Outer Automorphism Groups of Bieberbach Groups

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Abstract

Let Γ be a Bieberbach group it is a fundamental group of a flat manifold. In this paper we give necessary and sufficient conditions on Out(Γ) to be infinite. We also compare our result with the previous one ([8], Theorem 7.1) about Anosov automorphisms of flat manifolds. We give several examples of Bieberbach groups.

1 Introduction

Let Γ be a Bieberbach group. By definition it is the torsion - free group given by a short exact sequence

$$(*)\ 1 \to Z^n \to \Gamma \to G \to 1,$$

where G is a finite group (called the holonomy group of Γ) and Z^n is a free abelian group of rank n which is a maximal abelian subgroup of Γ. Let $T : G \to GL(n, Z)$ be the representation which the group Γ defines on Z^n by conjugations. The rank of Γ is n. It is well known that Γ is isomorphic to a uniform discrete subgroup of the group $E(n)$ of the rigid motions of the n - dimensional Euclidean space R^n. The orbit space $R^n/\Gamma = M$ is a closed flat manifold, for short a flat manifold, with fundamental group Γ.

In this note we are going to give necessary and sufficient conditions for Out(Γ) = Aut(Γ) / Inn(Γ) to be infinite.

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The group Out(Γ) is isomorphic to $Aff(M)/Aff_0(M)$, where $Aff(M)$ denotes the group of affine automorphisms of M and $Aff_0(M)$ denotes the identity component of $Aff(M)$ which is isomorphic to $(S^1)^{\beta_1}$, where β_1 is the first Betti number of M (see [2] p. 214).

If Γ has trivial centre, then Out(Γ) gives us complete information about extensions of Γ by integers. Using this information and the Calabi reduction theorem one can study the problem of the classification of flat manifolds. (See [10], [11].)

In this paper we shall prove:

Theorem A. *The following conditions are equivalent:*

(i) Out(Γ) is infinite.

(ii) The **Q***-decomposition of T has at least two components of the same isomorphism type or there is a* **Q***-irreducible component which is reducible over* **R**.

The main inspiration for us is [8], where flat manifolds admitting Anosov diffeomorphisms are characterized; the main result of that paper can be stated as:

Theorem B. ([8] Theorem 6.1) *The following conditions for a flat manifold M are equivalent:*

(i) M supports an Anosov diffeomorphism.

(ii) Each **Q***-irreducible component of T which occurs with multiplicity one is reducible over* **R**.

We illustrate the links between these results by a series of examples of Bieberbach groups. (See Section 3.) We also give an estimate for the magnitude of Out(Γ). In particular we shall prove:

Proposition. *For every $n \in N$ and every l = 1,2,...,n-1 there is a Bieberbach group Γ_l of rank n such that Out(Γ_l) is quasi - isometric to GL(l,Z).*

We recall the definition of quasi - isometry in Section 3.

The properties of the group Out(Γ) were studied in [3] and in Chapter 5 of [2]. We assume familiarity with those papers and freely use the methods described there.

It was pointed out to us by Wilhelm Plesken that Theorem **A** can be also proved by a different method (see Theorem 3.41 of [1]).

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2 Proof of Theorem A

We start with the observation that the group Out(Γ) is infinite if and only if the centralizer $C_{GL(n,Z)}(T(G))$ of the subgroup T(G) in GL(n,Z) is infinite. That is an easy exercise from Chapter 5 of [2].

We need the lemma:

Lemma 1. *Let S be a rational representation of a finite group of dimension m. Then S commutes with an element $K \in GL(m,Z)$ of infinite order if and only if S commutes with an element $K_1 \in GL(m,Q)$ of infinite order whose characteristic polynomial has integer coefficients with a unit constant term.*

Proof of Lemma 1. If K exists, it will do for K_1. If K_1 exists, its rational canonical form $R \in GL(m,Z)$. $R = P_1^{-1}K_1P_1$ for some $P_1 \in GL(m,Q)$. Let h be the product of the denominators of the elements in P_1 and P_1^{-1}.

Then there is an integer $l \geq 1$ such that h divides all the entries of $R^l - I$ - it is enough to consider the map $q : GL(m,Z) \to GL(m,Z_h)$. Then $K_1^l \in GL(m,Z)$ will do for K.

Proof of Theorem A. (i) $\Rightarrow$ (ii). If Out(Γ) is infinite then $C_{GL(n,Z)}(T(G))$ is infinite. We now change the basis of $\mathbf{Q}^n$ so that T splits up as $T_1 \oplus T_2 \oplus ... \oplus T_k$ with T_i $\mathbf{Q}$-irreducible for each i = 1,2,...,k.

We assume that $T_i \ncong T_j$ for $i \neq j$ and all the T_i are $\mathbf{R}$-irreducible. We shall prove that in this case the group $C_{GL(n,Z)}(T(G))$ is finite. Suppose $H \in C_{GL(n,Z)}(T(G)$. The new matrix $H' = P^{-1}HP$, where P is the matrix of the new basis, will have a single block corresponding to each T_i and commuting with it. The characteristic polynomial of this block will divide the characteristic polynomial of H, so this block will do for K_1 in Lemma 1. Hence, it is enough to prove our statement for T $\mathbf{Q}$-irreducible and $\mathbf{R}$-irreducible. We distinguish 3 cases.

1) T is absolutely irreducible. Then, by Schur's lemma, the only matrices commuting with T are scalar matrices. But the determinant of the matrix is +1 or -1 and a matrix comes from GL(n,Z). Hence scalars are roots of the unity and the matrix has finite order.

2) T decomposes over $\mathbf{C}$ with Schur index one. Choosing a suitable basis for K^n, where K is a minimal splitting field for T, the matrix of T will be

$$\begin{pmatrix} T_1 & 0 \\ 0 & \bar{T}_1 \end{pmatrix},$$

with T_1 absolutely irreducible and as Schur index is one, T_1 is not equivalent to $\bar{T}_1$. Again by Schur's lemma the only commuting matrices which come from GL(n,$\mathbf{Q}$) are of the form

$$\begin{pmatrix} \lambda I & 0 \\ 0 & \bar{\lambda} I \end{pmatrix},$$

with $\lambda \in K$. If the characteristic polynomial is in Z[x] with unit constant term then $\mid \lambda\bar{\lambda} \mid = 1$. Hence λ is on the unit circle. But $\lambda \in K$, a complex quadratic extension of $\mathbf{Q}$, and so, by Dirichlet's unit theorem, λ is a root of unity. Hence our matrix has finite order.

3) T decomposes over $\mathbf{C}$ with Schur index two. Let K be as in the case 2. Choosing a suitable basis for K^n, the matrix of T will be

$$\begin{pmatrix} T_1 & 0 \\ 0 & \bar{T}_1 \end{pmatrix},$$

with T_1 absolutely irreducible and $T_1 \sim \bar{T}_1$. Let $J \in GL(n/2, K)$ be such that $JT_1 = \bar{T}_1 J$. Then since $\bar{J}J$ commutes with T_1, $\bar{J}J = \kappa I$ with $\kappa \in \mathbf{Q} = K \cap \mathbf{R}$. If $\kappa > 0$ then

$$\begin{pmatrix} 0 & \bar{J} \\ J & 0 \end{pmatrix},$$

whose square is κI has real eigenvalues and commutes with T, which is irreducible over $\mathbf{R}$. As it is not a multiple of the identity, this contradicts Schur's lemma, and so $\kappa < 0$. Now, by Schur's lemma again, every matrix which commuts with T and comes from GL(n,$\mathbf{Q}$) must be of the form

$$\begin{pmatrix} \lambda I & \nu \bar{J} \\ \bar{\nu} J & \bar{\lambda} I \end{pmatrix}.$$

Note that the determinant of this matrix is $(\lambda\bar{\lambda} - \kappa\nu\bar{\nu})^{n/2}$. If its characteristic polynomial is in Z[x] with unit constant term then $\lambda\bar{\lambda} - \kappa\nu\bar{\nu} = 1$. It cannot be -1 as it is positive. Then $\mid \lambda \mid \leq 1$, so $\mid \lambda + \bar{\lambda} \mid \leq 2$. The characteristic polynomial of the above matrix is $(x^2 - (\lambda + \bar{\lambda})x + 1)^{n/2}$. Since the zeros of $x^2 + ax + 1$ are all roots of unity for $a = 0, -1, 1, -2, 2$, we can assume that $(\lambda + \bar{\lambda}) = 2$.

Hence $\lambda = 1$. Because any power of the above matrix has the same properties of the characteristic polynomial and eigenvalues, the only possibility is $\nu = 0$. Hence, again our matrix has finite order.

Now we can start the proof of the second part of the Theorem (ii)$\Rightarrow$ (i). If T has at least two components of the same isomorphism type it is easy to construct an integral matrix H_1 of infinite order which commutes with T (see [8] p.311) as $\mathbf{Q}$-representation. Then if we write $H_1^{'} = PH_1P^{-1}$ then $H_1^{'}$ satisfies the conditions of Lemma 1.

So, assume that T has one component which is $\mathbf{R}$-reducible. Let K be a minimal splitting field whose intersection with $\mathbf{R}$ is non-trivial. We may assume that K is a subfield of $\mathbf{Q}(\zeta)$, where ζ is a primitive $\mid G \mid$th root of unity, and so $\Gamma(K/\mathbf{Q})$ is abelian. K is not a complex quadratic extension of $\mathbf{Q}$, and so, by Dirichlet's unit theorem, there are in K algebraic units none of whose conjugates (in the Galois sense) are on the unit circle.

Let λ be such a unit and let $\{\sigma_i\}_i$ be the Galois group $\Gamma(K/\mathbf{Q})$. Then $\{\lambda^{\sigma_i}\}_i$ are the conjugates of λ. Write λ_i for λ^{σ_i}. If now we choose a basis for K^n so that T decomposes as the direct sum of T_i, where $T_i = T_1^{\sigma_i}$ then the block matrix $\lambda_i I$ in the diagonal blocks and zero elsewhere will commute with the image of T, it will come from GL(n,$\mathbf{Q}$), its characteristic polynomial will be in Z[x] and will have a unit for its constant term, and none of its eigenvalues will be on the unit circle (what is equivalent to infinite order). Next we apply Lemma 1. This finishes the proof of the theorem.

We have some immediately consequence.

Corollary 1. *If M supports an Anosov diffeomorphism then $Out(\pi_1(M))$ is infinite.*

Proof. The above corollary follows from Theorem **A** and Theorem **B**.

Corollary 2. *If $Out(\Gamma)$ is finite then $dim_{\mathbf{Q}}(\Gamma/[\Gamma,\Gamma]\otimes\mathbf{Q})$ = 0 or 1.*

Proof. Assume that $\mathrm{Out}(\Gamma)$ is finite and $dim_{\mathbf{Q}}(\Gamma/[\Gamma,\Gamma]\otimes\mathbf{Q})\geq 2$. The last means that the representation T has at least two trivial components (cf. [6]). Hence, by Theorem **A**, we obtain contradiction.

Remark. 1. We put as an exercise to the reader the observation that Theorem **A** stay valid for any crystallographic group (not only torsion free).

2. Theorem **A** clarifies some part of section 6 E_1 of the M. Gromov article [5]. (See [5] page 109.)

3 Examples and Applications

We keep the notation of the previous sections. We start with an example of an odd dimensional Bieberbach group whose outer automorphism group is finite.

Example 1. Let Γ be an n-dimensional generalised **Hantsche - Wendt** Bieberbach group defined by the short exact sequence

$$1\to Z^n\to\Gamma\to(Z_2)^{n-1}\to 1,$$

where $n\geq 3$ is an odd number and the action of $(Z_2)^{n-1}$ on Z^n is defined by the $(n\times n)$ matrices:

$$A_i=\begin{bmatrix}
-1 & 0 & ... & 0 & 0 & 0 & ... & 0 & 0\\
0 & -1 & ... & 0 & 0 & 0 & ... & 0 & 0\\
... & & & & & & & & \\
0 & 0 & ... & -1 & 0 & 0 & ... & 0 & 0\\
0 & 0 & ... & 0 & 1 & 0 & ... & 0 & 0\\
0 & 0 & ... & 0 & 0 & -1 & ... & 0 & 0\\
... & & & & & & & & \\
0 & 0 & ... & 0 & 0 & 0 & ... & -1 & 0\\
0 & 0 & ... & 0 & 0 & 0 & ... & 0 & -1
\end{bmatrix}, 1\leq i\leq n-1,$$

the "1" being in the i-th row and i-th column.
Moreover the cocycle $s^\star\in H^2((Z_2)^{n-1},Z^n)=H^1((Z_2)^{n-1},R^n/Z^n)$ corresponding to the above exact sequence is given by function $s:(Z_2)^{n-1}\to R^n/Z^n$; and

$$s(A_i) = u_i/2 + u_{i+1}/2 \text{ for } 1 \geq i \geq n-1$$

where $u_1, u_2, ..., u_n$ denotes a canonical base of Z^n. By [9] Γ is well defined and the first Betti number $\beta_1(R^n/\Gamma) = 0$. By Theorem **A** Out(Γ) is a finite group.

In the next example we shall see that for each positive integer $n \geq 3$ there is an n - dimensional Bieberbach group with nontrivial holonomy group and with infinite outer automorphism group. Moreover this outer automorphism group is quasi - isometric to GL(n-1,Z).

Definition 1. Let (X, d) and (X', d') be two metric spaces. A map $f : X \to X'$ is a *quasi - isometry* if there exist constants $\lambda > 0, C \neq 0$ such that

$$\tfrac{1}{\lambda}d(x,y) - C \leq d'(f(x), f(y)) \leq \lambda d(x,y) + C$$

for all $x, y \in X$. The spaces (X, d) and (X', d') are quasi - isometric if there exists a quasi - isometry $f : X \to X'$ and a constant $D \geq 0$ such that $d'(f(X), x') \leq D$ for all $x' \in X'$.

Let L be a finitely generated group. Choose a finite set S of generators for L, that does not contain the identity and such that $s^{-1} \in S$ the lenght $l_S(l)$ of any $l \in L$ to be the smallest positive integer n such that there exists a sequence $(s_1, s_2, ..., s_n)$ of generators in S for which $l = s_1 s_2 ... s_n$ and define the distance $d_S : L \times L \to R_+$ by $d_S(l_1, l_2) = l_S(l_1^{-1} l_2)$.

It is easy to check that d_S makes L a metric space. Two finitely generated groups are quasi - isometric if they are quasi - isometric as metric spaces. It is not difficult to see that quasi - iso relation and does not depend on the choice of the generating sets (see [4]).

Remark. Any finite extension of a finitely generated group L and any subgroup of a finite index in L is quasi - isometric to L.

Example 2. Let $n \geq 2$ and let Γ be a Bieberbach group defined by the short exact sequance:

$$1 \to Z^n \to \Gamma \to Z_2 \to 1,$$

where the action of Z_2 on Z^n is defined by the $(n \times n)$ matrix:

$$\begin{bmatrix} -1 & 0 & 0 & ... & 0 \\ 0 & -1 & 0 & ... & 0 \\ . & . & . & ... & \\ 0 & 0 & ... & -1 & 0 \\ 0 & 0 & ... & 0 & 1 \end{bmatrix}$$

We claim that Out(Γ) is quasi - isometric to GL(n-1,Z).

From definition, Out(Γ) is quasi - isometric to the centralizer C of the above matrix in GL(n,Z); (cf. chapter 5 of [2]). But C is obviously quasi - isometric to GL(n-1,Z).

By the above example and by [10] it is not difficult to prove:

Proposition 1. *For given positive integer n≥ 2 there exists a Bieberbach group Γ_l of rank n such that the group Out(Γ_l) is quasi - isometric to the general linear group over integers GL(l,Z) for l = 1,2,...,n-1.*

Proof. We use induction on n. If n = 2 the result follows by Example 2. Assume that there are Bieberbach groups Γ_l of dimension n with holonomy group G_l, such that Out(Γ_l) is quasi - isometric to $GL(l,Z)$ for l = 1,2,...,n-1. We shall define the groups $\Gamma_l^{'}$ of dimension n+1 with Out(Γ_l) = Out($\Gamma_l^{'}$) up to quasi - isometry for l = 1,2,...,n-1.

Let $T_l : G_l \rightarrow GL(n,Z)$ be the representation defined by Γ_l. By [10] we can define a new Bieberbach group $\Gamma_l^{'}$ of dimension n + 1 with holonomy group $G_l \oplus Z_2$, where the representation $T_l^{'} : G_l \oplus Z_2 \rightarrow GL(n+1,Z)$ defined by conjugation in $\Gamma_l^{'}$ can be written as

$$T_l^{'}(g,1) = \begin{bmatrix} T_l(g) & 0 \\ 0 & +1 \end{bmatrix}$$

and

$$T_l^{'}(g,-1) = \begin{bmatrix} T_l(g) & 0 \\ 0 & -1 \end{bmatrix}.$$

for $g \in G$ and $Z_2 = \{+1,-1\}$.

By arguments as in Example 2, Out(Γ_l) = Out($\Gamma_l^{'}$) up to quasi - isometry for l = 1,2,...,n-1. Hence we must only define the Bieberbach group $\Gamma_n^{'}$ of dimension n+1 with Out($\Gamma_n^{'}$) = GL(n,Z) up to quasi - isometry. But this is done in Example 2 and the proposition is proved.

Corollary 3. *For a given positive integer $n \geq 3$ there is a Bieberbach group Γ of rank n such that the group Out(Γ) is finite and Γ has finite abelianization.*

Proof. This follows from the proof of Proposition 1 and Example 1.

Our next examples show that there is a flat manifold $R^n/\Gamma = M$ with infinite Out(Γ) that does not support an Anosov diffeomorphism.

Example 3. Let M be a flat manifold of dimension $n \geq 3$, with $\pi_1(M)$ defined in Example 2. It is obvious that $\pi_1(M)$ has an infinite outer automorphism group and M has not Anosov automorphisms.

Example 4. Let M be a flat manifold with fundamental group Γ given by the short exact sequence:

$$1 \rightarrow Z^{12} \rightarrow \Gamma \rightarrow Z_5 \oplus Z_5 \oplus Z_2 \oplus Z_2 \rightarrow 1,$$

determined by an element $\alpha \in H^2(Z_5 \oplus Z_5 \oplus Z_2 \oplus Z_2, Z^{12})$.

Let $\alpha_1 \in H^2(Z_5 \oplus Z_5, Z^9)$ defines the Bieberbach group Γ_1 with holonomy group $Z_5 \oplus Z_5$ (see [6], remark on page 215) and $\alpha_2 \in H^2(Z_2 \oplus Z_2, Z^3)$ defines the Bieberbach group Γ_2 with holonomy group $Z_2 \oplus Z_2$. The action of $G = Z_5 \oplus Z_5 \oplus Z_2 \oplus Z_2$ on $Z^{12} = L$ in the above exact sequence is a direct product of the action of $Z_5 \oplus Z_5$ on Z^9 from the definition of Γ_1, and the action of $Z_2 \oplus Z_2$ on Z^3 from the definition of Γ_2.

We have (see [11] page 205)

$$H^2(G, L) = H^2(Z_2 \oplus Z_2, L^{Z_5 \oplus Z_5}) \oplus H^2(Z_5 \oplus Z_5, L)^{Z_2 \oplus Z_2} \text{ and}$$

$$H^2(G, L) = H^2(Z_5 \oplus Z_5, L^{Z_2 \oplus Z_2}) \oplus H^2(Z_2 \oplus Z_2, L)^{Z_5 \oplus Z_5}.$$

Hence, it is not difficult to verify that $\alpha = \alpha_1 + \alpha_2$.

Applying Theorem **A** we have that Γ has infinite outer automorphism group and M does not support an Anosov diffeomorphism (cf. [8]).

We must mention that the group $Z_5 \oplus Z_5 \oplus Z_2 \oplus Z_2$ has a **Q**-irreducible representation which is **R**-irreducible and has a **Q**-irreducible representation which is **R**-reducible.

We end with a problem.

Problem. *Describe the class $\Re_k$ of all finite groups with the following property: there exists a Bieberbach group Γ whose holonomy group G belongs to $\Re_k$ and Out(Γ) is quasi - isometric to SL(k,Z) for $k \geq 1$.*

For k = 1 this question is equivalent to: When is a finite group a holonomy group of a Bieberbach group with finite outer automorphism group ?
For a partial answer to this last question we refer to [7].

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