad \text{for} \quad \pi\sqrt{10}/2 \le v < +\infty\,, \\ &\frac{1}{\left(\dfrac{v^2}{\pi^2} - \dfrac{9}{2}\right)^{1/2}} \quad \text{for} \quad \pi\sqrt{18}/2 \le v < +\infty\,, \\ &\frac{1}{\left(\dfrac{v^2}{\pi^2} - \dfrac{13}{2}\right)^{1/2}} \quad \text{for} \quad \pi\sqrt{26}/2 \le v < +\infty\,, \end{aligned} \tag{2.19}$$

etc. Let us consider any such term in (1.10) with an acceptable (r,s) pair. For the sake of brevity, we introduce the short-hand notation

$$\left[\left(r-\frac{1}{2}\right)^2+\left(s-\frac{1}{2}\right)^2\right]^{1/2}\pi=\lambda \tag{2.20}$$

and examine how the general term

$$\frac{\pi}{(v^2-\lambda^2)^{1/2}} \tag{2.21}$$

contributes to $S_0(v)$ for $v\geq\pi\sqrt{10}/2$, in other words, what values get added to $-1/4$ (cfr. (2.18)) stemming from (2.21) with $\lambda=\pi\sqrt{10}/2$, $\pi\sqrt{18}/2$, $\pi\sqrt{26}/2$, ...? Making use of (2.13) with (2.21), the contribution to $S_0(v)$ is given by

$$\frac{1}{v}\left\{\frac{1}{(1-a^2)^{1/2}(v^2-\lambda^2)^{1/2}}\right.\\ \left.-\int_a^1 t\left[\frac{1}{(v^2t^2-\lambda^2)^{1/2}}-\frac{1}{(v^2-\lambda^2)^{1/2}}\right]\frac{dt}{(1-t^2)^{3/2}}\right\} \tag{2.22}$$

in which it is clear that a must be set equal to the positive lower bound of t for which $(v^2t^2-\lambda^2)^{-1/2}$ is real, hence $a=\lambda/v$. $a<1$ entails $v>\lambda$. (2.21) which only contributes to $S(v)$ for $v\geq\lambda$ when $v>0$ may be regarded as part of a discontinuous function on $\mathbb{R}^+$ which is zero in $[0,\lambda[$. This explains why there is no integral in (2.22) with bounds 0 and λ/v. (2.22) becomes

$$\frac{1}{v}\left\{\frac{v}{v^2-\lambda^2}-\int_{\lambda/v}^1 t\left[\frac{1}{(v^2t^2-\lambda^2)^{1/2}}-\frac{1}{(v^2-\lambda^2)^{1/2}}\right]\frac{dt}{(1-t^2)^{3/2}}\right\},\\ v>\lambda\,. \tag{2.23}$$

The integral can be evaluated as follows :

$$\begin{aligned}\int_{\lambda/v}^1\cdots &= \int_{\lambda/v}^1 t\frac{(v^2-\lambda^2)^{1/2}-(v^2t^2-\lambda^2)^{1/2}}{(v^2t^2-\lambda^2)^{1/2}(v^2-\lambda^2)^{1/2}}\frac{dt}{(1-t^2)^{3/2}}\\ &= \frac{1}{(v^2-\lambda^2)^{1/2}}\int_{\lambda/v}^1 t\,\frac{v^2(1-t^2)}{(v^2t^2-\lambda^2)^{1/2}[(v^2-\lambda^2)^{1/2}+(v^2t^2-\lambda^2)^{1/2}]}\\ &\qquad\times\frac{dt}{(1-t^2)^{3/2}}\\ &= \frac{v^2}{2(v^2-\lambda^2)^{1/2}}\int_{(\lambda/v)^2}^1\frac{1}{(1-u)^{1/2}(v^2u-\lambda^2)^{1/2}}\\ &\qquad\times\frac{du}{[(v^2-\lambda^2)^{1/2}+(v^2u-\lambda^2)^{1/2}]}\end{aligned}$$

$$
\begin{aligned}
&= \frac{v^2}{2(v^2-\lambda^2)^{1/2}} \int_0^{[1-(\lambda/v)^2]^{1/2}} \frac{1}{s[(v^2-\lambda^2)-v^2s^2]^{1/2}} \\
&\qquad\qquad \times \frac{2s\,ds}{\{(v^2-\lambda^2)^{1/2}+[(v^2-\lambda^2)-v^2s^2]^{1/2}\}} \\
&= \frac{v^2}{(v^2-\lambda^2)^{1/2}} \int_0^1 \frac{[1-(\lambda/v)^2]^{1/2}}{[(v^2-\lambda^2)(1-w^2)]^{1/2}} \\
&\qquad\qquad \times \frac{dw}{\{(v^2-\lambda^2)^{1/2}+(v^2-\lambda^2)^{1/2}(1-w^2)^{1/2}\}} \\
&= \frac{v}{v^2-\lambda^2} \int_0^1 \frac{dw}{(1-w^2)^{1/2}[1+(1-w^2)]^{1/2}} \\
&= \frac{v}{v^2-\lambda^2} \int_0^1 \frac{dz}{(1+z)(1-z^2)^{1/2}} \\
&= \frac{v}{v^2-\lambda^2} \int_0^{\pi/2} \frac{d\phi}{(1+\cos\phi)} = \frac{v}{v^2-\lambda^2} \tan\frac{\phi}{2}\bigg|_0^{\pi/2} = \frac{v}{v^2-\lambda^2}\,.
\end{aligned}
$$

Inserted into (2.23), this result yields zero for $v > \lambda$. Applied with $\lambda = \pi\sqrt{2}/2$ in (2.23), it shows that $S_0(v) = -1/4$ cannot be modified by the second term in (1.10), i.e., the first expression listed in (2.19), from $v > \pi\sqrt{2}/2$ onward. Similarly, with $\lambda = \pi\sqrt{10}/2$ in (2.23), it shows that again $S_0(v) = -1/4$ cannot be modified by the third term in (1.10), i.e., the second expression listed in (2.19), from $v > \pi\sqrt{10}/2$ onward. Clearly, the same holds for all the terms which can appear in (1.10). It is quite important to note that the real v-abscissae in which $S_0(v)$ remains unmodified do not include the points (2.20) because, as stated already before, $a < 1$ in (2.13) entails $v > \lambda$. Hence, the previous calculations prove that the even function

$$
\begin{aligned}
&S_0(v) = \sum_{m=0}^{+\infty}\sum_{n=1}^{+\infty}(-1)^{m+n}J_0(2v\sqrt{m^2+n^2}) = -\frac{1}{4}\,, \\
&\forall v \in \mathbb{R} \setminus \left\{ \pm\pi\sqrt{(r-\frac{1}{2})^2+(s-\frac{1}{2})^2}\,|\, r\in\mathbb{N}_0,\ s\in\{1,2,\ldots,r\} \right\}. \qquad (2.24)
\end{aligned}
$$

I readily admit that my method to study $S_0(v)$ exhibits the weakness that it does not provide information on $S_0(v)$ at the abscissae just excluded. Such information is provided, however, by the more powerful method expounded by N. Ortner and P. Wagner (University of Innsbruck, Austria) in their forthcoming paper [3]. In every abscissa excluded in (2.24), $S_0(v)$ is infinite as it is described by a Dirac δ-function with a positive coefficient. For $0 < \nu < 1$ in (2.8), each term in the double sum on the right is infinite at one of the abscissae comprised in (2.20) and finite for larger v. When ν tends to 0 in a continuous manner, $1/\Gamma(\nu)$ tends to zero. Thus, in every abscissa where the terms in the double sum are finite, the limit value of that sum is zero and $S_0(v) = -1/4$. But, where infinity is present, an indeterminacy of the type $0 \times (+\infty)$ appears when $\nu = 0+$. According to Ortner and Wagner's paper, in the limiting process the infinity stemming from every term

$$\left[v^2-\left(r-\frac{1}{2}\right)^2\pi^2-\left(s-\frac{1}{2}\right)^2\pi^2\right]^{\nu-1}$$

gets the upper hand and since this occurs in a point, it is acceptable that a Dirac δ-function is involved.

A way to verify this consists in proceeding *formally* as follows. For $\nu = 1$, (2.8) yields :

$$S_1(v) = \sum_{m=0}^{+\infty}\sum_{n=1}^{+\infty}(-1)^{m+n}\frac{J_1(2v\sqrt{m^2+n^2})}{\sqrt{m^2+n^2}} = -\frac{v}{4}+\frac{\pi}{v}\sum_{r=1}^{j} l_v(r)\,, \qquad \forall v \in \mathbb{R}\,, \tag{2.25}$$

in which j is the largest integer smaller than or equal to

$$\frac{1}{2}+\left(\frac{v^2}{\pi^2}-\frac{1}{4}\right)^{1/2} \qquad \text{from } v=\pi\sqrt{2}/2 \text{ onward}\,,$$

and $l_v(r)$ is the largest integer smaller than or equal to

$$\frac{1}{2}+\left[\frac{v^2}{\pi^2}-\left(r-\frac{1}{2}\right)^2\right]^{1/2}.$$

$S_1(v)$ is an odd function of v on $\mathbb{R}$. For $v > 0$, it exhibits an infinite number of finite jumps at the respective abscissae

$$\left[\left(r-\frac{1}{2}\right)^2+\left(s-\frac{1}{2}\right)^2\right]^{1/2}\pi\,, \quad \forall r\in\mathbb{N}_0\,, \quad \forall s\in\{1,2,\ldots,r\}\,.$$

Making use of

$$\frac{d}{dx}(xJ_1(x)) = xJ_0(x)$$

and assuming that in

$$\frac{d}{dv}v\sum_{m=0}^{+\infty}\sum_{n=1}^{+\infty}(-1)^{m+n}\frac{J_1(2v\sqrt{m^2+n^2})}{\sqrt{m^2+n^2}}$$

the differentiation may be carried out behind the double sum sign (which makes the calculations formal, thus not providing a strict proof), we find

$$\begin{aligned}\frac{d}{dv}vS_1(v) &= \sum_{m=0}^{+\infty}\sum_{n=1}^{+\infty}(-1)^{m+n}\frac{d}{dv}\left(v\frac{J_1(2v\sqrt{m^2+n^2})}{\sqrt{m^2+n^2}}\right)\\ &= 2v\sum_{m=0}^{+\infty}\sum_{n=1}^{+\infty}(-1)^{m+n}J_0(2v\sqrt{m^2+n^2})\\ &= 2vS_0(v)\,.\end{aligned}$$

Consequently,

$$
\begin{aligned}
S_0(v) &= \frac{1}{2v}\frac{d}{dv}\left(-\frac{v^2}{4}\right) = -\frac{1}{4}, \qquad 0 \le v < \pi\sqrt{2}/2\,, \\
S_0(v) &= \frac{1}{2v}\frac{d}{dv}\left[-\frac{v^2}{4} + \pi H(v-(\pi\sqrt{2}/2))\right] \\
&= -\frac{1}{4} + \frac{1}{\sqrt{2}}\delta(v-(\pi\sqrt{2}/2))\,, \qquad 0 \le v < \pi\sqrt{10}/2\,,
\end{aligned}
$$

where H denotes the Heaviside-function

$$
H(x) = \begin{cases} 0\,, & x < 0\,, \\ 1\,, & x \ge 0\,, \end{cases}
$$

$$
\begin{aligned}
S_0(v) &= \frac{1}{2v}\frac{d}{dv}\left[-\frac{v^2}{4} + \pi H(v-(\pi\sqrt{2}/2)) + 2\pi H(v-(\pi\sqrt{10}/2))\right] \\
&= -\frac{1}{4} + \frac{1}{\sqrt{2}}\delta(v-(\pi\sqrt{2}/2)) + \frac{2}{\sqrt{10}}\delta(v-(\pi\sqrt{10}/2))\,, \\
&\qquad 0 \le v < \pi\sqrt{18}/2\,, \qquad (2.26)
\end{aligned}
$$

etc. Since there is no simple general formula for the integer numerator of the coefficient of each Dirac δ-function, the easiest way to represent $S_0(v)$ is by means of a locally finite series as in (1.13) and (2.9). There comes :

$$
S_0(v) = -\frac{1}{4} + \sum_{r=1}^{+\infty}\sum_{s=1}^{+\infty} \frac{\delta\left[v - \frac{\pi}{2}\sqrt{(2r-1)^2+(2s-1)^2}\right]}{\sqrt{(2r-1)^2+(2s-1)^2}}\,,
$$

$$
\forall (r,s) : v^2 - \left(r-\frac{1}{2}\right)^2\pi^2 - \left(s-\frac{1}{2}\right)^2\pi^2 \ge 0\,, \qquad (2.27)
$$

complementing (2.24). For instance, when $0 \le v < \pi\sqrt{18}/2$, the (r,s)-couples in the two-dimensional grid which contribute to the double sum are (1,1), (1,2) and (2,1), and (2.26) results.

Extending the summations in (2.1) to the entire (m,n)-plane in the case of $\nu = 0$, (2.27) leads to

$$
\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty}(-1)^{m+n}J_0(2v\sqrt{m^2+n^2})
$$

$$
= 4\sum_{r=1}^{+\infty}\sum_{s=1}^{+\infty} \frac{\delta\left[v - \frac{\pi}{2}\sqrt{(2r-1)^2+(2s-1)^2}\right]}{\sqrt{(2r-1)^2+(2s-1)^2}}\,,
$$

$$
\forall (r,s) : v^2 - \left(r-\frac{1}{2}\right)^2\pi^2 - \left(s-\frac{1}{2}\right)^2\pi^2 \ge 0\,, \qquad (2.28)
$$

being a locally finite series of pure Dirac δ-functions (on the ground value zero).

Note

Other authors have considered Problems 1 and 2, and also related problems. Their names are cited in an editorial note on p.500 in Vol.35, No.3 of the SIAM Review. Besides this, also worth mentioning is a paper on m-dimensional Schlömilch series by A.R. Miller submitted to the Canadian Mathematics Bulletin.

References

[1] **G.N. Watson**, *A treatise on the theory of Bessel functions*, sec. ed., Cambridge University Press, 1948.

[2] **I.S. Gradshteyn** and **I.M. Ryzhik**, *Table of Integrals, Series and Products*, Acad. Press, New York, London, 4th ed., 1965.

[3] **N. Ortner** and **P. Wagner**, private communication.

3 A Finite Sum of Products of Binomial Coefficients

In Vol. 34, No. 4 (1992) of the SIAM Review, the following problem was posed :
– determine the sum

$$\sum_{m=0}^{n} \binom{-\frac{1}{4}}{m}^2 \binom{-\frac{1}{4}}{n-m}^2 .$$

This sum arises from the calculation of the shift of the frequency of an electromagnetic TM wave-mode caused by a small metallic cylinder in a resonant cavity.

The final result is

$$\sum_{m=0}^{n} \binom{-\frac{1}{4}}{m}^2 \binom{-\frac{1}{4}}{n-m}^2 = (-1)^n \binom{-\frac{1}{2}}{n}^3 . \tag{3.1}$$

I submitted it more for aesthetic reasons than for its degree of difficulty, but I never realized that the problem was that simple until Dr. Volker Strehl (Friedrich-Alexander-Universität Erlangen-Nürnberg) pointed out to me that it is a special case of Clausen's product identities. If the given sum is called S, one can write :

$$S = \sum_{m=0}^{n} \frac{\{(1/4)_m\}^2 \{(1/4)_{n-m}\}^2}{(m!)^2 (n-m)!^2}$$

and it is immediately clear that it is the coefficient of x^n in the series obtained by Cauchy multiplication of the power series expansion of the Gaussian hypergeometric function ${}_2F_1(1/4, 1/4; 1; x)$ by itself. Now, it is sufficient to make use of Clausen's identity

$$[{}_2F_1(a, b; a+b+(1/2); x)]^2 = {}_3F_2(2a, a+b, 2b; a+b+(1/2), 2(a+b); x)$$

with $a = b = 1/4$ to find out that S is also the coefficient of x^n in the expansion of ${}_3F_2(1/2, 1/2, 1/2; 1, 1; x)$, hence

$$S = \frac{\{(1/2)_n\}^3}{(n!)^3} = (-1)^n \binom{-\frac{1}{2}}{n}^3 .$$

Nevertheless, my little problem has drawn a lot of attention as appears from the discussion on pp. 645–646 in Vol. 35, No. 4 of the SIAM Review. Some authors even submitted remarkable identities generalizing (3.1) and being considerably more complicated to formulate as well as to prove.

Eq.(3.1) is of the form

$$\sum_{m=0}^{n} \binom{-a}{m}^2 \binom{-a}{n-m}^2 = (-1)^n \binom{-2a}{n}^3 \tag{3.2}$$

and I have been asking myself whether there exist other non-zero complex values of a than $1/4$ for which (3.2) is also satisfied. Here follows one way to treat this new problem.

Consider

$$\sum_{m=0}^{+\infty} \binom{-a}{m}^2 x^m , \qquad -1 < x < 1 .$$

By virtue of d'Alembert's ratio test, this series is absolutely convergent for any $x \in]-1, 1[$. Its sum is the Gaussian hypergeometric function ${}_2F_1(a, a; 1; x)$ which is a regular solution of the differential equation

$$x(1-x)y'' + [1 - (2a+1)x]y' - a^2 y = 0 .$$

By Cauchy multiplication, one obtains :

$${}_2F_1^2(a, a; 1; x) = \sum_{n=0}^{+\infty} \left[\sum_{m=0}^{n} \binom{-a}{m}^2 \binom{-a}{n-m}^2 \right] x^n. \tag{3.3}$$

If $y_1(x)$ and $y_2(x)$ are linearly independent solutions of

$$y'' + P(x)y' + Q(x)y = 0 ,$$

then $y_1^2(x), y_1(x)y_2(x)$ and $y_2^2(x)$ are linearly independent solutions of

$$\begin{aligned} z''' + 3P(x)z'' + [2P^2(x) + P'(x) + 4Q(x)]z' \\ +[4P(x)Q(x) + 2Q'(x)]z = 0 \end{aligned}$$

(see, for instance, [1], pp. 382–383). Applying this theorem with

$$P(x) = \frac{1 - (2a+1)x}{x(1-x)} , \qquad Q(x) = -\frac{a^2}{x(1-x)} ,$$

one finds

$$\begin{aligned} x^2(1-x)^2 z''' + 3x(1-x)[1 - (2a+1)x]z'' + [1 - 2(2a^2 + 4a + 1)x \\ +(12a^2 + 6a + 1)x^2]z' - 2a^2(1 - 4ax)z = 0 , \end{aligned} \tag{3.4}$$

having (3.3) as a regular solution. Solving eq. (3.4) by means of Frobenius's method in terms of a power series in x, one gets :

$$z=\sum_{n=0}^{+\infty} c_n x^n$$

with

$$\begin{cases} c_1-2a^2c_0=0\,, \qquad c_0\neq 0\,, \\ (n+1)^3c_{n+1}-[2n^3+6an^2+2a(2a+1)n+2a^2]c_n \\ \qquad\qquad +(n+2a-1)^3c_{n-1}=0\,, \qquad n=1,2,\ldots\,.. \end{cases} \tag{3.5}$$

For (3.2) to hold for any $n\in\mathbb{N}_0$, one should have that

$$\left(\frac{2a(2a+1)\ldots(2a+n-1)}{n!}\right)^3, \qquad n\in\mathbb{N}_0\,,$$

regarded as c_n, satisfies (3.5) with $c_0=1$. This leads to

$$8a^3-2a^2=0 \tag{3.6}$$

and

$$(n+1)^3\frac{(2a+n)^3}{(n+1)^3}-[2n^3+6an^2+2a(2a+1)n+2a^2]$$
$$+(n+2a-1)^3\frac{n^3}{(2a+n-1)^3}=0\,,$$

or equivalently,

$$(2n^3+6an^2+12a^2n+8a^3)-[2n^3+6an^2+2a(2a+1)n+2a^2]=0$$

or

$$(8a^2-2a)n+(8a^3-2a^2)=0\,, \qquad \forall n\in\mathbb{N}\,. \tag{3.7}$$

It is clear that (3.6) and (3.7) are satisfied by only one non-zero value of a, namely, $a=1/4$. (3.1) really is the sole way in $\mathbb{C}$ to satisfy (3.2) non-trivially.

References

[1] **C.C. Grosjean**, *On the analytical solution of certain linear differential equations of third and higher order*, Bull. Soc. Math. Belgique, T. XX, fasc. 4 (1968).

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