On the Killing Tensor Fields on a Compact Riemannian Manifold

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Abstract

Let (M, g) be a compact Riemannian manifold of dimension n. The aim of the present paper is to study the dimension of $K^q(M, R)$ in the connection with the Riemannian metric g on M.

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Key words: Riemannian manifold, Killing tensor field, Riemannian metric, harmonic q-form and Killing q-form.

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Let (M, g) be a compact Riemannian manifold of dimension n. Let $K^q(M, R)$ where $q = 2, ..., n-1$ be the vector space of Killing tensor fields of order q on M. The study of the dimension of $K^q(M, R)$ is an important problem. This importance comes from the fact there is a connection between q-harmonic forms and Killing tensor fields of order q. Let $H^q(M, R)$ be the vector space of harmonic q-forms. It is known that $dim(H^q(M, R)) = b_q$ is the q-Betti number of M, which is topological invariant. It is still open if $dim(K^q(M, R))$ for $q = 2, ..., n-1$ is also a topological invariant.

The aim of the present paper is to study this problem. We also improve Yano's results ([16]).

The whole paper contains three paragraphs. Each of them is analyzed as follows.

In the second paragraph we study differential operators of cross sections of a fibre bundle over a compact Riemannian manifold M. The Killing tensor fields of order q can be considered as special cross sections of the fibre bundle $\nabla^q(T(M))$ over M.

The space of Killing tensor fields $K^q(M, R)$ of order q with the connection of the Riemannian metric g on M is studied in the last paragraph. These results are an improvement Yano's results ([16]).

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Let (M,g) be a compact Riemannian manifold of dimension n without boundary. We denote by $\wedge^q(T(M))$ and $\wedge^q(T^\star M)$ the fibre bundles of antisymmetric convariant tensor fields of order q and antisymmetric contravariant tensor fields of order q respectively on the manifold M. It is known that the vector space $\wedge^q(T^\star M)$ coincide with the vector space $\wedge^q(M)$ of exterior q-forms.

We must notice that each exterior q-form w is a cross section of $\wedge^q(T^\star M) = \wedge^q(M)$. The same is true for each element $\lambda \in \wedge^q(TM)$. The Laplace operator Δ is a second order elliptic differential operator $C^\infty(\wedge^q(M))$, that is

$$\Delta = d\delta + \delta d : C^\infty(\wedge^q(M)) \to C^\infty(\wedge^q(M)),$$

$$\Delta = d\delta + \delta d : \alpha \to \Delta(\alpha) = d\delta(\alpha) + \delta d(\alpha), \quad \alpha \in C^\infty(\wedge^q(M)),$$

where α an exterior q-form and d, δ are first order differential operator defined by

$$d : C^\infty(\wedge^q(M)) \to C^\infty(\wedge^{q+1}(M)),$$
$$\delta : C^\infty(\wedge^q(M)) \to C^\infty(\wedge^{q-1}(M)).$$

These differential operators are related by

$$< \alpha, \delta\beta >=< d\alpha, \beta >, \quad \forall\alpha \in C^\infty(\wedge^q(M)), \quad \forall\beta \in C^\infty(\wedge^{q+1}(M)),$$

where $<>$ is the global inner product on $C^\infty(\wedge^q(M))$. The local inner product is defined by

$$< \alpha, \gamma >_1= \alpha_{i_1,\ldots,i_q} \quad \beta^{i_1,\ldots,i_q} = g^{j_1 i_1 \cdots} g^{j_q i_q} \quad \alpha_{i_1,\ldots,i_q} \quad \beta_{j_1,\ldots,j_q}.$$

Let $(x_1, ..., x_n)$ be a local coordinate system on the chart (U, φ) and let $\{e_1, ..., e_n\}$ be the associated local frame in M, that is

$$e_1 = \frac{\partial}{\partial x_1}, ..., e_n = \frac{\partial}{\partial x_n}.$$

If α is a q-form, which is a cross section of $\wedge^q(M)$, that is $\alpha \in C^\infty(\wedge^q(M))$, then α with respect to the local coordinate system can be expressed by

$$\alpha(e_{i_1}, e_{i_2}, ..., e_{i_q}) = \alpha_{i_1,\ldots,i_q}, \quad 1 \le i_1 < i_2 < ... < i_q \le n.$$

The following formulas are known

$$(d\alpha)_{i_1\ldots i_q j} = \frac{1}{q!}\varepsilon^{kj_1\ldots j_q}_{i_1\ldots i_q j}\nabla_k\alpha_{j_1\ldots j_q j} \tag{2.1}$$

$$(\delta\alpha)_{i_2\ldots i_q} = -\nabla_l\alpha^l_{i_2\ldots i_q} \tag{2.2}$$

$$(\Delta\alpha)_{i_1\ldots i_q} = -\nabla^k\nabla_k\alpha_{i_1\ldots i_q} +$$

$$+\frac{1}{(q-1)!}\,\varepsilon^{kj_2\dots j_q}_{i_1\dots i_q}(\nabla_l\nabla_k\alpha^l_{i_2\dots i_q}-\nabla_k\nabla_l\alpha^l_{i_2\dots i_q}), \tag{2.3}$$

where

$$\varepsilon^{i_1\dots i_r}_{j_1\dots j_r}=\begin{cases} 1 & \text{if} \quad (i_1 ... i_r) \quad \text{is even permutation of} \quad (j_1 ... j_r) \\ -1 & \text{if} \quad (i_1 ... j_r) \quad \text{is odd permutation of} \quad (j_1 ... j_r). \\ 0 & \text{if} \quad (i_1 ... i_r) \quad \text{is not permutation of} \quad (j_1 ... j_r) \end{cases}$$

The formula (2.3), by means of Ricci's formula, becomes

$$\nabla_l\nabla_k\alpha^l_{i_2\dots i_q}-\nabla_k\nabla_l\alpha^l_{i_2\dots i_q}=R^l_{rlk}\alpha^r_{i_2\dots i_q}-$$

$$-\sum_{s=2}^{q}R^r_{i_slk}\alpha^l_{i_2\dots i_{s-1}ri_{s+1}\dots i_q}, \tag{2.4}$$

and after some estimates, takes the form

$$(\Delta\alpha)_{i_1\dots i_q}=-\nabla^k\nabla_k\alpha_{i_1\dots i_q}+\frac{1}{(q-1)!}\,\varepsilon^{kj_2\dots j_q}_{i_1\dots i_q}R_{kl}\alpha^l_{j_2\dots j_q}-$$

$$-\frac{1}{2(q-2)!}\,\varepsilon^{klj_3\dots j_q}_{i_1\dots i_q}R_{klmn}\alpha^{mn}_{j_2\dots j_q}. \tag{2.5}$$

If α is a q-form, then we have

$$\frac{1}{2}\Delta(\mid\alpha\mid^2)=(\alpha,\Delta\alpha)-\mid\nabla\alpha\mid^2-\frac{1}{(q-1)!}L_q(\alpha), \tag{2.6}$$

where $q\geq 2$ and

$$\mid\nabla\alpha\mid^2=\frac{1}{q!}\nabla^k\alpha^{i_1\dots i_q}\,\nabla_k\alpha_{i_1\dots i_q}, \tag{2.7}$$

$$L_q(\alpha)=-(q-1)R_{klmn}\alpha^{kli_3\dots i_q}\,\alpha^{mn}_{j_3\dots i_q}+2R_{kl}\alpha^{ki_2\dots i_q}\alpha^l_{i_2\dots i_q}. \tag{2.8}$$

From (2.8) we can consider L_q as a quadratic form on the vector space $\wedge^q(M,R)$, that is

$$L_q:\wedge^q(M,R)\to R,\quad L_q:\alpha\to L_q(\alpha). \tag{2.9}$$

A q-form α is called killing q-form if its covariant derivative $\nabla\,\alpha$ is a $(q+1)$-form. This in local system $(x_1,...,x_n)$ can be expressed as follows

$$\nabla_j\alpha_{ii_2\dots i_q}+\nabla_i\alpha_{ji_2\dots i_q}=0, \tag{2.10}$$

which is equivalent to

$$q\nabla_j\alpha_{i_1i_2\dots i_q}+\nabla_{i_1}\alpha_{ji_2\dots i_q}+...+\nabla_{i_q}\alpha_{i_1i_2\dots i_{q-1}j}=0. \tag{2.11}$$

If α is a killing q-form, then from (2.11) we obtain

$$\nabla_j \alpha^j_{i_2 \dots i_q} = 0. \tag{2.12}$$

The killing q-form α satisfies the equations

$$q g^{jk} \nabla_k \nabla_j \, \alpha_{i_1 \dots i_q} + \sum_{s}^{1 \dots q} \alpha_{i_1 \dots i_{s-1}} r \, i_{s+1} \dots i_q R^r_{i_s} +$$

$$+ \sum_{s<t}^{1 \dots q} \alpha_{i_1 \dots i_{s-1}} r \, i_{s+1} \dots i_{t-1} \, \mu \, i_{t+1} \dots i_q \, R^{r\mu} \, i_s \, i_t = 0 \tag{2.12}$$

Hence if we consider the second order elliptic differential operator

$$D_q : C^\infty(\wedge^q(M,R)) \to C^\infty(\wedge^q(M,R))$$

$$D_q : \alpha \to D_q \, \alpha,$$

where

$$(D_q \, \alpha)_{i_1 \dots i_q} = q \, g^{jk} \nabla_k \nabla_j \, \alpha_{i_1 \dots i_q} + \sum_{s}^{1 \dots q} \alpha_{i_1 \dots i_{s-1} \, r \, i_{s+1} \dots i_q} \, R^r_{i_s} +$$

$$+ \sum_{s<t}^{1 \dots q} \alpha_{i_1 \dots i_{s-1} \, r \, i_{r+1} \dots i_{t-1} \, \mu \, i_{t+1} \dots i_q} \tag{2.13}$$

Therefore the $Ker(D_q)$ of D_q, that is

$$\mathrm{Ker}(D_q) = \{\alpha \in \wedge^q(M,R)/D_q(\alpha) = 0\}$$

consists of the killing q-forms, whose space is denoted by $K_q(M,R)$, that means $K_q(M,R) = \mathrm{Ker}(D_q)$.

Proposition 2.1.. *There is an isomorphism between the vector spaces $AD_q(M,R)$ and $AD^q(M,R)$, where $AD_q(M,R)$ and $AD^q(M,R)$ are the vector spaces of antisymmetric convariant tensor fields of order q, that is q-forms, and antisymmetric contravariant tensor fields of order q respectively.*

Proof. Let (U,ϕ) be a chart of M with local coordinate system $(x_1,...,x_n)$. If w is a q-form on M, then w has the following components

$$\{w_{i_1 \dots i_q}/1 \le i_1 < i_2 < \dots < i_q \le n\},$$

with respect to the local coordinate system $(x_1,...,x_n)$. We consider the following linear mapping

$$F : AD_q(M,R) = \wedge^q(M,R) \to AD^q(M,R)$$

$$F : w \to F(w)$$

whose component of $F(w)$ with respect to $(x_1, ..., x_n)$ are the following

$$F(w)^{i_1 \ldots j_q} = g^{i_1 j_1} \ldots g^{i_q j_q}\, w_{i_1 \ldots i_q}.$$

It can be easily proved that F is bijective. Therefore the vector spaces $AD_q(M, R)$ and $AD^q(M, R)$ are isomorphic, q.e.d.

Remark 2.2. If w is a killing q-form, then $F(w)$, which is an antisymmetric contravariant tensor field of order q, has the property $\nabla F(w) = 0$. An antisymmetric contravariant tensor field β of order q with the property $\nabla\beta = 0$, is called killing tensor field of order q. Due to isomorphism F we can use the notion killing tensor field of order q instead of killing q-form and conversely.

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The set of killing tensor fields of order q is denoted by $K^q(M, R)$, which is isomorphic onto $K_q(M, R)$.

In this paragraph we shall study the $\dim(K^q(M, R))$ with respect to some properties of the Riemannian metric g on M.

If α is a killing q-form, then

$$(\alpha, \Delta\alpha) - (\Delta\alpha)_{i_1 \ldots i_q} \alpha^{i_1 \ldots i_q}, \tag{3.1}$$

which by means of (2.5) and after some estimates and taking under to consideration (2.8) we obtain

$$(\alpha, \Delta\alpha) = -\frac{(q+1)}{q!} L_q(\alpha). \tag{3.2}$$

The equation (2.6) by means of (3.2) becomes

$$\frac{1}{2}\Delta(\mid \alpha \mid^2) = - \mid \nabla\alpha \mid^2 + \frac{(q+1)}{q!} L_q(\alpha). \tag{3.3}$$

From the second order elliptic differential operator D_q we obtain and endomorphism $(D_q)_x$ of the fibre $\wedge^q(M, R)_x$ in x, that is

$$(D_q)_x : \wedge^q(M, R)_x \to \wedge^q(M, R)_x, \tag{3.4}$$

which satisfies the relation

$$< (D_q)_x\, u, v > = < u, (D_q)_x v >, \quad \forall u, v \in \wedge^q(M, R),$$

where $<>$ is the inner product on $\wedge^q(M, R)_x$ induced by the inner product on $T^\star M$.

Now, we define

$$R(x) = Sup\,\{< (D_q)_x v, v > / v \in \wedge^q(M, R), < v, v >= 1\} \tag{3.5}$$

$$R_{max} = Sup\{R(x)/x \in M\} \tag{3.6.}$$

Now, we shall prove the theorem

Theorem 3.1. *Let (M, g) be a compact Riemannian manifold of dimension n. If $R(x) \leq 0$ and there exists an x_0 such that $R(x_0) < 0$, then $K^q(M, R) = \{0\}$.*

If $R_{max} = 0$, then $dim K^q(M, R) \leq 1 = rank\{\wedge^q(M, R)\}$.

Proof. If we integrate (3.3) on the manifold M, we obtain

$$\int_M \left[- \mid \nabla\alpha \mid^2 + \frac{q+2}{2q!} L_q(\alpha) \right] dM = 0. \tag{3.7}$$

From the inequalities

$$- \mid \nabla\alpha \mid^2 \leq 0 \tag{3.8}$$

and the assumptions that $R(x) \leq 0, \quad \forall x \in M - \{x_0\}$ and $R(x_0) < 0$, which imply

$$L_q(x) \leq 0, \quad \forall x \in M - \{x_0\} \quad \text{and} \quad L_q(x_0) < 0, \tag{3.9}$$

we conclude that

$$\nabla\alpha = 0 \quad \text{and} \quad \alpha/x = 0, \quad \forall x \in M, \tag{3.10}$$

which yields

$$\alpha = 0.$$

This proves that $K^q(M, R) = \{0\}$

If $R_{max} = 0$, then the formula (3.7) implies

$$\int_M [- \mid \nabla\alpha \mid^2] dm + \frac{q+2}{q!} \int_M L_q(\alpha) dM \leq 0 \tag{3.11}$$

which implies $\mid \nabla\alpha \mid = 0$, that means α is a parallel tensor field. Hence every killing tensor field of order q on M is parallel. Since the maximal number of independent parallel killing tensor fields on M is less or equal than the $rank(E)$, where E is the vector bundle of exterior q-forms, then we have

$$dim(K^q(M, R) \leq 1 = rank\{\wedge^q(M, R)\} \quad q.e.d.$$

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