Convergence of Rational Interpolants*

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Abstract

The convergence of (diagonal) sequences of rational interpolants to an analytic function is investigated. Problems connected with their definition are shortly discussed. Results about locally uniform convergence are reviewed. Then the convergence in capacity is studied in more detail. Here, a central place is taken by a theorem about the convergence in capacity of rational interpolants to functions with branch points. The notion of a symmetric domain plays a fundamental role. Apart from very special situations, proofs of the existence of such domains are known so far only for two types of interpolation schemes.

1 Rational Interpolation

Interpolating and approximating an analytic function by polynomials or rational functions with prescribed poles is rather well understood and has been studied in great detail by J.L. Walsh (cf. his book [Wal]). In many respects interpolation by rational functions with preassigned poles leads to a theory very similar to that of polynomial interpolation. A rather different situation arises if one considers interpolation by rational functions with free poles. Free poles means here that both, the numerator and the denominator polynomial, are determined by the interpolation conditions, while in case of preassigned poles this is true only for the numerator polynomial. The theoretical background of rational interpolation with free poles is very similar to that of Padé approximants. Actually, Padé approximants are a special type of rational interpolants, they are linearized (also called generalized) rational interpolants with all its interpolation points identical. In the literature rational interpolants with free poles are also known under the name of *multi-point Padé approximants.* There are good reasons for this somewhat strange terminology. Problems connected with the definition of rational interpolants will be addressed in the next paragraphs.

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Contrary to the situation in polynomial interpolation, for rational interpolation there is no definition that satisfies all wishes a potential user naively could have; some compromises are always necessary, and they result in a definition that may surprise at first glance. The definition will be discussed in the present section. The material is based on the excellent survey paper [Me] by J. Meinguet, which covers many other aspects of the problem including a very interesting and detailed review of the historic development.

Let an infinite triangular matrix of *interpolation points* $a_{ij} \in \overline{\mathbb{C}}$ (called *interpolation scheme*) be given:

$$\mathcal{A} := \begin{pmatrix} a_{00} & & & \\ \dots & \dots & & \\ a_{0n} & \dots & a_{nn} & \\ \dots & \dots & \dots & \dots \end{pmatrix} \tag{1.1}$$

Each row

$$A_n := \{a_{0n}, \dots, a_{nn}\} \tag{1.2}$$

of the matrix $\mathcal{A}$ defines an *interpolation set* with $n+1$ interpolation points. It is not excluded that some or all points are identical. Hence, in (1.2) we have in general a multiset with multiplicities of elements taken account of by repetition. With each interpolation set A_n a polynomial

$$w_n(z) := \prod_{x \in A_n} (z - x) = \prod_{j=0}^{n} (z - a_{jn}). \tag{1.3}$$

is associated.

By f we denote the function which will be interpolated. In the sequel it is assumed that this function is analytic at each point $z \in A_n$, $n \in \mathbb{N}$. By $\mathcal{P}_n$ and $\mathcal{R}_{mn}$ we denote the set of all complex polynomials of degree at most n and the set of rational functions of numerator and denominator degree at most m and n, respectively.

Definition 1.1: A rational function $r_{mn} = r_{mn}(f, A_{m+n}; \cdot) = r_{mn}(f, \mathcal{A}; \cdot) \in \mathcal{R}_{mn}, m, n \in \mathbb{N}$, is called *rational interpolant* of degree m, n to the function f at the $m + n + 1$ interpolation points of the set A_{m+n} if the quotient

$$\frac{f - r_{mn}}{w_{m+n}} \quad \text{is bounded at each} \quad x \in A_{m+n}. \tag{1.4}$$

Remarks: (1) Condition (1.4) implies that at each zero of the polynomial w_{m+n} the interpolation error $f - r_{mn}$ has a zero of at least the same order. Thus, $f - r_{mn}$ has a zero at each point $x \in A_{m+n}$ of at least the same order as the frequency of the point x in the set A_{m+n}, or in other words, the interpolant r_{mn} and its derivatives $r_{mn}^{(k)}$ coincide with the function f and its derivatives $f^{(k)}$ at the point x up to an order determined by the frequency of x in A_{m+n}. Relation (1.4) therefore defines interpolation in Hermite's sense.

(2) As the next example will show, the existence of a rational function $r_{mn} \in \mathcal{R}_{mn}$ satisfying (1.4) is in general not guaranteed. If for $m, n \in \mathbb{N}$ a rational function r_{nm}

exists that satisfies (1.4), then one says that the *Cauchy interpolation problem* is solvable (cf. [Me], Introduction).

(3) If the interpolation problem is solvable, then the solution is *unique*, which can easily be verified by comparing two potential candidates.

Example 1.1: We choose $m = n = 1$, $A_2 := \{-1, 0, 1\}$, and as function to be interpolated $f(z) := z^2$. Any rational function $r \in \mathcal{R}_{1,1}$ is either a Moebius transform or a constant. If r is a Moebius transform, then it is univalent in $\overline{\mathbb{C}}$ and therefore cannot interpolate the value 1 at the two different points -1 and 1. If r is a constant function, then it cannot interpolate the two different values 0 and 1. Hence, already in this very simple situation a rational function $r_{1,1}(f, A_2; \cdot)$ satisfying (1.4) does not exist.

Comparing rational interpolation with interpolation by polynomials, or by elements of any other family of functions forming a Chebychef systeme, shows that the main reason for the non-existence in case of rational interpolants is caused by the non-linearity of the parametrisation of the interpolants. In order to circumvent the difficulties one uses a linearized version of Definition 1.1. Actually, this has already been done by Cauchy [Ca] and Jacobi [Ja], but both authors do not mention (or possibly did not realize) that there is a prize to pay, namely the possibility of interpolation defects. Apparently, the first one who mentioned the possibility of non-existence of rational interpolants was Kronecker [Kr] (cf. [Me], Section 4).

Definition 1.2: The rational function

$$r_{mn} = r_{mn}(f, A_{m+n}; \cdot) = r_{mn}(f, \mathcal{A}; \cdot) = \frac{P_{mn}}{Q_{mn}} \in \mathcal{R}_{mn} \tag{1.5}$$

with $P_{mn} \in \mathcal{P}_m$, $Q_{mn} \in \mathcal{P}_n$, and $Q_{mn} \not\equiv 0$, is called *multi-point Padé approximant* or *linearized rational interpolant* of degree m, n to the function f at the $m + n + 1$ points of the interpolation set A_{m+n} if the quotient

$$\frac{Q_{mn} f - P_{mn}}{w_{m+n}} \quad \text{is bounded at each point } \; x \in A_{m+n}. \tag{1.6}$$

Remarks: (1) In the Definitions 1.1 and 1.2 the same symbol r_{mn} has been used on purpose, since if $r_{mn} \in \mathcal{R}_{mn}$ satisfies (1.4) then it automatically also satisfies (1.6) with an appropriate choice of the numerator and denominator polynomials $P_{mn} \in \mathcal{P}_m$ and $Q_{mn} \in \mathcal{P}_n$. Note that it may be necessary that the two polynomials P_{mn} and Q_{mn} contain common factors.

(2) The linerized version of the rational interpolant r_{mn} always exists. Indeed, relation (1.6) is equivalent to a systeme of $m + n + 1$ linear, homogenous equations for the $m + n + 2$ unknown parameters (coefficients) in the two polynomials P_{mn} and Q_{mn}. Hence, a non-trivial solution always exists, and it is not difficult to verify that for such a solution $Q_{mn} \equiv 0$ is impossible.

(3) It is easy to verify that the rational function r_{mn} is uniquely determined by (1.6). The same is not true for the pair of polynomials $(P_{mn}, Q_{mn}) \in \mathcal{P}_m \times \mathcal{P}_n \setminus \{0\}$. In any case the polynomials P_{mn} and Q_{mn} can be multiplied by a common non-zero constant, but there may exist more essential non-uniqueness.

The next lemma is a rather immediate consequence of (1.6) and (1.4).

Lemma 1.1: *If there exists a pair of polynomials* $(P_{mn}, Q_{mn}) \in \mathcal{P}_m \times \mathcal{P}_n \setminus \{0\}$ *such that (1.6) holds true and* $Q_{mn}(z) \neq 0$ *for all* $z \in A_{m+n}$*, then the Cauchy interpolation problem* $((m,n), f, A_{m+n})$ *is solvable, i.e., there exists* $r_{mn}(f, A_{m+n}; \cdot) \in \mathcal{R}_{mn}$ *satisfying (1.4).*

What happens if the Cauchy interpolation problem is not solvable? As we know, the linearized rational interpolant $r_{mn} = r_{mn}(f, A_{m+n}; \cdot)$ always exists and is unique. Consequently, if the Cauchy interpolation problem is not solvable, then there have to exist *interpolation defects*, i.e., for some elements $z_j = z_{j,m+n} \in A_{m+n}$, $j \in \{0, \ldots, m+n\}$, there exist points of the form $(z_j, f(z_j)) \in \overline{\mathbb{C}} \times \mathbb{C}$ or $(z_j, f^{(k)}(z_j)) \in \overline{\mathbb{C}} \times \mathbb{C}$ that do not lie on the graph of r_{mn} or the graph of the derivative $r_{mn}^{(k)}$, respectively, where $k \in \mathbb{N}$ is smaller than the frequency of the point z_j in the set A_{m+n}. These points $(z_j, f(z_j))$ or $(z_j, f^{(k)}(z_j))$ are called *unattainable*.

As already mentioned earlier an exellent survey about the solvability of the Cauchy interpolation problem is contained in [Me]. There, a unified approach to the analysis of the problem is given, which includes elements from the theory of continued fractions, and special matrices and determinants which have been introduced in connection with the interpolation problem, are discussed there. Efficient numerical algorithms that can be applied also in the presence of interpolation defects are discussed in [Gu].

In the present paper we are not really concerned with properties of rational interpolants for fixed degrees $m, n \in \mathbb{N}$, often called the algebraic aspect of the problem; our interest is the investigation of the convergence behavior of interpolants as $m + n \to \infty$, the so-called analytic aspect. The diagonal case $m = n$ is of main interest for us. In the next section the possibility of locally uniform convergence will be reviewed. There are two classes of functions, for which positive results have been proved. In many respects these are only island in a large sea of interesting functions, for which convergence results in the uniform norm are not available. Counterexamples constructed for Padé approximants show that locally uniform convergence can often not be proved since the possibility of spurious poles of the approximants cannot be ruled out. To circumvent these difficulties a weaker form of convergence, namely convergence in capacity, has been introduced, which allows for spurious poles as long as there are not too many or they are not too dense. This convergence will also be introduced here in Section 2. Section 2 is closed by the multi-point version of the Nuttall-Pommerenke Theorem. In Section 3 results concerning rational interpolants to functions with branch points are the main topic. A key role in the proof of these results is played by special domains, which are called *symmetric domains*. In Section 4 the exisence of such domains is considered in two special situations. A more general existence theorem for such domains is still missing.

2 The Convergence Problem

If the numerator and denominator degrees of the interpolants $r_{mn}(f, A; \cdot)$ grow, then the questions arize whether and where the interpolants converge to the function f that has been interpolated. Orientation can be obtained from convergence results proved for Padé approximants. There locally uniform convergence has been proved for certain classes of functions. In the present section we discuss corresponding results for rational interpolation. In comparison to later topics of the paper, we shall do this in a rather compressed and summerizing form. The discussion is followed by the introduction of convergence in capacity, and the section is closed by the Nuttall-Pommerenke Theorem for rational interpolants. In all cases our interest is restricted to diagonal or close-to-diagonal sequences of interpolants, *i.e.*, interpolants with numerator degree m equal or almost equal to the denominator degree n.

It is certainly not surprizing that the convergence behavior of the interpolants $r_{mn} = r_{mn}(f, \mathcal{A}; \cdot)$ depends on the asymptotic distribution of the interpolation points a_{jn} in the scheme $\mathcal{A}$ as $n + m \to \infty$.

Definition 2.1: A probability measure α is called *asymptotic distribution* of the interpolation scheme $\mathcal{A}$ (written $\alpha = \alpha(\mathcal{A})$) if

$$\frac{1}{n+1} \sum_{j=0}^{n} \delta_{a_{jn}} \xrightarrow{\ *\ } \alpha \quad \text{as } n \to \infty, \tag{2.1}$$

where $\xrightarrow{\ *\ }$ denotes *weak convergence* in the space of Borel measures, and δ_z is the Dirac measure at the point z. The *support* of an interpolation scheme $\mathcal{A}$ is defined as

$$\operatorname{supp}(\mathcal{A}) := \text{Closure}\{z \in \mathbb{C} \mid z \in A_n, n \in \mathbb{N}\}. \tag{2.2}$$

We always have $\operatorname{supp}(\alpha) \subseteq \operatorname{supp}(\mathcal{A})$. If $B \subseteq \overline{\mathbb{C}}$ is a Borel measurable set with $\alpha(\partial B) = 0$, then (2.1) implies that

$$\lim_{n\to\infty} \frac{1}{n+1} \operatorname{card}\{j \in \{0, \ldots, n\} \mid a_{jn} \in B\} = \alpha(B). \tag{2.3}$$

In the special case of Padé approximants all interpolation points a_{jn} are identical, say, $a_{jn} = x$, and therefore $\alpha = \delta_x$. Of course, in this case an asymptotic distribution always exists.

In the convergence theory of Padé approximants functions of the form

$$f(z) = \int \frac{d\mu(x)}{x - z} \tag{2.4}$$

with μ a positive measure supported on $\mathbb{R}$ play a prominent role. They are known as Markov, Stieltjes, or Hamburger functions, depending on whether $\operatorname{supp}(\mu)$ is (i) compact, (ii) contained in one of the two halfaxis $\mathbb{R}_-$ or $\mathbb{R}_+$, or (iii) unbounded and intersecting with both sets $\mathbb{R}_-$ and $\mathbb{R}_+$, respectively. Diagonal Padé approximants developed at infinity to functions f of type (2.4) converge locally uniformly in the domain $\overline{\mathbb{C}} \setminus I$ with I the smallest interval containing $\operatorname{supp}(\mu)$. In case of Stieltjes or Hamburger functions f it is necessary in addition that the moment problem associated with the measure μ is determinate (cf. for instance [BaGM], Chapter 5). These results are classical.

Analoguous results for rational interpolants to functions f of type (2.4) have been proved in [Go], [GoLo], [Lo1-4], [LoRa], [StTo], Chapter 6. However, in order to exploit the special structure of the functions (2.4), it is necessary that the interpolation scheme $\mathcal{A}$ is symmetric with respect to $\mathbb{R}$ and that all its points stay away from the intervall I, *i.e.*,

$$\overline{A_n} = A_n \quad \text{and} \quad \text{supp}(\mathcal{A}) \subseteq \overline{\mathbb{C}} \setminus I, \tag{2.5}$$

where the overline $\overline{}$ on A_n denotes the complex conjugation and I is the smallest interval containing $\text{supp}(\mu)$. Condition (2.5) is satisfied if the polynomials w_n, $n = 1, 2, \ldots$, introduced in (1.3) have real coefficients and are different from zero in a neighborhood of I. Condition (2.5) is automatically satisfied if all interpolation points a_{jn} lie in a closed subinterval of $\overline{\mathbb{R}} \setminus I$. In [Go] this case has been studied. Subsequently, his results have been extended to more general interpolation schemes $\mathcal{A}$.

If Q_n is the denominator polynomial of the Padé approximant $[n/n]$ to a function f of type (2.4) developed at infinity, then the polynomial Q_n is orthogonal with respect to the measure μ (cf. [StTo], Lemma 6.3.3). The denominator polynomial Q_n is characterized by this orthogonality up to a constant factor. A similar characterisation of the denominator holds in case of a rational interpolant $r_{nn} = r_{nn}(f, \mathcal{A}; \cdot)$ (cf. [StTo], Lemma 6.1.2), however, now the denominator polynomial Q_n is orthogonal with respect to a weighted orthogonality relation, we have

$$\int x^l Q_n(x) \frac{d\mu(x)}{w_{2n}(x)} = 0 \quad \text{for} \quad l = 0, \ldots, n-1, \tag{2.6}$$

where w_{2n} is the polynomial defined in (1.3). Thus, Q_n is orthogonal with respect to the measure

$$\frac{d\mu(x)}{w_{2n}(x)}, \quad x \in \text{supp}(\mu), \tag{2.7}$$

which is a measure depending on n. The measure (2.7) is real and has no sign-change on $\text{supp}(\mu)$ if the sets A_n and the scheme $\mathcal{A}$ satisfy the assumptions made in (2.5). The measures in (2.7) are known as *varying* or also as *'weighted' measures* (cf. [Lo4], [StTo], Chapter 3.3). In nearly all respects the convergence theory of rational interpolants to functions f of type (2.4) is a direct generalization of that of Padé approximants. The convergence domain is $\overline{\mathbb{C}} \setminus I$ in both cases, and for the interpolation error the asymptotic estimate

$$\limsup_{n\to\infty} |\left(f - [n/n]\right)(z)|^{1/2n} \leq \exp\left(-\int g_{\overline{\mathbb{C}}\setminus I}(z, x) d\alpha(x)\right) \tag{2.8}$$

holds for $z \in \overline{\mathbb{C}} \setminus I$, where $g_{\overline{\mathbb{C}}\setminus I}(z, x)$ is the Green function of the domain $\overline{\mathbb{C}} \setminus I$. Under certain conditions the estimate (2.8) is sharp (cf. [StTo], Chapter 6.1).

Another class of functions f, for which locally uniformly convergence of Padé approximants has been proved, are the Polya frequency functions

$$f(z) = e^{\gamma z} \frac{\prod_j (1 + \alpha_j z)}{\prod_j (1 + \beta_j z)}, \tag{2.9}$$

where $\gamma, \alpha_j, \beta_j \geq 0$ and $\sum_j (\alpha_j + \beta_j) < \infty$. It has been shown in [ArEd] that diagonal Padé approximants to these functions converge locally uniformly in $\overline{\mathbb{C}} \setminus$

$\{\beta_1^{-1}, \ldots, -\alpha_1^{-1}, \ldots\}$. In [BSW] this result has been extended to rational interpolation with interpolation schemes that contain only real interpolation points $a_{jn} \in [-\zeta, \zeta] \subseteq \mathbb{R}$ and the functions f can have only finitely many factors in definition (2.9). The general problem is still open.

From counterexamples involving Padé approximants we know that analyticity of the function f is not sufficient for guaranteeing locally uniform convergence of rational interpolants. In [Wa1] it has been shown that it is possible to construct an entire function f such that the diagonal sequence of Padé approximants $[n/n]$, $n = 1, 2, \ldots$, developed at the origin diverges at each point of $\overline{\mathbb{C}} \setminus \{0\}$. Thus, this counterexample underlines that in the convergence results for the classes of functions (2.4) and (2.9) the special structure of these functions is crucial.

Having the difficulties with locally uniform convergence in mind, it is certainly interesting to realize that convergence can be proved for large classes of functions, which are defined mainly by analyticity properties, if a weaker type of convergence is considered. Especially successful has proved convergence in capacity.

By $\mathrm{cap}(\cdot)$ we denote the (logarithmic) capacity of
(capacitable) subsets of $\mathbb{C}$ (for a definition see [Ts], [La], or [StTo], Appendix I). The notion of capacity zero can be extended to subsets of $\overline{\mathbb{C}}$ by Moebius transforms. For any Borel set $B \subseteq \mathbb{C}$ we have

$$m(B) \leq \pi \mathrm{cap}(B)^2, \tag{2.10}$$

where $m(\cdot)$ denotes the planar Lebesgue measure. This inequality shows that sets that are small in capacity are also small in planar Lebesgue measure.

Definition 2.1: A sequence of functions f_n, $n = 0, 1, 2, \ldots$, is said to *converge in capacity* to f in a domain $D \subseteq \overline{\mathbb{C}}$ if for every $\varepsilon > 0$ and every compact set $V \subseteq D \cap \mathbb{C}$ we have

$$\lim_{n \to \infty} \mathrm{cap}\{z \in V \mid |(f_n - f)(z)| > \varepsilon\} = 0. \tag{2.11}$$

The first result about convergence in capacity and Padé approximation was proved in [Po] after preparations in [Nu]. In [Wa2] the Nuttall-Pommerenke Theorem has been extended to rational interpolants.

Theorem 2.1 ([Wa2], Theorem 4) **:** *Let the function f be analytic (and single-valued) in the domain $\overline{\mathbb{C}} \setminus E$ with E a compact set of* $\mathrm{cap}(E) = 0$*, let $\mathcal{A}$ be an interpolation scheme as in (1.1) with* $\mathrm{supp}(\mathcal{A}) \cap E = \emptyset$*, and let $r_{nn} = r_{nn}(f, \mathcal{A}; \cdot)$, $n = 1, 2, \ldots$, be the (linearized) rational interpolant to the function f in the points of the set A_{2n}. Then for every compact set $V \subseteq \mathbb{C}$ and every $\varepsilon > 0$ we have*

$$\lim_{n \to \infty} \mathrm{cap}\{z \in V \mid |(f - r_{nn})(z)| > \varepsilon^n\} = 0. \tag{2.12}$$

Remarks: (1) From (2.12) it follows that the sequence of rational interpolants r_{nn}, $n = 1, 2, \ldots$, converges in capacity to f in $\mathbb{C}$. But even more, we see that the convergence speed is faster than geometric with possible exceptions on sets that become small in capacity as $n \to \infty$.

(2) Note that it is not necessary to exclude the compact set E with singularities of the function f from the convergence domain since $\text{cap}(E) = 0$.

All meromorphic functions f satisfy the assumptions of Theorem 2.1, but the functions covered by the theorem form a much larger class. For instance, the functions f may have essential singularities as long as there are not too many of them. Of course, any entire function is covered by Theorem 2.1, and consequently we now know that in case of Wallin's counterexample in [Wa1] convergence in capacity holds true, while point-wise convergence does not hold at any point of $\mathbb{C} \setminus \{0\}$. This is no contradiction since the divergence can be caused by a different subsequence at each point. Moreover, in [Mey] it has been shown that convergence in capacity implies point-wise convergence quasi everywhere for an appropriately chosen infinite subsequences. Thus, in Wallin's counterexamples there exist infinite subsequences with point-wise convergence quasi everywhere. In analogy to the notion 'almost everywhere' a property is said to hold 'quasi everywhere' on a set S if it holds for every $z \in S$ with possible exceptions on sets of outer capacity zero.

In [Wa2] Theorem 2.1 has been proved not only for the diagonal sequence $\{r_{nn}\}_{n\in\mathbb{N}}$, but for arbitrary sectorial sequences, *i.e.*, for sequences with an $\lambda > 0$ such that

$$\lambda n \leq m \leq \frac{n}{\lambda} \quad \text{as} \quad m, n \to \infty. \tag{2.13}$$

The assumption $\text{cap}(E) = 0$ is essential for the proof of Theorem 2.1. In [Lu] and [Ra] it has been shown by counterexamples that if the function f has a set of singularity $\tilde{E}$ of positive capacity, then convergence in capacity can no longer be guaranteed for diagonal Padé approximants in any subdomain of $\overline{\mathbb{C}}$.

Inspecting the proof of Theorem 2.1 in [Wa2] shows that the convergence speed faster than geometric plays a key role in the analysis. The fast speed is a consequence of the assumption that f is analytic outside of a compact set E of capacity zero. If the function f has branch points, but its singularities are still contained in a compact set E of capacity zero, for instance if the function f is an algebraic one, then the situation becomes totally different. It is no longer possible that rational interpolants converge (in capacity) throughout $\mathbb{C}$, since rational functions are single-valued and the function f is not. As a consequence convergence faster than geometric is no longer possible. But nevertheless, as the results in the next section will show, convergence in capacity can be proved for such functions. The convergence will no longer hold true throughout the whole complex plane $\mathbb{C}$; instead special subdomains will come up as convergence domains, and they will play a major role in the analysis.

3 Rational Interpolants to Functions with Branch Points

In the present section the convergence in capacity of rational interpolants to functions with branch points is studied. Similar problems have been investigated in [St4], [St5] and [GoRa]. In [St4] the convergence has been proved for rational interpolants if the asymptotic distribution $\alpha = \alpha(\mathcal{A})$ of the interpolation scheme $\mathcal{A}$ is connected in a special way with the equilibrium measures of a condenser. In [GoRa] rational interpolants have been considered in two situations, firstly in a situation related to the rational approximation of the exponential function on $\mathbb{R}_-$ and the

'1/9'-conjecture, and then secondly in a rather general setting. From this last result convergence in capacity can be deduced for rational interpolants if the existence of a symmetric domain corresponding to the asymptotic distribution $\alpha = \alpha(\mathcal{A})$ and the function f is known. Here, we give proofs that are based on the method developed in [St5]. There the convergence of close-to-diagonal sequences of Padé approximants has been investigated. In all methods available so far an a priori knowledge of the existence of a symmetric domain is essential for the proof. The investigations of the section are started by a definition of these domains, given in two steps.

Definition 3.1: A domain $D \subseteq \overline{\mathbb{C}}$ is said to be *symmetric* (or to posses the *symmetry property*) with respect to an asymptotic distribution $\alpha = \alpha(\mathcal{A})$ of an interpolation scheme $\mathcal{A}$ if the following three assertions hold true:
(i) $\operatorname{supp}(\alpha) \subseteq D$.
(ii) The complement $F := \overline{\mathbb{C}} \setminus D$ is of the form

$$F = F_0 \cup \bigcup_{j \in I} J_j \tag{3.1}$$

with $F_0 \subseteq \overline{\mathbb{C}}$ a compact set of $\operatorname{cap}(F_0) = 0$, the J_j, $j \in I$, open, analytic arcs, and $\bigcup_{j \in I} J_j \neq \emptyset$.
(iii) Let $g_D(z, w)$ denote the Green function of the domain D and

$$g(z) = g(\alpha, D; z) := \int g_D(z, x) d\alpha(x) \tag{3.2}$$

the Green potential of the measure α, then

$$\frac{\partial}{\partial n_+} g(z) = \frac{\partial}{\partial n_-} g(z) \quad \text{for all } z \in J_j,\ j \in I, \tag{3.3}$$

with $\partial/\partial n_+$ and $\partial/\partial n_-$ denoting the normal derivatives to both sides of the arcs J_j.

Most important in Definition 3.1 is property (3.3), called the *symmetry property*, which states that the slope of the Green potential $g = g(\alpha, D; \cdot)$ is identical on both sides of the arcs J_j, $j \in I$. Because of the condition $\operatorname{cap}(F_0) = 0$, the union $\bigcup_{j \in I} J_j$ is the dominant part of the complementary set $F = \overline{\mathbb{C}} \setminus D$. Note that the assumed analyticity of the arcs J_j implies that the potential g can be continued harmonicly accross the arcs J_j, and consequently the normal derivatives in (3.3) exist.

In the convergence results of the present section the symmetric domain D depends not only on the asymptotic distribution α of the interpolation scheme $\mathcal{A}$, but also on the function f to be interpolated.

Definition 3.2: Let $\mathcal{A}$ be an interpolation scheme with asymptotic distribution α, let f be a function with all its singularities in a compact set $E \subseteq \overline{\mathbb{C}}$ of $\operatorname{cap}(E) = 0$, and let K be a continuum such that

$$\operatorname{supp}(\mathcal{A}) \subseteq K \subseteq \overline{\mathbb{C}} \setminus E. \tag{3.4}$$

Then a domain $D = D_{\mathcal{A},f} \subseteq \overline{\mathbb{C}}$ is said to be a *symmetric domain* associated with the interpolation scheme $\mathcal{A}$ and the function f if the following three conditions are satisfied:

(i) D is a symmetric domain with respect to the asymptotic distribution $\alpha = \alpha(\mathcal{A})$ in the sense of Definition 3.1.

(ii) There exists a continuum K_0 with

$$\operatorname{supp}(\mathcal{A}) \subseteq K_0 \subseteq D \cap K. \tag{3.5}$$

(iii) The function f has a single-valued meromorphic continuation to D, and on each arc J_j the *jump function*

$$g_j := f_{j+} - f_{j-}, \quad j \in I, \tag{3.6}$$

does not vanish identically. By f_{j+} and f_{j-} we denote the boundary values of f from both sides of the arc J_j.

Remarks: (1) Condition (iii) implies that the function f has branch points, since otherwise all jump functions g_j, $j \in I$, would be identical zero.

(2) The existence of symmetric domains in the sense of Definition 3.2 has been proved in [St1] for two situations; in the first one all interpolation points of the scheme $\mathcal{A}$ have to be identical (the Padé approximation case) and in the second one the asymptotic distribution $\alpha = \alpha(\mathcal{A})$ of the scheme is one of the two measures of the equilibrium distribution of a condenser. This later case is connected with the problem of best rational approximation of the function f on a given continuum V. These results will be discussed in more detail in the next section. Existence proofs for more general situations are very desirable, but the theory in this area is still in a rather dissatisfactory state.

(3) Definition 3.2 could be formulated in a way that demands less analyticity from the function f (cf. for instance Definition 1.3 in [St5] for the case of interpolation schemes with all interpolation points identical).

The central result of the present section is contained in the next theorem. There the function

$$G(z) = G_D(z) := \exp\left(-g(\alpha, D; z)\right) \tag{3.7}$$

is used with $g(\alpha, D; \cdot)$ denoting the Green function introduced in (3.2), and D is a symmetric domain as defined in Definition 3.2.

Theorem 3.1: *Let the function f have all its singularities in a compact set $E \subseteq \overline{\mathbb{C}}$ with $\operatorname{cap}(E) = 0$ and assume that f has branch points such that Theorem 2.1 is not applicable. Let further $\mathcal{A}$ be an interpolation scheme with asymptotic distribution $\alpha = \alpha(\mathcal{A})$, and let $K \subseteq \overline{\mathbb{C}} \setminus E$ be a continuum with $\operatorname{supp}(\mathcal{A}) \subseteq K$. If there exists a symmetric domain $D = D_{\mathcal{A},f} \subseteq \overline{\mathbb{C}}$ associated with $\mathcal{A}$ and f such that*

$$\operatorname{supp}(\mathcal{A}) \subseteq K \subseteq D, \tag{3.8}$$

then for the sequence of rational interpolants $r_n = r_{nn}(f, A_{2n}; \cdot)$, $n = 1, 2, \ldots$, interpolating f at the $2n+1$ points of the set A_{2n} in $\mathcal{A}$ the following limits hold: For any compact set $V \subseteq D \setminus \{\infty\}$ and any $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \operatorname{cap}\{z \in V \mid |(f - r_n)(z)| > (G(z) + \varepsilon)^{2n}\} = 0, \tag{3.9}$$

and for $0 < \varepsilon \leq \inf_{z \in V} G(z)$,

$$\lim_{n\to\infty} \operatorname{cap}\{z \in V \mid |(f - r_n)(z)| < (G(z) - \varepsilon)^{2n}\} = 0. \tag{3.10}$$

Remarks: (1) Since $G(z) < 1$ for all $z \in D$, the limit (3.9) implies that the sequence of interpolants r_n, $n = 1, 2, \ldots$, converge to f in capacity in the domain $D = D_{\mathcal{A},f}$. The function G can be zero only on a set of capacity zero. Hence, it follows from the limit (3.10) that quasi everywhere in D the interpolants r_n converge only with geometric speed. Both limits (3.9) and (3.10) together show that the degree of convergence at the point $z \in D$ is given by $G(z)$ except for subsets of D that are possible under convergence in capacity.

(2) The introduction of the continuum K in (3.8) is necessary in order to make sure that the same branch of the function f is interpolated at all interpolation points.

(3) The existence of a symmetric domain $D_{\mathcal{A},f}$ is a critical assumption for the proof of Theorem 3.1. In a certain sense such an a priori requirement runs contrary to the philosophy of Padé approximation or rational interpolation. If the function f is defined and sufficiently smooth at each interpolation point, then the interpolants $r_n = r_n(f, \mathcal{A}; \cdot)$ can be calculated; a symmetric domain plays no role in this process, and it should come up only as a result of the convergence investigations and not as an assumption. In our analysis the existence of a symmetric domain is needed only because of the method of proof. A more powerfull method, however, would have to cope with additional difficulties, which are not there if the existence of a symmetric domain is assumed and not proved. We will discuss only two of these difficulties:

(i) The first one becomes apparent if, for instance, one tries to achieve the symmetry (3.3) by a variation of the arcs J_j, $j \in I$. Then it cannot be excluded that certain arcs J_j may intersect with $\text{supp}(\mathcal{A})$ or even with $\text{supp}(\alpha)$, and thereby topological properties along the arcs J_j would be destroyed that are necessary in the proof of Theorem 3.1.

(ii) It cannot be excluded that a variation of the arcs J_j may lead to a convergence set D, which is no longer a domain. Indeed, D may turn out to be a patch work of open sets separated by analytic curves and arcs, and the interpolants r_n will converge in capacity to different branches of the function f in different components of the set D. The assumption of the existence of a symmetric domain excludes all these difficulties, which will be possible in the general case.

Before we come to the proof of Theorem 3.1 we discuss two concrete examples. In both cases the same function f is interpolated, but with different interpolation schemes $\mathcal{A}_1$ and $\mathcal{A}_2$ in each case.

Example 3.1: Let the function f be defined as

$$f(z) := \sqrt{1 - \frac{2}{z^2} + \frac{9}{z^4}} \tag{3.11}$$

with a positive square root at infinity. The function has the 4 branch points $z_{1,\ldots,4} = \pm\exp(\pm i\pi/6)$, and $E = \{z_1, \ldots, z_4, 0\}$ is the set of singularities. At the origin the function f has a double pole. The function f will be interpolated in points $a_{jn} \in I := [-\infty, 1] \cup [1, \infty]$, $j = 0, \ldots, n$, $n = 1, 2, \ldots$. The set I is considered as an interval on the Riemann sphere $\overline{\mathbb{C}}$.

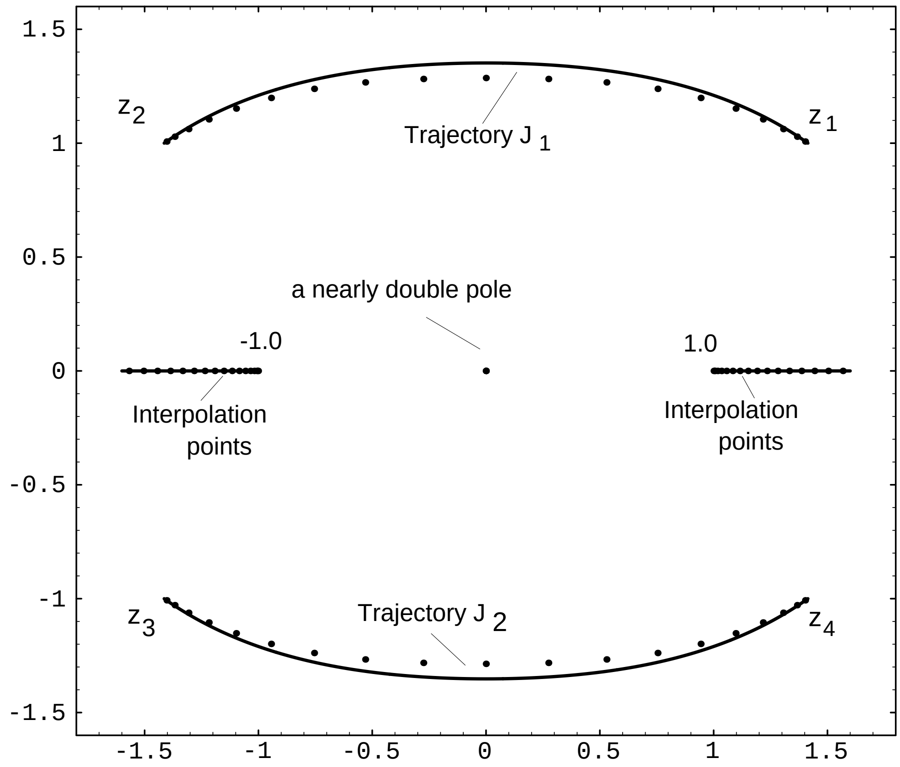

Figure 3.1: 40 poles of the rational interpolant $r_{40,40}(f, A_{80}; \cdot)$ together with the two arcs J_1, J_2 and parts of the interpolation points in A_{80}.

The interpolation scheme formed by the points a_{jn} is denoted by $\mathcal{A}_1$, and the interpolation points a_{jn} will be chosen in such a way that the interpolation scheme $\mathcal{A}_1$ has the asymptotic distribution $\alpha = \alpha(\mathcal{A}_1)$ defined by

$$d\alpha(x) := \frac{cxdx}{\sqrt{x^2-1}\sqrt{x^4-2x^2+9}} = \frac{cxdx}{\sqrt{(x^2-1)(x-z_1)\cdots(x-z_4)}}, \quad x \in I, \quad (3.12)$$

where the square root in (3.12) is chosen such that the measure is positive, and the constant $c = 0.45354\ldots$ has been chosen in such a way that $||\alpha|| = 1$. For a given $n \in \mathbb{N}$ the interpolation points a_{jn}, $j = 0, \ldots, n$, are defined by

$$\begin{aligned} \alpha([1, a_{jn}]) &= \frac{2j+1}{2n+2} \quad \text{for} \quad j = 0, \ldots, [n/2], \\ \alpha([-\infty, a_{jn}]) &= \frac{2j-n}{2n+2} \quad \text{for} \quad j = [n/2]+1, \ldots, n. \end{aligned} \quad (3.13)$$

The symmetric domain $D = D_{\mathcal{A}_1,f}$ associated with the scheme $\mathcal{A}_1$ and the function f in the sense of Definition 3.2 consists of the Riemann sphere $\overline{\mathbb{C}}$ minus two arcs J_1 and J_2, which connect the pairs of points $\{z_1, z_2\}$ and $\{z_3, z_4\}$, respectively. The two arcs J_1 and J_2 are trajectories of a quadratic differential, we have

$$\frac{z^2}{(z^2-1)(z-z_1)\cdots(z-z_4)}dz^2 \leq 0. \tag{3.14}$$

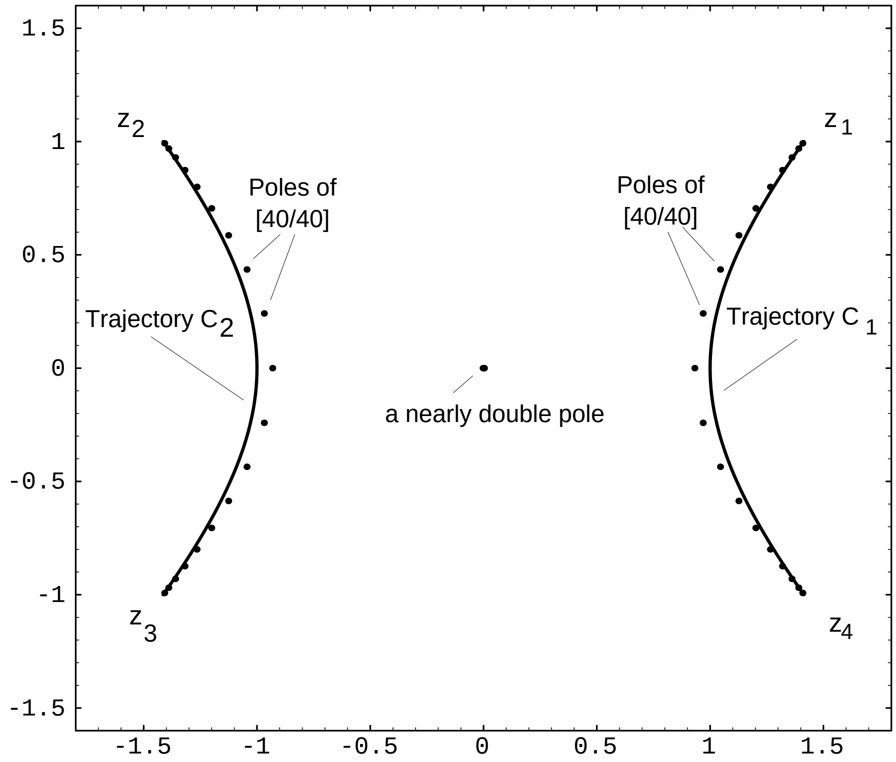

Figure 3.2: 40 poles of the Padé approximant [40/40] to the function (3.11) developed at infinity together with the two arcs C_1 and C_2 that form the complement of the convergence domain.

In Figure 3.1 the arcs J_1 and J_2 are shown together with the 40 poles of the rational interpolant $r_{40,40}(f, \mathcal{A}_1; \cdot) = r_{40,40}(f, A_{80}; \cdot)$ and parts of the 81 interpolation points of the set $A_{80} = \{a_{0,80}, \ldots, a_{80,80}\} \subseteq I$. Of the 40 poles of $r_{40,40}$ two subsets of 19 elements each cluster to the two arcs J_1 and J_2 and the two remaining poles form a nearly double pole close to the origin. These two poles approximate the double pole of f at the origin. From Theorem 3.1 we know that the sequence $r_{nn}(f, \mathcal{A}_1; \cdot)$, $n = 1, 2, \ldots$, converges to f in capacity in the domain $D = \overline{\mathbb{C}} \setminus (J_1 \cup J_2)$. The convergence of the rational interpolants $r_{nn}(f, \mathcal{A}_1; \cdot)$, $n = 1, 2, \ldots$, is fastest near the interval I and becomes slower the nearer one comes to the two arcs J_1 and J_2.

Example 3.2: The function f is again defined by (3.11), but now the function will be interpolated only at infinity. Thus, an interpolation scheme $\mathcal{A}_2$ is used that is defined by

$$a_{jn} = \infty \quad \text{for all} \;\; j = 0, \ldots, n, \;\; n = 0, 1, 2, \ldots \tag{3.15}$$

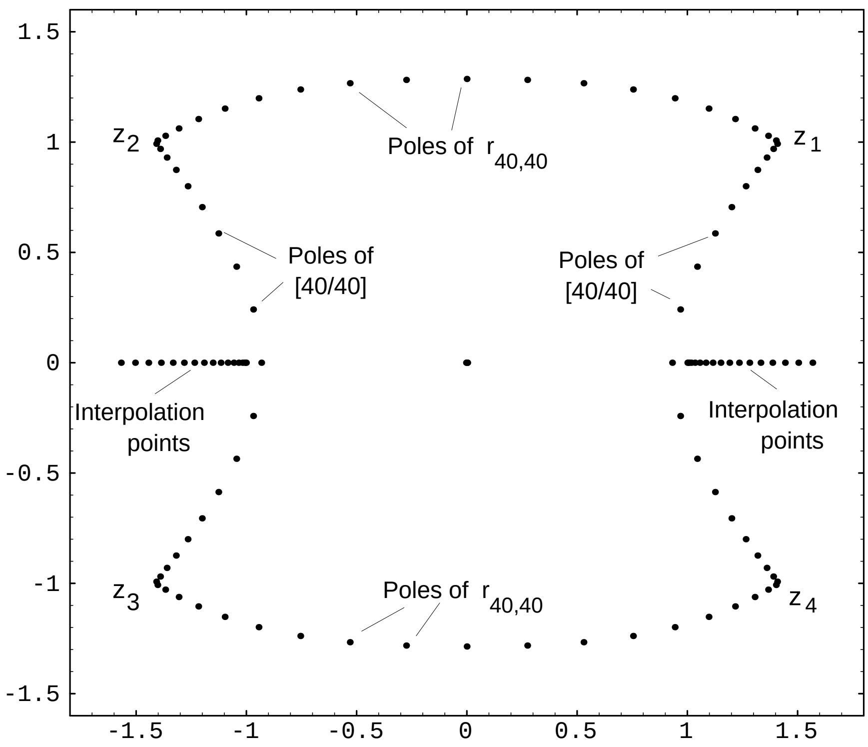

Figure 3.3: 40 poles of the Padé approximant [40/40] together with 40 poles of the rational interpolant $r_{40,40}(f, \mathcal{A}_1, \cdot)$ and two subsets of the 81 interpolation points used for the interpolant $r_{40,40}(f, \mathcal{A}_1; \cdot)$.

The rational interpolants $r_{nn}(f, \mathcal{A}_2; \cdot)$ are the Padé approximants $[n/n]$ developed at infinity. Again, the symmetric domain in the sense of Definition 3.2 consists of the Riemann sphere $\overline{\mathbb{C}}$ minus two arcs C_1 and C_2, but now the two arcs connect the pairs of branch points $\{z_4, z_1\}$ and $\{z_2, z_3\}$, respectively. The two arcs are again trajectories of a quadratic differential, they satisfy

$$\frac{z^2}{z^4 - 2z^2 + 9} dz^2 \leq 0. \tag{3.16}$$

The arcs C_1 and C_2 can be characterized by a principle of minimal capacity (cf. [St5], Theorem 1.3). In Figure 3.2 the arcs C_1 and C_2 are shown together with the 40 poles of the Padé approximant [40/40]. Again, 19 of these poles cluster at each of the two arcs C_1 and C_2, and two poles form a nearly double pole close to the origin.

Note that the different choices of interpolation points in the schemes $\mathcal{A}_1$ and $\mathcal{A}_2$ result in rather different shapes of the convergence domains. This in turn results in different convergence behaviors, which can be seen best in a neighborhood of the origin, where different branches of the function f are approximated by the sequences $\{r_{nn}(f, \mathcal{A}_1; \cdot)\}$ and $\{[n/n]\}$. In Figure 3.3 all 40 poles of the Padé approximant [40/40] and the 40 poles of the rational interpolant $r_{40,40}(f, A_1; \cdot)$ are plotted together with two subsets of the 81 interpolation points used for the interpolant $r_{40,40}(f, A_1; \cdot)$. The two sets of poles mark the boundary of the area on which different branches of the function f are approximated by the two sequences $\{r_{nn}(f, \mathcal{A}_1; \cdot)\}$ and $\{[n/n]\}$.

Proof of Theorem 3.1: Important aspects of the proof are practically copies of the proof of the Theorems 1.2 and 1.7 in [St5]. The complete proof is rather complex, and compared with this complexity the changes necessary for Theorem 3.1 are only minor. In any case, a complete reproduction of the full proof would be too long for the present paper. Therefore we will only discuss the main changes and shall not try to give a discription that can be understood independently of [St5].

A core piece of the method used in [St5] is to show that the limit

$$\frac{1}{n}\nu_{Q_n} \xrightarrow{\ *\ } \omega = \omega_{D,\alpha} \quad \text{as} \quad n \to \infty \tag{3.17}$$

holds true, where Q_n is the denominator polynomial of the (linearized) rational interpolant $r_n = r_{nn}(f, \mathcal{A}; \cdot)$, ν_{Q_n} the counting measure that places unit mass at each zero of Q_n (taking account of multiplicities), $\xrightarrow{\ *\ }$ denotes weak convergence in the space of measures, and $\omega_{D,\alpha}$ is the measure defined by

$$\omega_{D,\alpha} := \int \omega_{D,x} d\alpha(x) \tag{3.18}$$

with $\omega_{D,x}$ the harmonic measure on ∂D representing the point $x \in D$. As in Theorem 3.1 D denotes the symmetric domain $D_{\mathcal{A},f}$ and $\alpha = \alpha(\mathcal{A})$ the asymptotic distribution of the interpolation scheme $\mathcal{A}$.

The measure $\omega_{D,\alpha}$ appeared already in the Green potential $g(\alpha, D; \cdot)$ introduced in (3.2). The potential can be represented as

$$\int g_D(z,x)d\alpha(x) = \int \log \frac{1}{|z-x|} d(\alpha - \omega_{D,\alpha})(x) + \int g_D(x,\infty)d\alpha(x). \tag{3.19}$$

As in [St4] and [St5] we have to use a definition of logarithmic potentials, which takes special care of masses near infinity. The potentials are defined by

$$p(\mu; z) := \int \log frac1|H(z,x)|d\mu(x) \tag{3.20}$$

with a normalized linear factor

$$H(z,x) := \begin{cases} z - x & \text{for } x \in \overline{\mathbb{D}} \\ (z-x)/|x| & \text{for } x \in \mathbb{C} \setminus \overline{\mathbb{D}} \\ 1 & \text{for } x = \infty. \end{cases} \tag{3.21}$$

The use of the normalization implied by (3.21) is necessary since there may be sequences of potentials $p(\mu_1; \cdot), p(\mu_2; \cdot), \ldots$ with measures that contain masses such that $\mu_1, \mu_2, \ldots$ tend to infinity in a rather uncontrolled way; the sequence of measures in (3.17) is an example for such a situation. In Section 2 of [St5] the potential-theoretic consequences of definition (3.20) are studied in a sequence of lemmas.

In the proofs of the Theorems 1.2 and 1.7 in [St5] two Lemmas 3.1 and 3.2 play a fundamental role. There appears a logarithmic potential $p(\mu + \nu_1 + \nu_2; \cdot)$ with ν_1, ν_2, and μ measures with certain properties. In the new setting this potential has to be replaced by a potential of the form $p(\mu + \nu_1 + \nu_2 - 2\alpha; \cdot)$, *i.e.*, the role of infinity is now played by the measure α. In order to avoid complications one should assume $\infty \in D \setminus \mathrm{supp}(\mathcal{A})$, which always can be achieved, without loss of generality, by transforming the whole problem by a Moebius transform.

An important ingredient in the proof of the Lemmas 3.1 and 3.2 of [St5] and also a major tool in the proof of Theorem 1.8 of [St5] is the reflection function Φ introduced in (2.29) of [St5]. It is an anti-analytic, conformal mapping of neighborhoods of the arcs J_j, $j \in I$. The arcs J_j are invariant under this map. The existence of the map is a consequence of the symmetry property (3.3), only that in the new setting this property holds with respect to the Green potential $g(\alpha, D; \cdot)$, and not with respect to the Green function $g_D(z, \infty)$, as is the case in [St5].

The form of the identities (4.2) and (4.3) in Lemma 4.1 of [St5] reflects interpolation at infinity. In case of a general interpolation scheme $\mathcal{A}$ the identity (4.2) of [St5] has to be replaced by

$$\oint_C \zeta^k Q_n(\zeta) \frac{f(\zeta) d\zeta}{w_{2n}(\zeta)} = 0 \quad \text{for} \quad k = 0, \ldots, n-1, \tag{3.22}$$

where C is an integration path separating $\mathrm{supp}(\mathcal{A})$ from $F = \overline{\mathbb{C}} \setminus D$, w_{2n} is the polynomial introduced in (1.3), and Q_n is the denominator polynomial of the rational interpolant $r_n = r_{nn}(f, \mathcal{A}; \cdot)$. The formula (4.3) of [St5] for the interpolation error has to be replaced by

$$(f - r_n)(z) = \frac{1}{2\pi i} \frac{1}{(Q_n P)(z)} \oint_C \frac{(Q_n P f)(\zeta)}{w_{2n}(\zeta)} \frac{d\zeta}{\zeta - z}, \quad P \in \mathcal{P}_n \setminus \{0\}. \tag{3.23}$$

Note that the orthogonality relation (3.22) is the analogue of relation (2.5), which holds in case of rational interpolation of Markov-, Stieltjes-, or Hamburger functions. As a consequence of (3.22) the integral (4.39) in [St5], which is of central interest in the proof of Theorem 1.7 in [St5], has to be replaced by

$$\oint_C (P_n Q_n f)(\zeta) \frac{d\zeta w_{2n}(\zeta)}{\cdot} \tag{3.24}$$

There are more technical details that have to be changed, however all these changes follow rather immediately if one follows the logic that has governed the replacement of the formulas (4.2) and (4.3) in [St5] by the formulas (3.22) and (3.23). The reflection on the arcs J_j, $j \in I$, by the function Φ has to be done in exactly the same way as before. ∎

4 The Existence of Symmetric Domains

We have seen in the last section that the existence of a symmetric domain is a necessary condition for the validity of the proof of Theorem 3.1. Existence theorems for symmetric domains are known in two situations. These cases are reviewed in the Theorems 4.1 and 4.4, below. The section is closed by a discussion of the difficulties that arise in a proof of a more general existence result.

Let the special interpolation scheme with all interpolation points equal to a fixed point $z \in \overline{\mathbb{C}}$ be denoted by $\mathcal{A}_z$. In this case an asymptotic distribution always exist, and we have $\alpha(\mathcal{A}_z) = \delta_z$. The rational interpolants defined by such a scheme are the Padé approximants $[n/n]$ developed at the point z.

Theorem 4.1: *Let the function f have all its singularities in a compact set $E \subseteq \overline{\mathbb{C}}$ with $\operatorname{cap}(E) = 0$, and among the singularities there should be branch points. Let further $\mathcal{A} = \mathcal{A}_z$ with $z \in \overline{\mathbb{C}} \setminus E$. Then there exists a symmetric domain $D = D_z := D_{f,\mathcal{A}_z}$ in the sense of Definition 3.2. The domain is unique up to a set of capacity zero.*

Remarks: (1) The assumption that the function f has to have branch points implies that $\operatorname{cap}(\overline{\mathbb{C}} \setminus D_z) > 0$.

(2) It is immediate that a Moebius transform maps a symmetric domain again into a symmetric domain. Hence, without loss of generality we can assume in all proofs that $z = \infty$ in Theorem 4.1.

Theorem 4.1 is an immediate consequence of the Theorems 1 and 2 in [St1] and the Corollary to Theorem 1 in [St2] in case of the special interpolation scheme $\mathcal{A}_\infty$. From remark 2 we then know that the theorem holds in general. The next theorem follows also from the Theorems 1 and 2 in [St1].

Theorem 4.2: *The symmetric domain $D = D_\infty$ of Theorem 4.1 is uniquely determined up to a set of capacity zero by the following two conditions:*

(i) $\infty \in D$, and the function f has a single-valued meromorphic continuation in D.

(ii)$\operatorname{cap}(\overline{\mathbb{C}} \setminus D) = \inf_{\tilde{D}} \operatorname{cap}(\overline{\mathbb{C}} \setminus \tilde{D})$, where the infimum extends over all domains $\tilde{D} \subseteq \overline{\mathbb{C}}$ that satisfy condition (i).

We note that in [St1] and [St2] domains of single-valued analytic continuation were considered with respect to analyticity and not with respect to meromorphy as in the Definitions 3.1 and 3.2. However, the difference consists only of a denumerable set of isolated points, and therefore this set is of capacity zero and can be neglected. Further, we note that in [St1] and [St2] the domain D exists uniquely, which is the consequence of a third condition in Theorem 1 of [St1] that has not been applied in Theorem 4.2.

The second type of interpolation schemes $\mathcal{A}$, for which the existence of symmetric domains has been proved, is connected with the equilibrium distributions of condensers (V, F) and the *condenser capacity* $\operatorname{cap}(V, F)$ (for a definition of the condenser capacity see [Ba]). In the Theorems 1 and 2 of [St1] the following result has been proved:

Theorem 4.3: *Let the function f have all its singularities in a compact set $E \subseteq \overline{\mathbb{C}}$ with $\operatorname{cap}(E) = 0$, and among the singularities there should be branch points. Let further $V \subseteq \overline{\mathbb{C}} \setminus E$ be a continuum. Then there exists uniquely up to a set of capacity zero a domain $D = D_{f,V} \subseteq \overline{\mathbb{C}}$ such that*
(i) $V \subseteq D$, and f has a single-valued meromorphic continuation in D.
(ii)$\operatorname{cap}(V, \overline{\mathbb{C}} \setminus D) = \inf_{\tilde{D}} \operatorname{cap}(V, \overline{\mathbb{C}} \setminus \tilde{D})$, where the infimum extends over all domains $\tilde{D} \subseteq \overline{\mathbb{C}}$ that satisfy condition (i).

In the sequel the complement of the domain D will be denoted by F, *i.e.*,

$$F := \overline{\mathbb{C}} \setminus D. \tag{4.1}$$

We will discuss some notions connected with the condenser (V, F). Since the function f is assumed to have branch points, we have $\operatorname{cap}(F) = \operatorname{cap}(\overline{\mathbb{C}} \setminus D) > 0$. For any continuum V we have $\operatorname{cap}(V) > 0$. Consequently, there exists a condenser potential p_{VF}, which is defined by the following four properties: (i) In the domain $R := D \setminus V$ the potential p_{VF} is harmonic, (ii) it is lower semicontinuous in a neighborhood of V and upper semicontinuous in a neighborhood of F, (iii) we have $p_{VF}(z) = 0$ for quasi every $z \in F$ and $p_{VF}(z) = c_{VF}$ for all $z \in V$, where c_{VF} is a positive constant, and (iv) we have

$$\frac{1}{2\pi} \oint_C \frac{\partial}{\partial n} p_{VF}(\zeta) ds_\zeta = 1, \tag{4.2}$$

where C is a smooth integration path in the domain R separating the two sets V and F, $\partial/\partial n$ and ds are the normal derivative and the line element on C, respectively. The condenser capacity then is defined as

$$\operatorname{cap}(V, F) := \frac{1}{c_{VF}} \tag{4.3}$$

(cf. [Ba]). There exists a probability measure ω_{VF} on V such that

$$p_{VF}(z) = g(\omega_{VF}, D; z) = \int g_D(z, x) d\omega_{VF}(x). \tag{4.4}$$

The pair of probability measures (ω_{VF}, μ_{VF}) with μ_{VF} defined by

$$\mu_{VF} := \int \omega_{D,x} d\omega_{VF}(x) \tag{4.5}$$

is called the *equilibrium distribution* of the condenser (V, F). In (4.5) $\omega_{D,x}$ denotes the harmonic measure on F representing the point $x \in D$; consequently μ_{VF} is a probability measure on F.

The measure $\omega_{V,F}$ is fundamental for the second situation, in which we have an existence proof for symmetric domains. For each $n \in \mathbb{N}$ we can select $n + 1$ interpolation points $a_{jn} \in F$, $j = 0, \ldots, n$, such that

$$\frac{1}{n+1} \sum_{j=0}^{n} \delta_{a_{jn}} \xrightarrow{\ *\ } \omega_{VF} \quad \text{as } n \to \infty. \tag{4.6}$$

The triangular matrix $(a_{jn})_{j=0,\ldots,n,\ n=1,2,\ldots}$ forms an interpolation scheme, which we denote by $\mathcal{A} = \mathcal{A}_{f,V}$. By construction $A_{f,V}$ has ω_{VF} as asymptotic distribution, *i.e.*, $\alpha(\mathcal{A}_{f,V}) = \omega_{VF}$.

From the Theorems 1 and 2 of [St1] together with the Corollary to Theorem 1 of [St2] the following theorem follows.

Theorem 4.4: *Let the function f and the continuum V satisfy the assumptions of Theorem 4.3. Then there exists a symmetric domain $D = D_{f,\mathcal{A}}$ in the sense of Definition 3.2 that is associated with the function f and the interpolation scheme $\mathcal{A} = \mathcal{A}_{f,V}$, and the domain is identical to the domain $D_{f,V}$ in Theorem 4.3 up to a set of capacity zero.*

The Theorems 4.1 and 4.4 are so far the only results about the existence of symmetric domains. The existence of symmetric domains in a more general setting is still an open problem. The present section will be closed by an example, which allows to discuss and illustrate some of the difficulties that have to be dealt with in a more general existence proof.

Example 4.1: Let the function f be defined as

$$f(z) := \frac{1}{\sqrt{z^2 - 1}}, \tag{4.7}$$

and let $y > 1$. An interpolation scheme $\mathcal{A}_1$ with only two different interpolation points is defined by

$$a_{jn} := i(-1)^j y, \quad j = 0, \dots, n, \ n = 1, 2, \dots \tag{4.8}$$

If a branch of the function f is interpolated that is analytic on the continuum (in $\overline{\mathbb{C}}$) $K = [iy, i\infty] \cup [-i\infty, -iy]$, then by symmetry consideration it is rather immediate that the symmetry domain in the sense of Definition 3.2 is given by

$$D = D_{f,\mathcal{A}_1} := \overline{\mathbb{C}} \setminus [-1/1]. \tag{4.9}$$

If we move the two points iy and $-iy$ closer to the origin, say $0 < y < 1$, then the idea of minimal capacity, used in Theorem 4.2 for the characterization of symmetric domains with interpolation at infinity, could suggest that in the new situation the symmetric domain is equal to

$$D = \overline{\mathbb{C}} \setminus ([1, \infty] \cup [-\infty, -1]) . \tag{4.10}$$

but this is not the case. Actually, both domains (4.9) and (4.10) are symmetric in the sense of Definition 3.1, and the set $[1, \infty] \cup [-\infty, -1]$ lies even further away from iy and $-iy$ than $[-1, 1]$. But since it has been assumed that a branch of f is interpolated that is analytic on the continuum $K = [iy, i\infty] \cup [-i\infty, -iy]$, from the two domains (4.9) and (4.10) only the domain (4.9) is symmetric in the sense of Definition 3.2.

Next, we consider an interpolation scheme $\mathcal{A}_2$ with four different interpolation points. Let two real numbers y_1 and y_2 be given with $y_1 > 1$ large and $0 < y_2 < 1$, let further $N \in \mathbb{N}$ be large, odd, and define the interpolation points of $\mathcal{A}_2$ by

$$a_{jn} := \begin{cases} i(-1)^j y_1 & \text{if} \quad j \not\equiv 0 \bmod(N) \\ i(-1)^j y_2 & \text{if} \quad j \equiv 0 \bmod(N), \end{cases} \tag{4.11}$$

$j = 0, \dots, n$, $n = 0, 1, 2, \dots$. If a branch of f is interpolated that is analytic on $K = [iy_2, i\infty] \cup [-i\infty, -iy_2]$ considered as a continuum in $\overline{\mathbb{C}}$, then again (4.9) is the symmetric domain in the sense of Definition 3.2 associated with f and the interpolation scheme $\mathcal{A}_2$.

If, however, we interpolate a branch of the function f that is analytic on the continuum $K = [iy_1, i\infty] \cup [-i\infty, iy_2]$, then the situation becomes more complicated: If $N \in \mathbb{N}$ is sufficiently large, then there exists no symmetric domain in the sense of Definition 3.2, but symmetric open sets that are defined analoguously. Their union is of the form $\overline{\mathbb{C}} \setminus (C_1 \cup C_2)$, where C_1 is an arc connecting the two points -1 and 1, and separating the point iy_2 from the origin, and C_2 is a closed curve surrounding the point iy_2. In the two domains $\mathrm{Int}(C_2)$ and $\overline{\mathbb{C}} \setminus (C_1 \cup C_2 \cup \mathrm{Int}(C_2))$ different branches of the function f are approximated by the interpolants.

If on the other hand $y_2 > 0$ is small and $N \in \mathbb{N}$ is not too large, then there exists a symmetric domain in the sense of Definition 3.2 associated with f and $\mathcal{A}_2$. The domain is of the form $\overline{\mathbb{C}} \setminus C$ where C is an arc connecting the two points -1 and 1 and the arc intersects the y-axis between iy_2 and iy_1.

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