Finsler Connections in Generalized Lagrange Spaces

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Abstract

The Chern–Rund connection from Finsler geometry is settled in the generalized Lagrange spaces. For the geometry of these spaces, we refer to [5].

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Introduction

In a recent paper, [1], we showed that in a Finsler space the connection introduced by S.S. Chern in 1948 is the same with the connection proposed by H. Rund ten years later and bearing his name. Accordingly, we proposed the name of Rund be replaced with that of Chern, but several geometers including S.S. Chern himself, suggested to call it from now on a Chern–Rund connection.

As S.S. Chern and D. Bao showed in [2], the Chern–Rund connection is very convenient in treating of many global problems in Finsler geometry. This fact determined us to come back to the subject.

The efforts made in defining a covariant derivative and accordingly, a parallel displacement in *Finsler space* led to a concept generically called a Finsler *connection.* Among the Finsler connections there exist four, which are remarkable by their properties named the *Cartan, Berwald, Chern–Rund and Hashiguchi connections*, respectively. These are usually put together in a nice commutative diagram (cf. [3, Ch. III]).

The most utilized is the Cartan connection because it is fully metrical i.e. $h-$ and v–metrical, in spite of the fact it has torsion.

But there are some problems involving the Berwald connection which is by no means metrical or the Hashiguchi connection which is only v–metrical.

The Chern–Rund connection being h–metrical and free of torsion is the nearest to the Levi–Civita connection a fact which explains its adequacy for global problems in Finsler geometry.

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The Finsler connections are also suitable for the geometries more general than the Finslerian one as the Lagrange geometry or generalized Lagrange geometry. Our purpose is to review Finsler connections and to settle the Chern–Rund connection in this more general framework.

First, we give in §1 a definition of Finsler connection by local components and introduce its compatibility with a generalized Lagrange metric. Then, in §2, a Finsler connection is defined as a pair (N, ∇), where N is a nonlinear connection on TM and ∇ is a linear connection in the pull–back bundle $\tau^{-1}: M \longrightarrow TM$ with $\tau : TM \longrightarrow M$, the tangent bundle over a manifold M. These definitions are equivalent. The four remarkable connections mentioned above are characterized. A special attention is paid to the possibility of determining N from ∇.

1 Finsler connections. A definition by local components

Let M be a smooth i.e. C^∞ manifold of finite dimension n and $\tau : TM \to M$ its tangent bundle. A local chart $(U, (x^i))$ on M induces a local chart $(\tau^{-1}(U), (x^i, y^i))$ on TM, where $x^i \equiv x^i \circ \tau$ and (y^i) are provided by $u = y^i \frac{\partial}{\partial x^i}\big|_p$, $p = \tau(u)$.

A change of coordinates $(x^i, y^i) \longrightarrow (\tilde{x}^i, \tilde{y}^i)$ on TM has the form

$$\begin{aligned} &\tilde{x}^i = \tilde{x}^i(x^1, ..., x^n), \ \text{rank}\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) = n \\ &\tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j}(x) y^j. \end{aligned} \tag{1.1}$$

The indices $i, j, k, ...$ will run from 1 to n and Einstein's convention on summation is implied.

Let $L : TM \longrightarrow R$ be a scalar function on TM. Then $\widetilde{L}(\tilde{x}(x), \tilde{y}(y)) = L(x, y)$, from which, taking partial derivatives and using (1.1), one gets

$$\frac{\partial L}{\partial y^i} = \frac{\partial \tilde{x}^k}{\partial x^i} \frac{\partial L}{\partial \tilde{y}^k}, \tag{1.2}$$

$$\frac{\partial L}{\partial x^i} = \frac{\partial \tilde{x}^k}{\partial x^i} \frac{\partial L}{\partial \tilde{x}^k} + \frac{\partial^2 \tilde{x}^k}{\partial x^j \partial x^i} y^j \frac{\partial L}{\partial \tilde{y}^k}. \tag{1.3}$$

According to (1.2), the set of functions $\left(\frac{\partial L}{\partial y^i}(x, y)\right)$ may be regarded as the components of a covector field on TM. From (1.2), it follows that $\left(\frac{\partial^2 L}{\partial y^i \partial y^j}(x, y)\right)$ may be also viewed as the components of a (symmetric) tensor field on TM.

Thus on TM there exist geometrical objects whose law of transformation under (1.1) is the same as of the corresponding objects on M. These were called d–objects (d is from distinguished) in [4], Finsler objects in [3] and sometimes M–objects.

The geometry of d–objects is essentially involved in the study of those metrical structures which are more general than Riemannian structures i.e. Finsler structures, Lagrange structures, generalized Lagrange structures (see [4]).

Coming back to (1.3), we see that the behaviour of the operators $\dfrac{\partial}{\partial x^i}$ is drastically different from that of $\dfrac{\partial}{\partial y^i}\cdot$ Let us introduce a correction of $\dfrac{\partial}{\partial x^i} := \partial_i$,

$$\delta_i L = \partial_i L + N_i^k(x,y)\dot{\partial}_k,\ \dot{\partial}_k := \frac{\partial}{\partial y^k}, \tag{1.4}$$

such that, with respect to (1.1):

$$\delta_i L = \frac{\partial \tilde{x}^k}{\partial x^i}\,\tilde{\delta}_k L, \tag{1.5}$$

i.e. $(\delta_i L)$ to appear as the components of a covector field on TM. Then the functions $(N_i^k(x,y))$ have to satisfy

$$\frac{\partial \tilde{x}^j}{\partial x^i}\,\tilde{N}_j^h = N_i^j\,\frac{\partial \tilde{x}^h}{\partial x^j} + \frac{\partial^2 \tilde{x}^h}{\partial x^i \partial x^j}\,y^j. \tag{1.6}$$

Note that $(N_i^j(x,y))$ are not the components of a $(1,1)$–tensor field on TM but the difference of two sets of this type is so.

As it is well–known, when M is paracompact, there exists on M a linear connection, say of local coefficients $(\Gamma^i_{jk}(x))$. Then $N^i_k(x,y) = \Gamma^i_{jk}(x)y^j$ verify (1.6). This example assures also the existence of a nonlinear connection within a generally accepted hypothesis on M.

The local vector fields (δ_i), $i = 1,2,...,n$, given by (1.4) are linearly independent and in a point $u \in TM$ they span an n–dimensional subspace $H_u TM$ of $T_u TM$.

Let $\tau_{*,u}$ be the tangent mapping (the Jacobian) of τ. Then $V_u TM = \ker \tau_{*,u}$ is called the vertical subspace of $T_u TM$. A vertical vector is of the form $X^k(x,y)\dot{\partial}_k$ such that under (1.1) one has

$$\widetilde{X}^k = \frac{\partial x^k}{\partial x^i}\,X^i. \tag{1.7}$$

We immediately have

$$T_u TM = V_u TM \oplus H_u TM. \tag{1.8}$$

Furthermore, $\tau_{*,u}$ restricted to $H_u TM$ gives an isomorphism of it with $T_{\tau(u)}M$ such that $\tau_{*,u}(\delta_i) = \partial_i\big|_{\tau(u)}$.

Conversely, if a supplement of $V_u TM$ in $T_u TM$ is specified by a basis (δ_i), $i = 1,2,...,n$, which is carried by τ_* to (∂_i), then letting $\delta_i = \partial_i - N_i^k\dot{\partial}_i$, the condition $\delta_i = \dfrac{\partial \tilde{x}^k}{\partial x^i}\widetilde{\delta}_k$ implies (1.6) for (N^k_i). One says that $(N^k_i(x,y))$ are the *coefficients* of a nonlinear connection.

A reason for this term is that when (N^k_i) are linear with respect to (y) i.e. $N^k_i(x,y) = G^k_{ji}(x)y^j$, then (G^k_{ji}) are the coefficients of a linear connection on M.

Summarizing the foregoing discussion we may formulate the following two equivalent definitions for a nonlinear connections.

Definition 1.1. A nonlinear connection is a set of functions $(N^i_j(x,y))$ defined on each domain of local chart on TM such that an overlaps, (1.6) holds good.

Definition 1.2. A nonlinear connection is a smooth distribution $u \longrightarrow H_uTM$ supplementary to the vertical distribution $u \longrightarrow V_uTM$ i.e. (1.7) holds good for every $u \in TM$.

Let $(v^i(x,y))$ be the components of a d–vector field. Then $\left(\dfrac{\partial v^i}{\partial y^j}(x,y)\right)$ are the components of a d–tensor field of type $(1,1)$. In other words the partial derivatives with respect to (y^i) are covariant. However, in some circumstances, these have to be replaced by

$$v^i{}_{|j} = \frac{\partial v^i}{\partial y^j} + C^i_{kj}(x,y)v^k, \tag{1.9}$$

where $(C^i_{kj}(x,y))$ are the components of a d–tensor field. One of them is as follows.

First, we introduce

Definition 1.3. A d–tensor field of type $(0,2)$ of components $(g_{ij}(x,y))$ which is *a*) symmetric, i.e. $g_{ij} = g_{ji}$, *b*) nondegenerate i.e. $\det(g_{ij}(x,y)) \neq 0$ and *c*) the quadratic form $g_{ij}(x,y)\xi^i\xi^j$ $(\xi \in I\!R^n)$ has constant signature is called *a generalized Lagrange metric (GL–metric for brevity).*

Extending (1.9), the covariant derivative of (g_{ij}) is given by

$$g_{ij}\Big|_k = \partial_j v^i - C^h_{ik}g_{hj} - C^h_{jk}g_{ih}. \tag{1.10}$$

One says that the GL–metric $(gij(x,y))$ is *v–covariant constant* if $g_{ij}\Big|_k = 0$. For the general $v^i\Big|_j$, the condition $g_{ij}\Big|_k = 0$ can be fulfilled with

$$\overset{c}{C}{}^h_{ij} = \frac{1}{2}g^{hk}(\dot{\partial}_i g_{kj} + \dot{\partial}_j g_{ik} - \dot{\partial}_k g_{ij}). \tag{1.11}$$

The partial derivatives with respect to (x^i) are far to be covariant derivatives. A correction of them could be $\partial_j v^i + H^i_{kj}(x,y)v^k$, but $(H^i_{kj}(x,y))$ have a complicated law of transformation A better one is

$$v^i{}_{|j} = \delta_j v^i + F^i_{kj}(x,y)v^k, \tag{1.12}$$

since then $(F^i_{kj}(x,y))$ changes under (1.1) as the local coefficients of a linear connection on M. These derivatives can be extended to any d–tensor field. For instance, the v–covariant derivative of $(g_{ij}(x,y))$ is given by (1.10) and its h–covariant derivative is

(1.13) $$g_{ij|k} = \delta_k g_{ij} - F^h_{ik} g_{hj} - F^h_{jk} g_{ih}.$$

The GL–metric $(g_{ij}(x,y))$ is said to be h–covariant constant if $g_{ij|h} = 0$. It is easy to check that the equation $g_{ij|h} = 0$ is satisfied with

(1.14) $$\overset{c}{F}{}^k_{ij} = \frac{1}{2} g^{kh} (\delta_i g_{hj} + \delta_j g_{ih} - \delta_h g_{ij}).$$

The foregoing discussions suggest

Definition 1.4. A Finsler connection is a triad $F\Gamma = (N^i_j(x,y), F^i_{jk}(x,y), C^i_{jk}(x,y))$, where $N^i_j(x,y)$ are the coefficients of a nonlinear connection, $F^i_{jk}(x,y)$ are like the coefficients of a linear connection on M and $C^i_{jk}(x,y)$ are the components of a d–tensor field.

We have also got a first example of Finsler connection $C\Gamma = (N^i_j(x,y), \overset{c}{F}{}^i_{jk}(x,y), \overset{c}{C}{}^i_{jk}(x,y))$.

Definition 1.5. Let $F\Gamma$ be a Finsler connection and $(g_{ij}(x,y))$ a GL–metric. $F\Gamma$ is said to be h–metrical if $g_{ij|h} = 0$, v–metrical if $g_{ij}\big|_h = 0$ and metrical if the both equations hold.

In the above we have proved

Proposition 1.1. *The Finsler connection* $C\Gamma$ *is metrical.*

The following d–tensor fields are called the *torsions* of $F\Gamma$:

(1.15) $$\begin{aligned} T^i_{jk} &= F^i_{jk} - F^i_{kj}, & R^i_{jk} &= \delta_k N^i_j - \delta_j N^i_k, C^i_{jk}, \\ P^i_{jk} &= \dot{\partial}_k N^i_j - F^i_{kj}, & S^i_{jk} &= C^i_{jk} - C^i_{kj}. \end{aligned}$$

Remark 1.1. R^i_{jk} is the integrability tensor of the horizontal distribution. It measures also the curvature of the nonlinear connection N.

The d–tensor fields

(1.16) $$D^i_j = F^i_{kj} y^k - N^i_j, \; d^i_j = \delta^i_j + C^i_{kj} y^k,$$

where (δ^i_j) is Kronecker' symbol, are called *h–deflection* and *v–deflection* of $F\Gamma$, respectively.

From (1.6) we infer that $G^i_{jk} = \dot{\partial}_j N^i_k$ transform under (1.1) as F^i_{jk}. Thus $B\Gamma = (N^i_j, G^i_{jk}, 0)$ is a Finsler connection. It will be called the *Berwald connection.* This connection is neither v–metrical nor h–metrical and is free of torsions if and only if N is integrable ($R^i_{jk} = 0$) and symmetric ($\dot{\partial}_j N^i_k = \dot{\partial}_k N^i_j$).

The connection $C\Gamma$ will be called the *Cartan connection.* It is h–metrical, h–symmetric ($\overset{c}{F}{}^i_{jk}(x,y) = \overset{c}{F}{}^i_{kj}(x,y)$), v–metrical and v–symmetric. The Finsler connection $H\Gamma = (N^i_j, G^i_{jk}(x,y), \overset{c}{C}{}^i_{kj}(x,y))$ will be called the *Hashiguchi connection.* This is v–metrical, no h–metrical and has torsion. The Finsler connection $CR\Gamma = (N^i_j, \overset{c}{F}{}^i_{jk}(x,y), 0)$ will be called the *Chern–Rund connection.* This is h–metrical but not v–metrical.

Summarizing, for a fixed nonlinear connection N and a GL–metric $(g_{ij}(x,y))$ we have four typical Finsler connections: $B\Gamma, C\Gamma, H\Gamma$ and $CR\Gamma$.

Let us replace TM by $T_0M = TM \setminus 0$.

A GL–metric $(g_{ij}(x,y))$ on T_0M reduces to a Finsler metric if there exists a fundamental Finsler function $F : T_0M \longrightarrow R_+$ such that $g_{ij}(x,y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2(x,y))$. Taking as N the Cartan nonlinear connection of coefficients $\overset{c}{N}{}^i_j = \frac{1}{2}\dot{\partial}_j\gamma^i_{oo}$, $\gamma^i_{oo} = \gamma^i_{jk}y^jy^k$, $\gamma^i_{jk} = \frac{1}{2}g^{ih}(\partial_j g_{hk} + \partial_k g_{jh} - \partial_h g_{jk})$, the afore mentioned Finsler connections reduce to the four remarkable connections in Finsler geometry (3, [Ch. III]).

The form of D^i_j in (1.6) shows that one may associate to any $F\Gamma$ a new Finsler connection $(F^i_{kj}y^k - D^i_j, F^i_{kj}, C^i_{kj})$ whose h–deflection is just D^i_j, when this is prescribed. In particular, for $D^i_j = 0$ a Finsler connection without h–deflection is obtained. In Finsler geometry $B\Gamma, C\Gamma, H\Gamma$ and $CR\Gamma$ are h–deflection free. So we have an explanation why the nonlinear connection was noted quite late in Finsler geometry.

2 Another definition of Finsler connections

Let be $\tau^{-1}TM = \{(u,v) \in TM \times TM, \tau(u) = \tau(v)\}$ fibered over TM by $\pi(u,v) = u$. The local fiber in (u,v) is $T_{\tau(u)}M$. A section in $(\tau^{-1}TM, \pi, TM)$ is locally of the form $\bar{X} = \bar{X}^i(x,y)\bar{\partial}_i$ with $(\bar{\partial}_i)$ the natural basis in $T_{\tau(u)}M$. It follows that under (1.1) we have

$$\tilde{\bar{X}}^i = \frac{\partial \tilde{x}^i}{\partial x^k}\,\bar{X}^k. \tag{2.1}$$

$\bar{X}$ will be called a τ–vector field on TM. It can be identified with the d–vector field $(\bar{X}^i(x,y))$. More general, the tensorial algebra of the pull–back bundle $\tau^{-1}TM$ can be thought of as algebra of d–tensor fields on TM. There exists a remarkable τ–vector field $\mathbb{C} : u \longrightarrow (u,u)$, which locally is $y^i\bar{\partial}_i$ and so it can be identified to the Liouville vector field $\mathbb{C} = y^i\dot{\partial}_i$.

Theorem 2.1. *There exists a one–to–one correspondence between the set of Finsler connections $F\Gamma$ and the set of pairs (N, ∇) with N a nonlinear connection on TM and ∇ a linear connection in the pull–back bundle $\tau^{-1}TM$.*

Proof. If $F\Gamma$ is specified by $(N^i_j, F^i_{jk}, C^i_{jk})$, we take $N = (N^i_j)$ and define ∇ by

$$\nabla_{\delta_k}\bar{\partial}_i = F^i_{jk}\bar{\partial}_i,\ \nabla_{\dot{\partial}_k}\bar{\partial}_j = C^i_{jk}\bar{\partial}_i. \tag{2.2}$$

In the natural basis ∇ takes the form

$$\nabla_{\partial_k}\bar{\partial}_j = \Gamma^i_{jk}\bar{\partial}_i, \quad \nabla_{\dot{\partial}_k}\bar{\partial}_i = C^i_{jk}\bar{\partial}_i. \tag{2.3}$$

$$\Gamma^i_{jk} = F^i_{jk} + N^h_k C^i_{jh}. \tag{2.4}$$

Conversely, given $N = (N^i_j)$ and ∇ specified by (2.3) it results that $(N^i_j, F^i_{jk}, C^i_{jk})$ with F^i_{jk} given by (2.4) is a Finsler connection.

A GL–metric $(g_{ij}(x,y))$ defines a metrical structure g in the bundle $\tau^{-1}TM$:

$$g = g_{ij}(x,y)dx^i \otimes dx^j. \tag{2.5}$$

Conversely, any metrical structure in the bundle $\tau^{-1}TM$ defines by (2.5) a GL–metric.

One easily checks

Theorem 2.2. *In the correspondence* $F\Gamma \longleftrightarrow (N,\nabla)$ *we have*

a) $F\Gamma$ *is h–metrical if and only if* $\nabla_{hX}g = 0$,

b) $F\Gamma$ *is v–metrical if and only if* $\nabla_{vX}g = 0$,

c) $F\Gamma$ *is metrical if and only if* $\nabla_X g = 0$,

for every $X \in \mathcal{X}(TM)$.

Let $\rho : TTM \longrightarrow \tau^{-1}TM$ be the morphism of vector bundles given by $\rho(X_u) = (u, \tau_{*,u}X_u)$, $X_u = T_uTM$, $u \in TM$. It follows that $\ker \rho_u = V_uTM$ i.e. $\rho(\dot{\partial}_i) = 0$ and $\rho(\delta_i) = \bar{\partial}_i$. Alternatively, we may define a morphism $\sigma : TTM \longrightarrow \tau^{-1}TM$ on basis by $\sigma(\delta_i) = 0$, $\sigma(\dot{\partial}_i) = \bar{\partial}_i$. We say that

$$\begin{aligned} \mathbb{T}_\rho(X,Y) &= \nabla_X \rho(Y) - \nabla_Y \rho(X) - \rho[X,Y], \\ \mathbb{T}_\sigma(X,Y) &= \nabla_X \sigma(Y) - \nabla_Y \sigma(X) - \sigma[X,Y], \ \ X,Y \in \mathcal{X}(TM), \end{aligned} \tag{2.6}$$

are *torsions* of ∇.

The following characterizations of the Finsler connections $B\Gamma, H\Gamma, CR\Gamma$ and $C\Gamma$ follow.

Theorem 2.3. *In the correspondence* $F\Gamma \longleftrightarrow (N,\nabla)$ *we have*

a) $B\Gamma \longleftrightarrow (N,\nabla)$ *with* $\mathbb{T}_\sigma(hX,vY) = 0,\ \mathbb{T}_\rho(hX,vY) = 0$;

b) $H\Gamma \longleftrightarrow (N,\nabla)$ *with* $\mathbb{T}_\sigma(hX,vY) = 0,\ \mathbb{T}_\sigma(vX,vY) = 0,\ \nabla_{vX}g = 0$;

c) $CR\Gamma \longleftrightarrow (N,\nabla)$ *with* $\mathbb{T}_\rho(hX,vY) = 0,\ \mathbb{T}_\rho(hX,hY) = 0,\ \nabla_{hX}g = 0$;

d) $C\Gamma \longleftrightarrow (N,\nabla)$ *with* $\mathbb{T}_\rho(hX,vY) = 0,\ \mathbb{T}_\sigma(vX,vY) = 0,\ \nabla_X g = 0$.

Proof. The local expressions of $\mathbb{T}_\rho$ and $\mathbb{T}_\sigma$ in conjunction with Theorem 2.2 give the desired results.

Now the following question appears. Which conditions sould satisfy ∇ in order to determine N such that the pair (N,∇) to correspond to a Finsler connection. An answer is as follows.

Definition 2.1. A linear connection ∇ in the pull–back bundle $\tau^{-1}TM$ is said to be regular if the subspace $\{X_u \mid \nabla_{X_u}\mathbb{C} = 0,\ X \in \mathcal{X}(TM)\}$ of T_uTM is supplementary to V_uTM for every $u \in TM$.

By the definition, every regular connection ∇ induces a nonlinear connection N on TM. The pair (N,∇), as we have seen before, corresponds to a Finsler connection $F\Gamma$. This $F\Gamma$ has to be of a particular form. Indeed, one has

Theorem 2.4. *There exists a bijection between the set of regular connections in* $\tau^{-1}TM$ *and the set of Finsler connections* $F\Gamma = (N^i_j, F^i_{jk}, C^i_{jk})$ *satisfying* $D^i_j = 0$ *and* $\det(d^i_h) \neq 0$.

Proof. Let ∇ be specified by (2.3). Using $N = (N^i_j)$ provided by the regularity of ∇, we define F^i_{jk} as in (2.4). Then $0 = \nabla_{\delta_k}\mathbb{C} = (y^jF^i_{jk} - N^i_k)\bar{\partial}_k$ implies $D^i_k = 0$. Contracting (2.4) by y^j we get $N^h_k(d^i_h) = y^j\Gamma^i_{jk}$ and as (N^h_k) is specified this equation has to have an unique solution. Hence with necessity $\det(d^i_h) \neq 0$.

Conversely, let (N,∇) be in correspondence with $F\Gamma$. The condition $D^i_j=0$ assures that the subspace $\{X_u|\nabla_{X_u}\mathbb{C}=0, X\in\mathcal{X}(TM), u\in TM\}$ is contained in the horizontal subspace H_uTM of N. The condition $\det(d^i_k)\neq 0$ implies that this subspace is supplementary to V_uTM. Thus ∇ is regular and the nonlinear connection derived from it coincides with N.

Let us assume that (g_{ij}) reduces to a Finsler metric on T_0M. Then $C\Gamma$ is characterized by the following Matsumoto's axioms:

$$(*)\qquad T^i_{jk}=0,\ g_{ij|k}=0,\ S^i_{jk}=0,\ g_{ij}\Big|_k=0,\ D^i_j=0.$$

It results $d^i_j=\delta^i_j$.

Combining these with Theorems 2.4 and 2.3, one obtains

Theorem 2.5. *Let $F^n=(M,F)$ be a Finsler space. There exists a unique regular connection ∇ in $\pi^{-1}T_0M$ satisfying the conditions*:

$$\mathbb{T}_\rho(hX,hY)=0,\ \mathbb{T}_\sigma(vX,vY)=0,\ \nabla_X g=0,\ X,Y\in\mathcal{X}(T_0M)$$

where h and v are projectors of N induced by ∇.

We note that ∇ is determined by F only.

According to 4 the Chern–Rund connection in a Finsler space is characterized by the following axioms:

$$T^i_{jk}=0,\ g_{ij|k}=0,\ C^i_{jk}=0,\ D^i_j=0.$$

We have again $d^i_j=\delta^i_j$. By Theorems 2.3 and 2.4 we have

Theorem 2.6. *Let $F^n=(M,F)$ be a Finsler space. There exists a unique regular connection ∇ in $\pi^{-1}T_0M$ satisfying the conditions:*

$$\mathbb{T}_\rho(hX,hY)=0,\ \mathbb{T}_\rho(hX,vY)=0,\ \nabla_{hX}g=0,\ X,Y\in\mathcal{X}(T_0M)$$

where h and v are projectors of N induced by ∇.

The systems of axioms for $H\Gamma$ and $B\Gamma$ discussed for minimality in 4 give similar results in view of Theorems 2.3 and 2.4.

The Finsler connections may be viewed also as special liner connections on TM or in the Finsler bundle $\pi^{-1}LM$, where LM is the principal bundle of linear frames on M. We refer to 4 and 3, respectively.

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