Representation of NonHamiltonian Vector Fields in the Coordinates of the Observer via the Isosymplectic Geometry

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Abstract

We study axiom-preserving isotopic liftings of the symplectic geometry which permit the representation of nonhamiltonian vector fields in the inertial frame of the observer without the need of Darboux's reduction to a Hamiltonian form in frames which are no longer inertial and not realizable in experiments.

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1 Introduction

Despite momentous advances, the symplectic geometry still remains with fundamental open problems particularly motivated by *physical needs*.

This is due to the fact that the symplectic geometry (see, e.g., ref.s[1,7] for technical presentations and ref.[10], Sect.2.3, for a review and literature) was historically build on systems entirely representable with a Hamiltonian [10]; these systems were originally called *exterior systems* [10, 11], are today called *(locally) Hamiltonian vector fields*, and represent a finite number of isolated point-like particles moving in vacuum under action-at-a-distance, potential interactions.

Physical systems of current interests are instead given by the more general *interior systems* [10,11], which are given by a finite set of extended particles moving within physical media. Unlike the former, the latter systems require $2n$ first-order differential equations which are *arbitrarily nonlinear in the velocities, integro-differential and variationally nonselfadjoint.*

As an example, missiles in atmosphere have nowadays reached such speeds to experience resistive forces proportional up to the *tenth power of the speed*; their equations of motion are characterized by ordinary differential equations representing the trajectory of the center-of-mass $x(t)$ plus corrective terms due to the *shape* of the satellite usually given by *surface integrals*, thus being in that sense ″integro-differential″;

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and, finally, they are ″variationally nonselfadjoint″ in the sense of violating the integrability conditions for the existence of a potential, as well as, more generally, of a Hamiltonian [10].

It is evident that interior systems are outside the representational capabilities of the symplectic geometry on a number of mathematical and physical grounds. In particular, they are *nonhamiltonian*, both locally or globally.

Mathematically, the topology of the symplectic geometry can only represent local-differential systems, thus requiring a suitable integro-differential broadening.

Even after approximating nonlocal-integral terms via power series expansions in the valocities, thus regaining the local-differential character, the symplectic geometry remains afflicted by the following problematic aspects of *physical* character related to Darboux's theorem [3].

As it is well known (see, e.g., [1,7,10]) Darboux's Theorem essentially states that, when a (local-differential and well behaved) systems $2n$ first-order ordinary differential equations in the vector field form $X(b)$ is not Hamiltonian in the $2n$-differential local coordinates $b = \{x, p\}$ of the cotangent bundle (phase space), there always exist a new coordinate system $b'(b)$ in which the system is Hamiltonian.

With the understanding that the mathematical correctness of Darboux's Theorem is beyond and possible doubt, and the theorem is now well established in the history of geometry, the physical problematic aspects are due to the fact that Darboux's transformations.

$$b = \{x, p\} \rightarrow b'(b) = \{x'(x, p), p'(x, p)\}, \tag{1.1}$$

are necessarily (noncanonical and *nonlinear*). This implies the *inapplicability* to a *Darboux's frame* b' *of contemporary relativities*, such as Galilei's relativity and Einstein's special relativity, because the latter only apply to *inertial frames*, while Darboux's frames b', being the nonlinear images of the inertial ones, are highly *noninertial.*

Even ignoring the abandonment of conventional relativities, *Darboux's frames are not realizable in actual experiments.* As an example, if the coordinates x are those of the experimenter, their Darboux's images are expressions, say, of the type $x' = \alpha\, exp(\beta x \times p)$, where α and β are suitable constants. Nonlinear expressions of the latter type are manifestly not realizable in an actual experiment, thus restricting Darboux's theorem to the sole mathematical significance.

This establishes the physical need of achieving a generalization/covering of the symplectic geometry which is ″directly universal″ for interior systems, that is, capable of representing all well behaved systems of the class considered (″universality″), directly in the inertial frame of the observer (″direct universality″), without any use of the transformation theory.

Note that all studies of direct universality are necessarily *local* as well as in *fixed* local coordinates, features which are tacitly assumed hereon.

A first form of direct universality of the conventional symplectic geometry was apparently reached for the first time in monograph [11]. Suppose that a given vector field $\Xi(b)$, is non Hamiltonian in b, i.e., there exist no function $H(b)$ such that $\Xi(b) \lrcorner \omega = dH(b)$, where ω is the *exact, nondegenerate, canonical, symplectic two-form* [1,7,10]. Then, it was proved in [11] that, under certain continuity and regularity conditions, there always exists a *general, exact, nondegenerate symplectic two-form* $\Omega(b)$ such that the following identity holds $\Xi(b) \lrcorner \Omega(b) = dH(b)$. In the latter case the system is derivable from a first-order Pfaffian action, the underlying equations are *Birkhoff's equations* [2] and $\Xi(b)$ is called a (locally) *Bikhoffian vector field* [11].

Note that the coordinates of the experimenters are preserved in the above direct universality, and the representation of nonhamiltonian systems is achieved via the use of the most general possible exact symplectic form $\Omega(b)$, rather than the canonical one ω.

Subsequent studies indicated that the above direct universality is still afflicted by problematic aspects, again, of physical nature. In fact, the quantization of interior systems via their Birkoffian representation (or, equivalently, the lifting of the symplectic quantization via the general, rather than the canonical, two-form), exhibits insurmountable difficulties in the physical interpretation of the emerging operator formalism (see [16], App.2.B).

At any rate, being based on the *conventional* symplectic geometry, the Birkoffian mechanics is strictly *local-differential* and, thus generally inapplicable to the *nonlocal-integral* interior *systems.*

The latter problems forced this author to seek yet another generalization of the symplectic geometry, this time, achieving *direct universality for interior systems via the sole use of the canonical two-form.*

Even though not necessarily unique, effective methods for the study of this problem are given by the so-called *isotopies*, which were first introduced in ref. [9] of 1978 in the form here need, and which are today defined as maps of linear, local-differential and Hamiltonian systems into their most general possible nonlinear, nonlocal-integral and nonhamiltonian form, yet capable of restoring linearity, locality and canonicity in certain generalized spaces over generalized fields.

The *isotopies of the symplectic geometry*, or *isosymplectic geometry* for short, were submitted, apparently for the first time, by the author in memoir [13] of 1988 and subsequently studied in various works (see monograph [15] for a recent account).

This first formulation of the isosymplectic geometry was based on the *isotopic degress of freedom of the product*, based on the lifting of the $2n$-dimensional trivial unit of the symplectic geometry, $I = diag.(1, 1, ..., 1)$, into a nondegenerate, real-valued and symmetric matrix $\hat{I}_{(1)} = \hat{T}_{(1)}^{-1}$ whose elements have a well behaved but otherwise arbitrary functional dependence. The lifting of the unit $I \rightarrow \hat{I}$ then implies the corresponding lifting of one-forms

$$\theta = p \times dx \rightarrow \hat{\theta} = p \hat{\times} dx = p_i \hat{T}^i_{(1)j} dx^j, \quad i, j = 1, 2, ..., n,$$

with corresponding liftings of the canonical two-form

$$\omega = d\theta = \frac{1}{2}\omega_{\mu\nu} db^\mu \wedge db^\nu \rightarrow \hat{\omega} = d\hat{\theta} = \frac{1}{2}\omega_{\mu\alpha}\hat{T}^\alpha_{(2)\nu} db^\alpha \wedge db^\nu, \mu, \nu, 1, 2, ..., 2n,$$

where $\hat{T}_{(2)}$ is a new matrix derivable from $\hat{T}_{(1)}$ via certain simple algebra [15].

The above isotopies permitted the first alternative formulation of Darboux's theorem for the representation of nonhamiltonian systems in the reference of their experimental observation and via the use of the canonical symplectic structure. However, the formalism was still afflicted by insufficiencies due to the *change of the unit in the transition from one- to two-forms* [15].

In this note we study a resolution of the latter, seemingly final difficulty as permitted by the *isotopies of the differential calculus* or *isodifferential calculus* for short, submitted by the author at the *International Workshop on Differential Geometry and Lie Algebras*, held at the Aristotle University in Thessaloniki in December 1994, and then published in ref.s [19].

More specifically, this note is devoted to the reformulation of the isosymplectic geometry via the isodifferential calculus for the representation of interior systems in which the coordinates of the observer $b = (x, p)$, the canonical symplectic structure *and* the generalized unit are kept unchanged. As we shall see, the emerging broadening of the conventional formulation of the symplectic geometry results to have a novel *integro-differential topology*, thus being naturally able to represent interior systems.

To render this note self-sufficient we shall briefly review in Sect.2 the main aspects of the isotopic methods, and then pass in Sect.3 to the new formulation of the isosymplectic geometry.

Our entire presentation is, not only *local*, but in the *fixed* local coordinates x of the observer without use of the transformation theory. To avoid misrepresentations, the study of global, coordinate-free formulations is recommended only *after* achieving the desired direct universality in the inertial frame of the observer.

As we shall see, the isotopic formulation of the symplectic geometry is such that the *abstract formulations of the conventional and symplectic geometries coincide.* We merely have *two* different *realizations* of the same abstract axioms: the conventional symplectic version [1,7,10], and the broader isotopic one.

It should be stressed that this note has been written by a theoretical physicist and, in any case, the studies are in their first infancy, thus requiring comprehensive mathematical reformulations by interested mathematicians.

2 Elements of isotopic methods

The fundamental isotopies from which all others can be uniquely derived are those of the *unit* [9], i.e., the liftings of the n-dimensional unit $I = diag.(1, 1, 1, ...)$ of the Euclidean geometry (in the same dimension) into real-valued and symmetric $n \times n$ matrices $\hat{I} = (\hat{I}^i_j) = \hat{I}^t$ whose elements $\hat{I}^i_j$ have an unrestricted functional dependence in coordinates x, velocities $v = \frac{dx}{dt}$, accelerations $a = \frac{dv}{dt}$, local density μ, local temperature τ, and any needed characteristics of the interior problem,

$$I \rightarrow \hat{I} = \hat{I}(x, v, a, \mu, \tau, ...) = \hat{I}^t. \tag{2.1}$$

The above liftings were classified by Kadeisvili [5] into: **Class I** (generalized units that are nondegenerate, Hermitian and positive-definite, characterizing the *isotopies* properly speaking); **Class II** (the same as Class I although $\hat{I}$ is negative-definite, characterizing the so-called *isodualities*; **Class III** (the union of Class I and II); **Class IV** (Class III plus the zeros of the generalized unit, $\hat{I} = 0$); and **Class V** (Class IV plus unrestricted generalized units, e.g., realized via discontinuous functions, distributions, lattices, etc).

All isotopic structures also admit the same classification which will be omitted for brevity. In this note we shall study isotopies of Classes I and II, at times treated in a unified way via those of Class III whenever no ambiguity arises. The isotopies of Classes IV and V are vastly unexplored at this writing.

The isotopies of the unit evidently imply corresponding compatible isotopies with the *totality* of conventional mathematical methods underlying the symplectic geometry. Regrettably, we cannot provide here a review of the isotopic methods and refer the reader to the recent treatment [19].

We merely recall that a field $F(n, +, \times)$ of real, complex of quaternionic numbers with element n, sum $+$ and multiplication $\times$, is lifted under isotopies in the *isofield* $\hat{F}(\hat{n}, +, \hat{\times})$ of the *isonumbers* $\hat{n} = n \times \hat{I}$ with conventional sum $+$ and isoproduct $\hat{\times} = \times \hat{T} \times$. Under the condition $\hat{I} = \hat{T}^{-1}, \hat{I}$ is then the correct left and right unit of $\hat{F}$. In this case $\hat{F}$ verifies all conventional axioms of a field (even though $\hat{I}$ is outside the original F) and the listing $F \to \hat{F}$ is isotopic.

Similarly, a metric space $S(x, g, R)$ with local chart x and metric $g(x)$ over thereals $R(n, +, \times)$ must be lifted, for evident reasons of compatibility, into the *isospace* $\hat{S}(\hat{x}, \hat{g}, \hat{R})$ of isocoordinates $\hat{x} = x$ (in their contravariant form) and isometric $\hat{g} = \hat{T}(x, \dot{x}, \ddot{x}, ...) \times g(x)$ over the isofield $\hat{R}$.

The ordinary differential calculus must also be lifted under isotopies into the *isodifferential calculus* which is characterized by the *isodifferentials*

$$\hat{d}\hat{x}^k = \hat{I}^k_i(x, ...)dx^i, \quad \hat{d}\hat{x}_k = \hat{T}^i_k(x, ...)dx_i. \tag{2.2}$$

and *isoderivatives*

$$\hat{f}'(\hat{q}^k) = \frac{\hat{\partial}\hat{f}(\hat{x})}{\hat{\partial}\hat{x}^k} \mid_{\hat{x}^k=\hat{q}^k} = \hat{T}^i_k \frac{\partial f(x)}{\partial x^i} \mid_{\hat{x}^k=\hat{q}} = Lim_{\hat{d}\hat{x}^k \to \hat{0}^k} \frac{\hat{f}(\hat{q}^k + \hat{d}\hat{x}^k) - \hat{f}(\hat{q}^k)}{\hat{d}x^k} \tag{2.3}$$

with properties

$$\hat{d}\hat{f}(\hat{x})_{|contrav.} = \frac{\hat{\partial}\hat{f}}{\hat{\partial}\hat{x}^k}\hat{d}\hat{x}^k = \hat{T}^i_k \frac{\partial f}{\partial x^i}\hat{I}^k_j dx^j = \frac{\partial f}{\partial x^k}dx^k = \frac{\partial f}{\partial x^i}\hat{T}^j_i dx^j,$$

$$\hat{d}f(x)_{|covar.} = \frac{\hat{\partial}\hat{f}}{\hat{\partial}\hat{x}_k}\hat{d}\hat{x}_k = \hat{I}^k_i \frac{\partial f}{\partial x_i}\hat{T}^j_k dx_j = \frac{\partial f}{\partial x_k}dx_k = \frac{\partial f}{\partial x_j}\hat{I}^i_j dx_i,$$

$$\frac{\hat{\partial}^{\hat{2}}\hat{f}(\hat{x})}{\hat{\partial}\hat{x}^{k\hat{2}}} = \hat{T}^i_k\hat{T}^j_k \frac{\partial^2 f(x)}{\partial x^i \partial x^j}, \quad \frac{\hat{\partial}^{\hat{2}}\hat{f}(\hat{x})}{\hat{\partial}\hat{x}_k{}^{\hat{2}}} = \hat{I}^k_i\hat{I}^k_j \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \text{ (no sums on k)}$$

$$\frac{\hat{\partial}\hat{x}^i}{\hat{\partial}\hat{x}^j} = \delta^i_j, \quad \frac{\hat{\partial}\hat{x}_i}{\hat{\partial}\hat{x}_j} = \delta^j_i, \quad \frac{\hat{\partial}\hat{x}_i}{\hat{\partial}\hat{x}^j} = \hat{T}^j_i, \quad \frac{\hat{\partial}\hat{x}^i}{\hat{\partial}\hat{x}_j} = \hat{I}^i_j. \tag{2.4}$$

The notion of *isocontinuity* on an isospace was first studied by Kadeisvili [5] and resulted to be easily reducible to that of conventional continuity for Class III isotopies because the *isomodulus* $\hat{|}\hat{f}(\hat{x})\hat{|}$ of a function $\hat{f}(\hat{x})$ on the isospace $\hat{E}(\hat{x}, \hat{\delta}, R)$ over the isofield $\hat{R}(\hat{n}, +, \hat{\times})$ is given by the conventional modulus
$| \hat{f}(\hat{x}) |$ multiplied by the a well behaved isounit $\hat{I}$.

The notion topology of n-dimensional *isomanifold* was first studied by Tsagas and Sourlas [20] and it is today called the *Tsagas-Sourlas isotopology.*

The isotopies imply simple, yet nontrivial generalizations of *all* conventional mathematical structures, with no exception known to this author. This implies also a compatible lifting of functional analysis whose study was initiated by Kadeisvili in ref.[5] under the name of *functional isoanalysis* (see [15] for brevity).

3 Isosymplectic geometry

. We are now equipped to study the *isotopies of the symplectic geometry*, or *isosymplectic geometry* [13] for short, as characterized by the isodifferential calculus of the preceding section.

Unless otherwise stated, our formulation is local and in the fixed coordinates of the observer. All quantities are assumed to satisfy the needed continuity conditions, e.g., of being of class $\hat{C}^\infty$ and all neighborhoods of a point are assumed to be star-shaped or have an equivalent topology. For the conventional symplectic geometry we shall use the local formulation of ref.[7]. We shall first study the isosymplectic geometry of Class I representing matter and then study its antiautomorphic image under isoduality for the characterized of antimatter.

Let $\hat{M}(\hat{E}) = \hat{M}(\hat{E}(\hat{\delta}, \hat{R}))$ be an n-dimensional Tsagas-Sourlas isomanifold [20] on the isoeuclidean space $\hat{E}(\hat{x}, \hat{\delta}, \hat{R})$ over the isoreals $\hat{R} = \hat{R}(\hat{n}, +, \hat{\times})$ with $n \times n$-dimensional isounit $\hat{I} = (\hat{I}^i_j), i, j = 1, 2, ..., n$, of Kadeisvili Class I and local chart $\hat{x} = \{\hat{x}^k\}$. A *tangent isovector* $\hat{X}(\hat{m})$ at a point $\hat{m} \in \hat{M}(\hat{E})$ is an isofunction defined in the neighborhood $\hat{N}(\hat{m})$ of $\hat{m}$ with values in $\hat{R}$ satisfying the *isolinearity conditions*

$$\hat{X}_m(\hat{\alpha}\hat{\times}\hat{f} + \hat{\beta}\hat{\times}g) = \hat{\alpha}\hat{\times}\hat{X}_{\hat{m}}(\hat{f}) + \hat{\beta}\hat{\times}\hat{X}_{\hat{m}}(\hat{g}),$$

$$\hat{X}_{\hat{m}}(\hat{f}\hat{\times}\hat{g}) = \hat{f}(\hat{m})\hat{\times}\hat{X}_{\hat{m}}(\hat{g}) + \hat{g}(\hat{m})\hat{\times}\hat{X}_m(\hat{f}), \tag{3.1}$$

for all $\hat{f}, \hat{g} \in \hat{M}(\hat{E})$ and $\hat{\alpha}, \hat{\beta} \in \hat{R}$, where $\hat{\times}$ is the isomultiplication in $\hat{R}$ and the use of the symbol $\hat{}$ means that the quantities are defined on isospaces.

The collection of all tangent isovectors at $\hat{m}$ is called the *tangent isospace* and denoted $T\hat{M}(\hat{E})$. The *tangent isobundle* is the $2n$-dimensional union of all possible tangent isospaces when equipped with an isotopic structure (see below).

The *cotangent isobundle* $T^\star\hat{M}(\hat{E})$ is the $2n$-dimensional dual of the tangent isobundle with local coordinates $\hat{b} = \{\hat{b}^\mu\} = \{\hat{x}^k, \hat{p}_k\}, \quad \mu = 1, 2, ..., 2n$. Since $\hat{p}$ is independent of $\hat{x}$, the isounits of the respective differentials are generally different, i.e., we can have $\hat{d}x = \hat{I}dx$ and $\hat{d}p = \hat{W}dp, \quad \hat{I} \neq \hat{W}$, in which case the total isounit of $T^\star\hat{M}(\hat{E})$ is the $2n$-dimensional Cartesian product $\hat{I}_2 = \hat{I} \times \hat{W}$.

For reasons which will be clarified later on, in this note we assume the following particular form of the *isounit of the cotangent isobundle*

$$\hat{I}_2 = (\hat{I_2}^{\mu}{}_{\hat{\nu})} = \begin{pmatrix} \hat{I}_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \hat{T}_{n\times n} \end{pmatrix} = \hat{T}_2^{-1} = (\hat{T}_{2\mu}{}^{\nu})^{-1} \quad \hat{I} = \hat{T}^{-1}, \tag{3.2}$$

where $\hat{I}$ is the isounit of the coordinates $\hat{d}x = \hat{I}dx$, and $\hat{T}$ is the isounit of the momenta, $\hat{d}p = \hat{T}p = \hat{I}^{-1}dp$. In different terms, we select the particular case in which $\hat{W} = \hat{I}^{-1}$.

An *isobasis* of $T^\star\hat{M}(\hat{E})$ is, up to equivalence, the (ordered) set of isoderivatives $\hat{\partial} = \{\frac{\hat{\partial}}{\hat{\partial}\hat{b}^\mu}\} = \{\hat{T}_{2\mu}{}^{\nu}\frac{\partial}{\partial b^\nu}\}$. A generic elements $\hat{X} \in T^\star\hat{M}(\hat{E})$, called *vector isofield*, can then be written $\hat{X} = \hat{X}^\mu(\hat{m})\frac{\hat{\partial}}{\hat{\partial}\hat{b}^\mu} = \hat{X}^\mu\hat{T}_{2\mu}{}^{\nu}\frac{\partial}{\partial b^\mu}$.

The *fundamental one-isoform* on $T^\star\hat{M}(\hat{E})$ is given in the local chart $\hat{b}$ by

$$\hat{\theta} = \hat{R}^\circ_\mu(\hat{b})\hat{d}\hat{b}^\mu = \hat{R}^\circ_\mu(b)\hat{I}^{\mu}_{2}{}_{\nu}d\hat{b}^\nu = \hat{p}_k\hat{d}\hat{x}^k = \hat{p}_k\hat{I}^k_i d\hat{x}^i, \quad \hat{R}^\circ = \{\hat{p}, \hat{0}\}. \tag{3.3}$$

The above expression, which can be written $\hat{\theta} = p\hat{d}x = p_i \hat{I}^i_j dx^j$ to emphasize the differential origin of the isotopies, should be compared with the originally proposed one-isoform $\hat{\theta} = p\hat{\times}dx = p_k \hat{T}^k_i dx^i$ [13] obtained via the isotopic degrees of freedom of the product. The preference of the isodifferential calculus over the isomultiplication is then evident for a geometric unity of the conventional and isotopic formulations.

The space $T^\star \hat{M}(\hat{E})$, when equipped with the above one-form, is an *isobundle* denoted $T^\star_1 \hat{M}(\hat{E})$. The *isoexact, nowhere degenerate, isocanonical isosymplectic two-isoform* is given by

$$\hat{\omega} = \hat{d}\hat{\theta} = \frac{1}{2}\hat{d}(\hat{R}^\circ_\mu \hat{d}\hat{b}^\mu) = \frac{1}{2}\omega_{\mu\nu}\hat{d}\hat{b}^\mu \wedge \hat{d}\hat{b}^\nu =$$

$$= \hat{d}\hat{x}^k \wedge \hat{d}\hat{p}_k = \hat{I}^k_i d\hat{x}^i \wedge \hat{T}^j_k d\hat{p}_j \equiv d\hat{x}^k \wedge d\hat{p}_k. \tag{3.4}$$

The isomanifold $T^\star \hat{M}(\hat{E})$, when equipped with the above two-isoform, is called *isosymplectic isomanifold* in isocanonical realization and denoted $T^\star_2 \hat{M}(\hat{E})$. The *isosymplectic geometry* is the geometry of the isosymplectic isomanifolds.

The last identity in (3.4) show that the *two-isoform $\hat{\omega}$ formally coincides with the conventional symplectic canonical two-form ω*, and this illustrates the selection of isounit (3.2). The abstract identity of the symplectic and isosymplectic geometries is then evident. However, one should remember that: the underlying metric is isotopic; $\hat{p}_k = \hat{T}_k{}^i p_i$, where p_i is the variable of the conventional canonical realization of the symplectic geometry; and identity $\hat{\omega} \equiv \omega$ no longer holds for the more general isounit $\hat{I}_2 = \hat{I} \times \hat{W}, \quad \hat{I} \neq \hat{W}^{-1}$.

Note that the *isosymplectic geometry has the Tsagas-Sourlas Integro - differential topology* and, as such, it can characterize interior systems when all nonlocal- integral terms are embedded in the isounit.

A *vector isofield* $\hat{X}(\hat{m})$ defined on the neighborhood $\hat{N}(\hat{m})$ of a point $\hat{m} \in T^\star_2 \hat{M}(\hat{E})$ with local coordinates $\hat{b}$ is called (locally) *isohamiltonian* when there exists an isofunction $\hat{H}$ on $\hat{N}(\hat{m})$ over $\hat{R}$ such that

$$\hat{X} \lrcorner \hat{\omega} = \hat{d}\hat{H}, \quad \text{i.e.,}$$

$$\omega_{\mu\nu}\hat{X}^\nu(\hat{m})\hat{d}\hat{b}^\mu = \hat{d}\hat{H}(\hat{m}) = \frac{\hat{\partial}\hat{H}}{\hat{\partial}\hat{b}^\mu}\hat{d}\hat{b}^\mu. \tag{3.5}$$

We are now equipped to present the main result of this note, the isotopic alternative to Darboux's Theorem for the representation of nonlinear, nonlocal-integral and nonhamiltonian interior systems within the fixed coordinates of their experimental observation, which can be formulated as follows.

Theorem 1.Direct Universality of the Isosymplectic Geometry for Interior Systems: *Under sufficient continuity and regularity conditions, all possible vector fields which are not (locally) Hamiltonian in the given coordinates are always isohamiltonian in the same coordinates, that is, there exists a neighborhood $N(\hat{m})$ of a point $\hat{m}$ of their variable $\hat{b} = (\hat{x}, \hat{p})$ under which Eq.s (3.5) hold.*

Proof. Let $\hat{X}^\mu(\hat{b})$ be a vector field which is nonhamiltonian in the chart $\hat{b}$, and consider the decomposition

$$\hat{X}(b) = \hat{\Gamma}^{\mu}{}_{\alpha}(b)\hat{X}_0^{\alpha}(b), \tag{3.6}$$

where the $2n \times 2n$ matrix $(\hat{\Gamma}^{\mu}{}_{\alpha})$ is nowhere degenerate and $\hat{X}_0^{\alpha}$ is the maximal, local-differential and Hamiltonian sub-vector field, i.e., there exists a function $H(b)$ and a neighborhood $N(m)$ of a point m of $b = (x, p)$ such that

$$\omega_{\alpha\beta}\hat{X}_0^{\beta}(m)db^{\alpha} = dH(m) = (\frac{\partial H}{\partial b^{\alpha}})\hat{d}\hat{b}^{\alpha}, \tag{3.7}$$

and all nonlocal-integral and nonhamiltonian terms are embedded in $\hat{\Gamma}$. Then, there always exists an isotopy such that

$$\omega_{\mu\nu}\hat{X}^{\nu}(\hat{m})\hat{d}\hat{b}^{\mu} = \omega_{\mu\alpha}\hat{\Gamma}^{\alpha}{}_{\beta}(\hat{m})\hat{X_0}^{\beta}(\hat{m})\hat{d}\hat{b}^{\mu} =$$

$$= \hat{d}\hat{H}(\hat{m}) = \frac{\hat{\partial}\hat{H}}{\hat{\partial}\hat{b}^{\mu}}\hat{d}\hat{b}^{\mu} = \hat{T}_{\mu}{}^{\beta}\frac{\partial H}{\partial b^{\beta}}\hat{d}b^{\mu}. \tag{3.8}$$

In fact, the script $\hat{X}^{\mu}$ is only a unified formulation in $2n$ dimension of two separate each in n-dimension. Therefore, the quantify $\hat{\Gamma}$ has the structure

$$\hat{\Gamma} = \begin{pmatrix} \hat{A}_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \hat{B}_{n\times n} \end{pmatrix}. \tag{3.9}$$

The identification

$$\hat{I} = \begin{pmatrix} \hat{B}^{-1}_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \hat{A}^{-1}_{n\times n} \end{pmatrix}, \tag{3.10}$$

then implies

$$\hat{I}^{\mu}{}_{\alpha}\omega_{\mu\nu}\hat{\Gamma}^{\nu}{}_{\rho} \equiv \omega_{\alpha\rho}, \tag{3.11}$$

and identities (3.8) always exist. **q.e.d.**
Corollary 1.A: *For all Newtonian systems we have $\hat{A} = \hat{B}^{-1}$, i.e., the $2n$-dimensional isounit of the cotangent isobundle has the structure (3.2).*
Proof. All Newtonian systems in the $2n$-dimensional, first-order, vector field form can be written in disjoint n-component

$$\begin{pmatrix} dx/dt \\ dp/dt \end{pmatrix} = \begin{pmatrix} p/m \\ F^{SA} + F^{NSA} \end{pmatrix} = \hat{X}(b) = (\hat{X}^{\mu}(b)), \tag{3.12}$$

where SA(NSA) stands for variational selfadjointness (nonselfadjointness), i.e., the integrability conditions for the existence (lack of existence) of a Hamiltonian. Thus $F^{SA} = -\partial H/\partial x$, with $H = p^2/2m + V(x)$, while there is no such Hamiltonian for F^{NSA}.

Then, isohamiltonian representation (3.8) explicitly reads

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p/m \\ F^{SA} + F^{NSA} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} p/m \\ F^{SA} \end{pmatrix} =$$

$$= \begin{pmatrix} -B\,F^{SA} \\ A\,p/m \end{pmatrix} = \begin{pmatrix} \hat\partial H/\hat\partial x \\ \partial \hat H/\hat\partial p \end{pmatrix} = \begin{pmatrix} B\,\partial H/\partial x \\ A\,\partial H/\partial p \end{pmatrix}. \tag{3.13}$$

From which we have the general solution.

$$\hat T = B = 1 + F^{NSA}/F^{SA} = A^{-1} = \hat I^{-1}, \tag{3.14}$$

where the last identity follows from the fact that, since $\partial H/\partial p = p/m$, A remains arbitrary and can be therefore assumed to be $A = B^{-1}$. **q.e.d.**

It is now important to verify that the above isotopies do indeed preserve all remaining axiomatic properties of the symplectic geometry. For this it is sufficient to prove the preservation under isotopies of the Poincaré Lemma and of Darboux's Theorem [1,7,10].

To prove the preservation of the Poincaré Lemma one can easily construct isoforms $\hat\Phi_p$ of arbitrary order p. The proof of the following property is a simple isotopy of the conventional proof (see, e.g. [7]) via the use of the isodifferential calculus.

Lemma 1 (Isopoincaré Lemma): *Under the assumed smoothness and regularity conditions, isoexact p-isoforms are isoclosed, i.e.,*

$$\hat d\hat\Phi_p = \hat d(\hat d\hat\phi_{p-1}) \equiv 0. \tag{3.15}$$

The nontriviality of the above result is illustrated by the following

Corollary 1.A: *Isoexact p-isoform are not necessarily closed, i.e., their projection in the original tangent bundle does not necessarily verify the Poincare Lemma.*

By comparison, we should mention that the original formulation of the isopoincar'e lemma [13,15], that via the isotopic degress of the product did verify the Poincaré lemma in both the conventional and isotopic bundle.

To prove the preservation of the Darboux's Theorem, consider the *general one-isoform* in the local chart $\hat b$

$$\hat\Theta(\hat b) = \hat R_\mu(\hat b)\hat d\hat b^\mu = \hat R_\mu(b)\hat I_2{}^\mu{}_\nu(t, b, db/dt, ...)db^\nu, \tag{3.16}$$

where

$$\hat R = \{\hat P(\hat x, \hat p), \hat Q(\hat x, \hat p)\}. \tag{3.17}$$

The *general isosymplectic isoexact two-isoform* in the same chart is then given by

$$\hat\Omega(\hat b) = \frac{1}{2}\hat d(\hat R_\mu(\hat b)\hat d\hat b^\mu) = \frac{1}{2}\hat\Omega_{\mu\nu}(\hat t, \hat b, \hat d\hat b/\hat d\hat t, ...)\hat d\hat b^\mu \wedge \hat d\hat b^\nu,$$

$$\hat\Omega_{\mu\nu} = \frac{\hat\partial \hat R_\nu}{\hat\partial \hat b^\mu} - \frac{\hat\partial \hat R_\mu}{\hat\partial \hat b^\nu} = \hat T_{2\mu}{}^\alpha \frac{\partial \hat R_\mu}{\partial \hat b^\alpha} - \hat T^\alpha_{2\nu}\frac{\partial \hat R_\mu}{\partial \hat b^\alpha}. \tag{3.18}$$

One can see that, while at the canonical level the exact two-form ω and its isotopic extension $\hat\omega$ formally coincide, *this is no longer the case for exact, but arbitrary two forms* Ω and $\hat\Omega$ in the same local chart.

Note that the isoform $\hat\Omega$ is isoexact, $\hat\Omega = \hat d\Theta$, and therefore isoclosed, $\hat d\hat\Omega \equiv 0$ (Lemma 1), in isospace over the isofield $\hat R$. However, if the same isoform $\hat\Omega$ is projected in ordinary space and called Ω, it is no longer necessarily exact, $\Omega \neq d\theta$ and, therefore, it is not generally closed, $d\Omega \neq 0$.

Recall that the Poincaré Lemma $d\Omega = d(d\Theta) = 0$ for the case of the two-form Ω provide the necessary and sufficient conditions for the tensor $\Omega^{\mu\nu} = [(\Omega_\alpha^{-1}]^{\mu\nu}$ to be Lie [11]. It is easy to prove that this basic property persist under isotopy, although it characterizes a generalization of Lie's theory proposed in [9] and today known as the *Lie-Santilli Theory* (see, e.q., [6,22] and references quoted therein). We therefore have the following

Theorem 2. (General Lie-Santilli Brackets): *Let*

$$\Omega(\hat{b}) = \hat{d}\hat{\Theta} = \hat{d}(\hat{R}_\mu \hat{d}\hat{b}^\mu) = \hat{\Omega}_{\mu\nu}\hat{d}\hat{b}^\mu \wedge \hat{d}\hat{b}^\nu$$

be a general exact two-isoform. Then the brackets among sufficiently smooth and regular isofunctions $\hat{A}(\hat{b})$ and $\hat{B}(\hat{b})$ on $T_2^\star M(\hat{E})$

$$[\hat{A}, \hat{B}]_{isot.} = \frac{\hat{\partial}\hat{A}}{\hat{\partial}\hat{b}^\mu}\hat{\Omega}^{\mu\nu}\frac{\hat{\partial}\hat{B}}{\hat{\partial}\hat{b}^\nu},$$

$$\Omega^{\mu\nu} = \left[\left(\frac{\hat{\partial}\hat{R}_\alpha}{\hat{\partial}\hat{b}^\beta} - \frac{\hat{\partial}\hat{R}_\beta}{\hat{\partial}\hat{b}^\alpha}\right)^{-1}\right]^{\mu\nu}, \tag{3.19}$$

satisfy the Lie-Santilli axioms [9,6,22] in isospace (but not necessarily the same axioms when projected in ordinary spaces).

The following additional property completes the axiom-preserving character of the isotopies of the symplectic geometry.

Theorem 3. (Isodarboux Theorem): *A 2n-dimensional cotangent isobundle $T_2^\star\hat{M}(\hat{E})$ equipped with a nowhere degenerate, exact, $\hat{C}^\infty$ two-isoform $\hat{\Omega}$ in the local chart $\hat{b}$ is an isosymplectic manifold if and only if there exist coordinate transformations $\hat{b} \to \hat{b}'(\hat{b})$ under which $\hat{\Omega}$ reduces to the isocanonical two-isoform $\hat{\omega}$, i.e.,*

$$\frac{\hat{\partial}\hat{b}^\mu}{\hat{\partial}\hat{b}'^\alpha}\hat{\Omega}_{\mu\nu}(\hat{b}(\hat{b}'))\frac{\hat{\partial}\hat{b}^\nu}{\hat{\partial}\hat{b}'^\beta} = \omega_{\alpha\beta}. \tag{3.20}$$

Proof. Suppose that the transformation $\hat{b} \to \hat{b}'(b)$ occurs via the following intermediate transform $\hat{b} \to \hat{b}''(\hat{b}) \to \hat{b}'(b''(b))$. Then there always exists a transform $\hat{b} \to \hat{b}''$ such that

$$(\hat{\partial}\hat{b}^\rho/\hat{\partial}\hat{b}''^\sigma)(\hat{b}'') = \hat{I}^\rho_\sigma(\hat{b}(\hat{b}'')), \tag{3.21}$$

under which the general isosymplectic tensor $\hat{\Omega}_{\mu\nu}$ reduces to the Birkhoffian form when recompute in the $\hat{b}$ chart

$$\frac{\hat{\partial}\hat{b}^\mu}{\hat{\partial}\hat{b}''^\alpha}\hat{\Omega}_{\mu\nu}(\hat{b}(\hat{b}''))\frac{\hat{\partial}\hat{b}^\nu}{\hat{\partial}\hat{b}''^\beta}\ |_{\hat{b}''} = \left(\frac{\partial\hat{R}_\nu}{\partial\hat{b}^\alpha} - \frac{\partial\hat{R}_\mu}{\partial\hat{b}^\nu}\right)|_{\hat{b}''} = \Omega_{\alpha\beta|\hat{b}''}. \tag{3.22}$$

The existence of a second transform $\hat{b}'' \to \hat{b}'$ reducing $\Omega_{\alpha\beta}$ to $\omega_{\alpha\beta}$ is then known to exist (see, e.g., [11]). This proves the necessity of the isodarboux transform. The sufficiency is proved as in the conventional case [7]. **q.e.d.**

The nonlinear, nonlocal and noncanonical character of the isotopies is evident from the preceding analysis. It is important to point out that linearity is reconstructed in isospace and called *isolinearity*, as shown in Eq.(3.1). Locality is equally reconstructed in isospace, and called *isolocality*, because one- and two-isoforms are based on the local isodifferentials $\hat{d}\hat{x}$ and $\hat{d}\hat{p}$. Similarly, canonicity is reconstructed in isospace, and called *isocanonicity*, because the canonical form $p_k dx^k$ is preserved by the isotopic form $\hat{p}_k \hat{d}\hat{x}^k$ in isospace. The nonlinear, nonlocal and noncanonical character of isotopic theories solely emerge when they are projected in the original spaces.

The isotopies of the remaining aspects of the symplectic geometry (Lie derivative, global treatment, symplectic group, etc.) can be constructed along the preceding lines and are omitted for brevity.

On closing we should mention that the preceding formulation of the isosymplectic geometry is solely restricted for the representation of *matter*. The characterization of antimatter is made via the antiautomorphic isodual map $\hat{I}_2 \to \hat{I}_2^d = -\hat{I}$. This results in the *isodual isosymplectic geometry* which is characterized by *isodual coordinates* $\hat{b}^d$*, isodual isodifferentials* $\hat{d}^d\hat{b}^d$*, isodual one-isoforms* $\hat{\theta}^d(\hat{b})^d$*, isodual two-isoforms* $\hat{\omega}^d$*, isodual cotangent isobundle* $T^\star \hat{M}^d(\hat{E}^d)$, and similar isodualities whose explicit construction id left to the interested reader for brevity.

It is evident that the isotopies and isodualities of the symplectic geometry imply corresponding liftings of classical mechanics, called by the author *isohamiltonian mechanics* and additional liftings of the symplectic quantization and related quantum mechanics called *hadronic mechanics*. For the latter aspects and related applications in classical and quantum mechanics, we refer the interested reader to monography [16].

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