The closeness of the range of a probability on a certain system of random events — an elementary proof

Vladimír Balek Ivan Mizera

Abstract

An elementary combinatorial method is presented which can be used for proving the closeness of the range of a probability on specific systems, like the set of all linear or affine subsets of a Euclidean space.

The motivation for this note came from the second author's research in statistics: high breakdown point estimation in linear regression. By a probability distribution P, defined on the Borel σ-field of $\mathbb{R}^p$, a collection of regression design points is represented; then, a system $\mathcal{V}$ of Borel subsets of $\mathbb{R}^p$ is considered. Typical examples of $\mathcal{V}$ are, for instance, the system $\mathcal{V}_1$ of all linear, or $\mathcal{V}_2$ of all affine proper subspaces of $\mathbb{R}^p$. The question (of some interest in statistical theory) is:

$$\text{Is there an } E_0 \in \mathcal{V} \text{ such that } P(E_0) = \sup\{P(E)\colon E \in \mathcal{V}\}? \tag{1}$$

For some of $\mathcal{V}$, the existence of a desired E_0 can be established using that (a) $\mathcal{V}$ is compact in an appropriate topology; (b) P is lower semicontinuous with respect to the same topology. The construction of the topology may be sometimes tedious; moreover the method does not work if, possibly, certain parts of $\mathcal{V}$ are omitted, making $\mathcal{V}$ noncompact. Also, a more general problem can be considered:

$$\text{Is the range } \{P(E)\colon E \in \mathcal{V}\} \text{ closed?} \tag{2}$$

The positive answer to (2) implies the positive one to (1). The method outlined by (a) and (b) cannot answer (2) — we have only lower semicontinuity, not full continuity.

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Nevertheless, an elementary method provides the desired answer, for general P and $\mathcal{V}$. The method does not require a topologization of $\mathcal{V}$, and it works also for various, possibly noncompact, subsets of $\mathcal{V}$. The main idea can be regarded as an extension of a simple fact that the probabilities of pairwise disjoint events cannot form a strictly increasing sequence. Linear subspaces are not disjoint; however, the intersection of two distinct ones with the same dimension is a subspace with a lower dimension. Iterating this process further, we arrive to the unique null-dimensional subspace. If, say, instead of linear subspaces the affine ones are considered, the method works in a similar way — only the terminal level is slightly different.

A well-known related property — to be found, for instance, in [1], Ch. II, Ex. 48–50 — says that the range $\{P(E)\colon E \in \mathcal{S}\}$ is closed for every probability space $(\Omega, \mathcal{S}, P)$. However, here the background is different: probabilities of general events can form an increasing sequence — this is not true in our setting.

Theorem. *Let $(\Omega, \mathcal{S}, P)$ be a probability space. If $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \cdots \subseteq \mathcal{A}_n$ are sets of events such that $\operatorname{card} \mathcal{A}_0 = 1$ and for every $k = 1, 2, \dots, n$, the intersection of two distinct events from $\mathcal{A}_k$ belongs to $\mathcal{A}_{k-1}$, then the set $\{P(E)\colon E \in \mathcal{A}_n\}$ is closed.*

Corollary. *Under the assumptions of Theorem,* (1) *is true with $\mathcal{V} = \mathcal{A}_n$.*

Applying Theorem for $\mathcal{V} = \mathcal{V}_1$, we set $n = p - 1$; $\mathcal{A}_k$ consists of all proper subspaces of dimension less or equal to k. Note that $\mathcal{A}_n = \mathcal{V}_1$ and $\mathcal{A}_0 = \{\mathbf{0}\}$; the other assumptions hold as well. According to Theorem, the range of P on $\mathcal{A}_n$ is closed and the supremum is attained. The cases of other $\mathcal{V}$ are treated in an analogous way.

We shall call a system $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_n$ satisfying the assumptions of Theorem an *intersection system.* Suppose that $\mathcal{B}$ is a set of events such that $\mathcal{B} \subseteq \mathcal{A}_n$. If $\mathcal{A}'_0, \mathcal{A}'_1, \dots, \mathcal{A}'_\nu$ is another intersection system such that $\mathcal{B} \subseteq \mathcal{A}'_\nu$, we can form an intersection system $\mathcal{A}''_0, \mathcal{A}''_1, \dots, \mathcal{A}''_m$ by taking consecutively $\mathcal{A}''_m = \mathcal{A}_n \cap \mathcal{A}'_\nu$, $\mathcal{A}''_{m-1} = \mathcal{A}_{n-1} \cap \mathcal{A}'_{\nu-1}$, $\dots$, identifying $\mathcal{A}''_0$ with the first set with cardinality 1 obtained in this process. As a result, we have $m \le \min(n, \nu)$ and $\mathcal{B} \subseteq \mathcal{A}''_m$. The similar construction can be carried out with more than two intersection systems; if there is any intersection system $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_n$ such that $\mathcal{B} \subseteq \mathcal{A}_n$, then the intersection of all intersection systems with this property will be called the *intersection system generated by* $\mathcal{B}$. Note that for all k, the set $\mathcal{A}_{k-1}$ contains exactly all pairwise intersections of events from $\mathcal{A}_k$. Hence if $\mathcal{A}_k$ is finite, so is $\mathcal{A}_{k-1}$. If $\mathcal{A}_k$ is (at most) countable, so is $\mathcal{A}_{k-1}$.

Let $1 \le k \le n$. An intersection system is said to satisfy a *finiteness condition at level* k, if any event from $\mathcal{A}_{k-1}$ is a subset of at most a finite number of events from $\mathcal{A}_k$. Note that if the finiteness condition is satisfied at level k and $\mathcal{A}_k$ is infinite, so is $\mathcal{A}_{k-1}$. As a consequence, an intersection system with infinite $\mathcal{A}_n$ cannot satisfy the finiteness condition at all levels $k = 1, 2, \dots, n$.

Lemma. *Suppose that the intersection system $\mathcal{A}_0$, $\mathcal{A}_1$, $\dots$, $\mathcal{A}_n$ generated by $\{E_1, E_2, \dots\}$ satisfies the finiteness condition at levels $k = 2, \dots, n$ and $\mathcal{A}_0 = \{\emptyset\}$. Then* $\lim_{i\to\infty} P(E_i) = 0$.

Proof. By assumptions, $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ are countably infinite. For any $F \in \mathcal{A}_k$, $k = 1, 2, \dots, n$, let $\tilde{F} = F \smallsetminus \bigcup \mathcal{A}_{k-1}$. Note that $\tilde{F} = F$ for $F \in \mathcal{A}_1$, since $\mathcal{A}_0 = \{\emptyset\}$.

For all k, the elements of $\{\tilde{F}\colon F \in \mathcal{A}_k\}$ are pairwise disjoint. Fix $\varepsilon > 0$. Pick $\mathcal{B}_1 \subseteq \mathcal{A}_1$ such that $\mathcal{A}_1 \setminus \mathcal{B}_1$ is finite and

$$P\Big(\bigcup_{F\in\mathcal{B}_1} F\Big) = P\Big(\bigcup_{F\in\mathcal{B}_1} \tilde{F}\Big) = \sum_{F\in\mathcal{B}_1} P(\tilde{F}) \le \varepsilon. \tag{3}$$

Given $\mathcal{B}_{k-1}$, and assuming that $\mathcal{A}_{k-1} \setminus \mathcal{B}_{k-1}$ is finite, we construct inductively a set $\mathcal{C}_k$ to be the set of all $F \in \mathcal{A}_k$ such that there is no $G \in \mathcal{A}_{k-1} \setminus \mathcal{B}_{k-1}$ which is a subset of F; then $\mathcal{B}_k \subseteq \mathcal{C}_k$ is picked in a way that $\mathcal{C}_k \setminus \mathcal{B}_k$ is finite and

$$P\Big(\bigcup_{F\in\mathcal{B}_k} \tilde{F}\Big) = \sum_{F\in\mathcal{B}_k} P(\tilde{F}) \le \varepsilon. \tag{4}$$

Since $\mathcal{A}_{k-1} \setminus \mathcal{B}_{k-1}$ is finite, by the finiteness condition (at level k) also $\mathcal{A}_k \setminus \mathcal{C}_k$ and hence $\mathcal{A}_k \setminus \mathcal{B}_k$ are finite. Starting from (3), we proceed inductively, using (4):

$$\begin{aligned} P\Big(\bigcup_{F\in\mathcal{B}_k} F\Big) &\le P\Big(\bigcup_{F\in\mathcal{B}_k}\Big(F \setminus \bigcup_{G\in\mathcal{B}_{k-1}} G\Big) \cup \bigcup_{G\in\mathcal{B}_{k-1}} G\Big) \\ &= P\Big(\bigcup_{F\in\mathcal{B}_k}\Big(F \setminus \bigcup_{G\in\mathcal{A}_{k-1}} G\Big)\Big) + P\Big(\bigcup_{G\in\mathcal{B}_{k-1}} G\Big) \\ &= P\Big(\bigcup_{F\in\mathcal{B}_k} \tilde{F}\Big) + P\Big(\bigcup_{G\in\mathcal{B}_{k-1}} G\Big) \le \varepsilon + (k-1)\varepsilon = k\varepsilon, \end{aligned} \tag{5}$$

the first equality due to the fact that $\mathcal{B}_k \subseteq \mathcal{C}_k$. Since (5) holds also for $k = n$ and ε was arbitrary, the statement follows: given $\delta > 0$, there is only a finite number of E_i for which

$$P(E_i) \le P\Big(\bigcup_{E_i\in\mathcal{B}_n} E_i\Big) \le \delta$$

does not hold. ■

Proof of Theorem. The statement holds if $\mathcal{A}_n$ is finite. Suppose that $\mathcal{A}_n$ is infinite. Fix a sequence $E_1, E_2, \dots$ of events from $\mathcal{A}_n$ such that $P(E_i)$ is convergent. Proving that there is an $E_0 \in \mathcal{B}$ such that $\lim_{i\to\infty} P(E_i) = P(E_0)$ is a trivial task if $\{E_1, E_2, \dots\}$ is finite; suppose that the events E_i are pairwise distinct. Let $\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_\nu$ be the intersection system generated by $\{E_1, E_2, \dots\}$. There is an $m \ge 1$, $m \le \nu$, such that the finiteness condition holds for $k = \nu, \nu-1, \dots, m+1$ and fails for $k = m$. Consequently, an infinite number of pairwise intersections of elements of $\mathcal{B}_m$ coincide — let the corresponding element of $\mathcal{B}_{m-1}$ be denoted by E_0. Let $\mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_{\nu-m+1}$ be the intersection system generated by the set $\{F_1, F_2, \dots\} \subseteq \{E_1, E_2, \dots\}$ consisting of those events from $\mathcal{B}_n$ which contain E_0 as a subset. Note that $\mathcal{C}_0 = \{E_0\}$ and $\mathcal{C}_k$ for $k \ge 1$ is the set of all events from $\mathcal{B}_{m+k-1}$ which contain E_0 as a subset. By the choice of E_0, $\mathcal{C}_1$ is countably infinite; hence so are $\mathcal{C}_2, \dots, \mathcal{C}_{\nu-m+1}$. Let $\mathcal{D}_k = \{F \setminus E_0\colon F \in \mathcal{C}_k\}$. The system $\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_{\nu-m+1}$ satisfies all assumptions of Lemma. Hence,

$$\lim_{i\to\infty} P(E_i) = \lim_{i\to\infty} P(F_i) = P(E_0) + \lim_{i\to\infty} P(F_i \setminus E_0) = P(E_0).$$

The statement follows, since $E_0 \in \mathcal{B}_{m-1} \subseteq \mathcal{A}_n$. ■

Reference

[1] A. Rényi: *Probability Theory*, Budapest, Akadémiai Kiadó, 1970.

Department of Theoretical Physics and
Department of Probability and Statistics
Comenius University, Bratislava
Slovakia
email : balek@fmph.uniba.sk, mizera@fmph.uniba.sk