A Note on two inventory models

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Introduction

In inventory models the major objective consists of minimizing the total inventory cost and to balance the economics of large orders or large production runs against the cost of holding inventory and the cost of going short.
In the present paper we analyse the fluctuations in the stock and starting from some basic assumptions we obtain bounds between which the stock varies. The main purpose and use of our results is that we are able to determine the exact upper and lower stockbounds. In the paper we formulate a deterministic and a stochastic version of our model.

1 The deterministic model

Consider an inventory process involving one item and suppose that the initial stock is equal to a, a positive real number. During regular time-intervals the stock decreases because of demand. We assume that the demand is in units of size b. When inventory is below the level b, the policy consists of ordering $p = a + b$ new units. This type of inventory process has also been analyzed by Andres and Emmons (1975) and by Zoller (1977). Formally the model is the following. Let $0 < b < a$ denote arbitrary real numbers. We define a sequence $\{x_n\}$ of stock-levels as follows :

$$x_1 = a;$$
$$x_{n+1} = x_n - b \quad \text{if} \quad x_n \geq b$$

and

$$x_{n+1} = x_n - b + p = x_n + a \quad \text{if} \quad x_n < b.$$

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We analyse the sequence $\{x_n\}$ and want to find sharp lower and upper bounds between which the sequence takes its values. Obviously we have to find an upper bound by examining those x-values greater than a. We shall denote by z_n the x-value for which $x_i \geq a$ for the n-th time. We can find an upperbound by studying $\{z_n, n \geq 1\}$; we can find a lower bound by studying $\{z_n - a, n \geq 2\}$.

Now suppose that $x_m \geq a$ and that $x_m = z_n$, the n-th time that we have a value $\geq a$. We determine $x_{m+1}, x_{m+2}, ..., x_{m+k}$ and k such that

$$x_{m+k} = x_m - kb \geq b \quad \text{and} \quad x_{m+k+1} = x_m - (k+1)b < b.$$

Obviously $k + 1 = [x_m/b]$ with $[x_m/b]$ the integer part of x_m/b. Using this value of k we find a new x-value $\geq a$:

$$z_{n+1} = x_{m+k+2} = x_m - (k+1)b + a = x_m - [x_m/b]b + a.$$

It follows that $z_1 = a$ and that $z_{n+1} = zn - [z_n/b]b + a, n \geq 1$.

Now divide by b and let $u_n := z_n/b$ so that

$$u_1 = d = a/b \;\text{ and }\; u_{n+1} = d + u_n - [u_n], n \geq 1.$$

A further simplification we get by replacing u_n by $y_n = u_n - [d]$:

$$y_1 = c = d - [d] \text{ and } y_{n+1} = c + y_n - [y_n], n \geq 1.$$

In the next lemma we find a simple expression for y_n :

Lemma 1.1 *For $n \geq 1 : y_n = nc - [(n-1)c] =: f(n)$.*

Proof. By induction on n. ∎

We study the sequence $\{y_n\}$ and distinguish two cases. In case 1 we assume that c is rational. The second case is devoted to irrational c.

1.1 Case 1 : c is rational

We suppose that $c = p/q$ where $p, q \in N, p < q$ and $GCD(p, q) = 1$. It is easy to see that $f(q + 1) = f(1)$ and consequently that $f(n + q) = f(n)$. Hence $\{f(n)\}$ is periodic with period not greater than q. Now suppose that $\{f(n)\}$ is periodic with period r, i.e. $f(r + 1) = f(1)$. From this it follows that $rc = [rc]$ is an integer. It follows that the period of $\{f(n)\}$ equals q because q is the first integer r for which rc is a natural number. Summarizing our findings we have

Lemma 1.2 a) *If $c = p/q$ with $GCD(p, q) = 1$, then $\{f(n)\}$ is periodic with period q.*
b) *$\{f(n)\}$ is periodic if and only if c is a rational number.* ∎

Under the conditions of Lemma 1.2 we can find max $\{f(n)\}$ by looking at $\{f(1), f(2), ..., f(q)\}$. Obviously all numbers $f(i)$ $(i = 1, ..., q)$ are different - since otherwise the period of $\{f(n)\}$ would be less than q. Moreover for each i we have $c \leq f(i) < c+1$ and $f(i)$ is of the form $f(i) = c + n_i/q$ where $0 \leq n_i \leq q-1$. Hence $\{f(i), i = 1, 2, ..., q-1\} = \{c + j/q, j = 0, ..., q-1\}$ and hence max $\{f(i)\} = c + (q-1)/q = (p+q-1)/q$.

For $\min\{x_n\}$ we study $\min\{z_n - a, n \geq 2\}$. Introducing y_n and $f(n)$ as before, we find that $\min\{f(i), i \geq 2\} = c + 1/q$. Summarizing, we have proved the following result.

Theorem 1.3 *If $c = p/q \in Q$, with $GCD(p, q) = 1$, then*

$$\max\{y_n, n \geq 1\} = (p+q-1)/q = 1 + c - 1/q.$$
$$\min\{y_n, n \geq 2\} = 1/q.$$

■

Returning to the starting point of our analysis, we obtain

Corallary 1.4 *Max$\{x_n\} = a + b - GCD(a, b)$ and min$\{x_n\} = GCD(a, b)$.*

Proof. For the first part, use Lemma 1.3 and $z_n = b(y_n + [d])$, where $d = a/b$. For the second part, recall that lower bound can be found by considering the minimum of the values $z_n - a$. ■

Remarks. 1. Consider the following extension of the previous model. Consider an inventory process involving m items. For each item j the inventory policy consists of the pair (b_j, a_j) as in the previous model. If $x_{n,j}$ denotes the inventory level of item j at stage n, then the total inventory level equals $I_n = x_{n,1} + ... + x_{n,m}$. Corollary 1.4 states that

$$\sum_{j=1}^{m} GCD(a_j, b_j) \leq I_n \leq \sum_{j=1}^{m} (a_j + b_j + GCD(a_j, b_j)).$$

Whereas the bounds in Corollary 1.4 are sharp, the question rises whether the bounds for I_n are sharp.

2. If we assume that the demand is not in units of b but in fractions of b (i.e. the demand is tb per time unit, where $1/t$ is an integer and $t < 1$), then the sequence of inventory levels (with $c = tb + a$) is defined as follows :

$$x_1 = a; x_{n+1} = x_n - tb \text{ if } x_n \geq b \quad \text{and} \quad x_{n+1} = x_n + a \text{ if } x_n < b.$$

Among similar lines the analogue of Corollary 1.4 follows. Using $u_n = z_n/tb$ and then $y_n = u_n - [a/tb]$ it follows that $y_1 = c = a/tb - [a/tb]$ and $y_{n+1} = y_n - [y_n] + c + 1 - 1/t$. We obtain

Corallary 1.5 *For the sequence $\{x_n\}$ we have*
$\max\{x_n\} = a + (2t-1)b - GCD(a, tb)$ and $\min\{x_n\} = (t-1)b + GCD(a, tb)$. ■

3. The results of remark 2. can be generalized as in remark 1.

1.2 Case 2 : c irrational

In the case where c is a irrational number we prove the following result, cf. lemma 1.3.

Lemma 1.6 *For irrational* $c, 0 < c < 1$, *there holds*

$$\sup\{nc - [nc], n \geq 1\} = 1 \quad \text{and} \quad \inf\{nc - [nc], n \geq 1\} = 0.$$

Proof. If c is irrational we represent c as

$$c = q_1 + \cfrac{1}{q_2 + \frac{1}{q_3 + \dots}}$$

or as $c = (q_1, q_2, q_3, \dots)$, where $q_1 = [c], q_2 = [1/(c - q_1)] > 0$, etc.

Using this notation, for each integer n we consider $A_n/B_n = (q_1, q_2, \dots, q_n)$. Obviously A_n and B_n are integers with the following properties :

(i) $A_1 = q_1, B_1 = 1, A_2 = q_1q_2 + 1, B_2 = q_2$;
(ii) for $n > 1 : A_{n+1} = A_nq_{n+1} + A_{n-1}$ and $B_{n+1} = B_nq_{n+1} + B_{n-1}$;
(iii) using (ii) :$A_nB_{n+1} - A_{n+1}B_n = (-1)^{n+1}$;
(iv) $\{B_n\}$ is strictly increasing with limit $+\infty$.

Now let $a = (q_{n+1}, q_{n+2}, \dots)$ so that $a > 0$ and $c = (q_1, \dots, q_n, a)$. From (ii) it follows that

(v) $c = (A_na + A_{n-1})/(B_na + B_{n-1})$.

A combination of (iii) and (v) yields

(vi) $A_{2n}/B_{2n} < c < A_{2n+1}/B_{2n+1}$.

Using (v) and (vi) we obtain $0 < B_{2n}c - A_{2n} < 1/B_{2n+1}$.
By using (iv), for each $\beta > 0$ we can find N sufficiently large so that

$$0 < B_{2n}c - A_{2n} < \beta, \text{ for all } n \geq N.$$

Since A_{2n} is an integer, it follows that $0 \leq B_{2n}c - [B_{2n}c] < \beta$.
Since B_{2n} is an integer also, it follows that $\inf\{nc - [nc], n \geq 1\} = 0$.
In a similar way from (v) and (vi) we obtain that $0 < A_{2n+1} - B_{2n+1}c < 1/B_{2n}$.
For arbitrary $\beta > 0$ it follows that $0 \leq 1 - (B_{2n+1}c - [B_{2n+1}c]) < \beta$, for all $n \geq N$.
Hence the other statement of the lemma follows. ■

For the original sequence $\{x_n\}$ we have the following analogue of Corollary 1.4.

Theorem 1.7 *If a/b is irrational, then sup$\{x_n\} = a + b$ and inf $\{x_n\} = 0$.* ■

2 The Stochastic model

Suppose a, b and $c(a > b)$ are positive real numbers and suppose that $X, X_1, X_2, ...$ are i.i.d. positive random variables (r.v.) with distribution function (d.f.) $F(x)$. A new sequence $Y_0, Y_1, .., Y_n$ of r.v. is constructed as follows. Let $Y_0 = a$ and for $n \geq 0$, let

$$Y_{n+1} = Y_n - X_{n+1} \quad \text{if } Y_n \geq b$$

and

$$Y_{n+1} = Y_n - X_{n+1} + c \quad \text{if } Y_n < b.$$

The interpretation of Y_n is clear : we start with an initial inventory (a) and demand is stochastic. When the inventory at time n is below level b the policy consists of ordering a fixed number c of new units during the next time interval.

Example 1. A waterreservoir is initially at level a; from time to time a random amount X_i of water is demanded. Once the level is below b, an amount of water a is added to the reservoir.

Example 2. A firm has an amount of cash a and at certain times bills have to be paid and the cash decreases with an amount X_i. Once below a level b, the cash is increased by an amount c.

For this sequence relevant questions are : what is the maximal value ? what is the smallest value ? What is the mean re-order time ?

Let $S_0 = 0$ and for each $n \geq 1$, let $S_n = X_1 + ... + X_n$. Clearly we have $Y_0 = a$. For $n = 0, 1, 2, ...$ we have $Y_n = a - S_n$ as long as $S_{n-1} \leq a - b$. The first time $S_{n-1} > a - b$ we have $Y_n = a - S_n + c$.

The corresponding value of n is related to the the renewal counting process for the process S_n. As usual we define $N(x)$, the renewal process, as follows :

$$N(x) = n \quad \text{if and only if} \quad S_{n-1} \leq x < S_n.$$

It follows that $S_{n-1} \leq a - b < S_n$ iff $N(a - b) = n$.

At the first renewal the inventory equals $m_1 = Y_{N(a-b)} = a - S_{N(a-b)}$ and at the next time point the inventory is $M_1 = Y_{N(a-b)+1} = a + c - S_{N(a-b)+1}$.

From this time on, the process starts again and a new renewal occurs at step $N(a + c - b)$. Then we have $m_2 = Y_{N(a+c-b)} = a + c - S_{N(a+c-b)}$ and $M_2 = a + 2c - S_{N(a+c-b)+1}$.

Continuing this way we find that at step k we have

$$mk = a + kc - S_{N(a+kc-b)}$$

and

$$M_k = a + (k + 1)c - S_{N(a+kc-b)+1}.$$

In the following discussion we will concentrate on m_k. We can do this because the event $\{N(x) = n\}$ does not depend on X_{n+1}, and this implies that the distributins of m_k and M_k are related by the following obvious relation :

(1) $$Mk =^d= mk + c - X.$$

Remark. If the r.v. X is degenerate and takes only the value b, then $S_n = nb$ and $N(x) = [x/b] + 1$. In this case we have $m_k = a + kc - N(a + kc - b)b$. If $c = a + b$, this leads to $m_k = a(k + 1) - [(k + 1)a/b]b$, a relation which was treated in section 1.

In order to find the mean values of m_k (and M_k) we use the renewal function. The renewal function $H(x)$ is defined as the mean of $N(x) : H(x) = E(N(x))$.

It is well-known (see e.g. Feller (1971) or Ross (1970)) that $H(x) = \sum_{n=0}^{\infty} F^{*n}(x)$ where * denotes convolution. Now Wald's identity states that $E(S_{N(x)}) = E(X)H(x) = \mu H(x)$ whenever $E(X) = \mu$ is finite. Using Wald's identity and (1), we find

Lemma 2.1 If $E(X) = \mu$ is finite, then $E(m_k) = a + kc - \mu H(a + kc - b)$ and $E(M_k) = a + (k + 1)c - \mu(H(a + kc - b) + 1)$. ∎

The asymptotic behaviour of $H(x)$ is well-known and can be used to determine the asymptotic behavior of $E(m_k)$ and $E(M_k)$ as k a tends to infinity.

Lemma 2.2 *(a) Suppose that $F(x)$ is not lattice and that* $\mathrm{Var}(X) < \infty$. *Then as k tends to infinity,* $\lim E(m_k) = b - E(X^2)/2\mu$ *and* $\lim E(M_k) = b + c - \mu - E(X^2)/2\mu$.

(b) If F is lattice with $GCD\{n|P\{X = n\} > 0\} = 1$ and if $\mathrm{Var}(X) < \infty$, then as k tends to infinity, $E(m_k) = b - E(X^2)/2\mu + (a + kc - b - [a + kc - b] - 1/2) + o(1)$.

Proof. (a) Since F is not lattice and X has finite variance, we have (Feller, 1971, IX.3)
$H(x) = x/\mu + E(X^2)/2\mu^2 + o(1)$. The result (a) now follows from Lemma 2.1.
(b) For lattice distributions with GCD $\{n|P\{X = n\} > 0\} = 1$ and with finite variance, it follows from Feller (1970, XIII,3 Theorem 3 and eq.(12.8),p.341) that $H(x) = [x]/\mu + (EX^2 + \mu)/2\mu^2 + o(1)$. Using Lemma 2.1 it follows that

$$E(m_k) = a + kc - \mu\{(E(X^2) + \mu)/2\mu^2 + [a + kc - b]/\mu + o(1)$$

where the $o(1)$-term tends to 0. Hence result (b) follows. ∎

In the next result we determine the d.f. of m_k and M_k. To this end let $Y(t) = S_{N(t)} - t$ denote the excess or residual life at time t. Clearly $m_k = b - Y(a + kc - b)$.

For $x \leq b$ it follows that

(2) $$P\{m_k < x\} = P\{Y(a + kc - b) > b - x\} = 1 - P\{Y(a + kc - b) \leq b - x\}.$$

Now the distribution function of $Y(t)$ is known (see e.g. Ross (1970), p.44):

$$P\{Y(t) \leq x\} = 1 - \int_0^t [1F(t\ x\ y)]dH(y),\ x, t \geq 0. \tag{3}$$

Furthermore, if F is not lattice and X has a finite mean μ, then the limit distribution of $Y(t)$ is known :

$$\lim_{t\to\infty} P\{Y(t) \leq x\} = \frac{1}{\mu}\int_0^x [1 - F(y)]dy =: m(x). \tag{4}$$

In the next result we obtain the limit distributions of m_k and M_k. Recall that f^*g denotes the (Stieltjes-)convolution of the two measures f and g.

Lemma 2.3 *If $F(x)$ is not lattice and X has a finite mean μ, then :*

$$(a)\qquad \lim_{k\to\infty} P\{m_k < x\} = 1 - m(b - x), \quad x \leq b$$

and

$$(b)\qquad \lim_{k\to\infty} P\{M_k < x\} = 1 - m * F(b - x + c), \quad x \leq b + c.$$

Proof. Result (a) follows from (2) and (4).
Result (b) follows from (a) and relation (1). ■

Remarks. 1. It follows from the result that $\lim P\{m_k < 0\} = 1 - m(b)$ and $\lim P\{M_k < b\} = 1 - m^*F(c)$. When b and c are sufficiently large, then these probabilities can be made as small as desired.
2. If X is concentrated in the interval $[0, b']$, then $1 - m(b) = 0$ for each $b > b'$.
3. Suppose X has the exponential distribution $F(x) = 1 - \exp(-x/\mu)$ for $x > 0$. In this case $H(x) = F^*(0) + x/\mu$ and the d.f. of $Y(t)$ equals $F(x)$. For this special case we have
$P(m_k \leq x) = \exp(-(b - x)/\mu)\quad (x \leq b)$ and $E(m_k) = b - \mu$.
4. If $E(X)$ is not finite, then the previous results do not apply. In this case instead of limit laws for $Y(t)$ one can consider limits for $Y(t)/t$, see e.g. Bingham et.al. (1987, p. 361-364). Now only limit results for m_k/k are available.

Finally we consider the rate of convergence in Lemma 2.3. To this end for $u, v \geq 0$ we define $R(u, v) = H(u + v) - H(u) - v/\mu$. Using (3) and the definition of $m(x)$ and $R(u, v)$ it is straightforward to prove the following identity :

$$P\{Y(t) \leq x\} - m(x) = \int_{z=0}^x R(t + x - z, z)dF(x) + R(t, x)(1 - F(x)). \tag{5}$$

For non-lattice d.f. $F(x)$ and for fixed $x, R(t, x)$ tends to zero as t tends to infinity. The rate at which this occurs determines the rate of convergence in Lemma 2.3. In order to formulate the next result, recall that a measurable positive function is regularly varying at infinity and with index α (notation RV_α) iff $f(tx)/f(t)$ tends to x^α for each $x > 0$.

From Frenk (1983) we recall the following result.

Lemma 2.4 *(Frenk (1983) Theorem 4.1.9) Suppose that X has an absolute continuous d.f. $F(x)$ and that $1 - F(x)$ is $RV_{-\alpha}$ with $\alpha > 1$.*
Then as $t \to \infty, R(t, x) \sim (1 - m(t))/\mu$ locally uniformly (l.u.) in x. ■

Under the conditions of Lemma 2.4, $1-m(t)$ is $RV_{1-\alpha}$ and $1-m(t+x) \sim 1-m(t)$ l.u. in x. It follows from (5), Lemma 2.4 and Lebesgue's theorem that as $t \to \infty$,

(6) $(P\{Y(t) \leq x\} - m(x))/(1-m(t)) \to F(x)/\mu + (1-F(x))/\mu = 1/\mu, l.u.$ in x.

Specializing to m_k and combining (2) and (6) we obtain the following result. A similar result holds for M_k.

Lemma 2.5 *Under the conditions of lemma 2.4, for $x \leq b$ as $k \to \infty$ we have*

$$P\{m_k < x\} = 1 - m(b-x) - (1/\mu + o(1))c^{1-\alpha}(1-m(k)).$$

■

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