Reflector Spaces Over the 4-Dimensional Kaneyuki-Kozai's Para-Hermitian Symmetric Spaces

P.M. Gadea and L. Hernández Encinas

Abstract

The reflector spaces over the 4-dimensional para-Hermitian symmetric spaces in Kaneyuki-Kozai's classification, are determined.

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1 Introduction

As is well known, the non-simplicity of $SO(4)$ enriches 4-dimensional Riemannian manifolds with some special features. Similarly, the non-simplicity of $SO(2,2)$ endows 4-dimensional neutral manifolds with special characteristics. For instance, in the almost para-Hermitian case (see [2, 3]), Jensen and Rigoli [9] have defined analogs to Riemannian twistors, called *reflectors*. In the case of a Riemannian 4-manifold (see [13, p. 97]), the twistor space can be defined as the set of oriented almost complex structures, and we thus have a bundle with fibre $SO(4)/U(2) \approx S^2$. The *reflector spaces* are defined as the sets of either positively or negatively oriented almost para-Hermitian structures on an oriented neutral 4-manifold. So, analogously to the complex case, we have almost paracomplex spaces of real dimension 6 as total spaces of bundles over such 4-manifolds, with fiber also an almost paracomplex space, specifically one of the homogeneous spaces $SO(2,2)/B_{\pm}(2) \approx S_1^2$, which are the factors of the decomposition of the Grassmannian $SO(2,2)/SO(1,1) \times SO(1,1)$ (see §2).

In the present paper we determine the reflector spaces over the 4-dimensional symmetric spaces corresponding to the symmetric pairs in Kaneyuki-Kozai's table in [10]. (We shall not consider here the 4-dimensional Kaneyuki-Kozai's symmetric spaces obtained as combinations—direct products or not—of two 2-dimensional spaces.) We give as a corollary the homotopy classification of oriented almost para-Hermitian structures on such spaces.

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2 Preliminaries

Definitions 1. An *almost paracomplex manifold* (M, J) is an almost product manifold (i.e. J is an $(1,1)$ tensor field on M with $J^2 = 1$) such that the two eigenbundles T^+M and T^-M associated to the two eigenvalues $+1$ and -1 of J, respectively, have the same rank. An *almost para-Hermitian manifold* (M, g, J) is a differentiable manifold M equipped with an almost paracomplex structure J and a neutral metric g, compatible in the sense that

$$g(JX, Y) + g(X, JY) = 0, \quad X, Y \in \mathcal{X}(\mathcal{M}).$$

For more details on (almost) paracomplex and (almost) para-Hermitian manifolds see [3] and references therein.

On the other hand, let $B_+(2)$ denote the usual almost para-Hermitian group in the 4-dimensional case

$$B_+(2) = \left\{ \begin{pmatrix} A & \\ & {}^tA^{-1} \end{pmatrix} : A \in GL(2,\mathbb{R}) \right\} \subset SL(4,\mathbb{R});$$

and $B_-(2)$ denote its conjugate subgroup [9, p. 430], defined by

$$B_-(2) = \{ A \in SO(2,2) : \ AI_- = I_-A, \ \ I_- = \mathrm{diag}(-1,1,1,-1) \}.$$

The groups $SO(2,2)/B_\pm(2)$ are ([9, p. 430]) diffeomorphic to the Lorentz space form $S_1^2 \approx S^1 \times \mathbb{R}$.

The nonsimplicity of $SO(2,2)$ induces the decomposition of the Grassmannian $G_{1,1}(2,2)$ of oriented neutral planes in $\mathbb{R}^{2,2}$ as

$$\begin{aligned} G_{1,1}(2,2) &\approx SO(2,2)/SO(1,1) \times SO(1,1) \approx \\ &\approx (SO(2,2)/B_+(2)) \times (SO(2,2)/B_-(2)) \approx S_1^2 \times S_1^2. \end{aligned}$$

This fact determined Jensen and Rigoli to define the reflector spaces on an oriented neutral manifold:

Definitions 2 ([9]). The *reflector spaces* $r_\pm : Z_\pm(M) \to M$ on an oriented neutral manifold M are the two bundles of positively (resp. negatively) oriented almost para-Hermitian structures on M, with respective total spaces

$$\begin{aligned} Z_\pm(M) &= \{(p, J) : J \text{ is a para-Hermitian tensor on } (T_pM, g|_p) \text{ of } \pm \text{ orientation}\} \\ &\approx \ \ \mathbf{SO}(M) \times_{SO(2,2)} (SO(2,2)/B_\pm(2)) \ \approx \ \mathbf{SO}(M)/B_\pm(2), \end{aligned}$$

where $\mathbf{SO}(M)$ stands for the total space of the bundle of oriented null frames over M; and the projections are given by $r_\pm(p, J) = p$.

The fact that the groups $SO(2,2)/B_\pm(2)$ are diffeomorphic to the Lorentz space form $S_1^2 \approx S^1 \times \mathbb{R}$, can be easily visualized at the Lie algebra level: in fact, since we have ([7, p. 520]) the isomorphism $\mathfrak{so}(2,2) \approx \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})$, from the Iwasawa decomposition of $\mathfrak{sl}(2,\mathbb{R})$ one has

$$\mathfrak{so}(2,2) \approx \mathfrak{so}(2) + \mathbb{R} + \mathbb{R} + \mathfrak{so}(2) + \mathbb{R} + \mathbb{R}, \tag{1}$$

in such a way that any element of $\mathfrak{so}(2,2)$ can be written as

$$\begin{pmatrix} & -s \\ s & \end{pmatrix} + \begin{pmatrix} a & \\ & -a \end{pmatrix} + \begin{pmatrix} & c \\ & \end{pmatrix} + \begin{pmatrix} & -t \\ t & \end{pmatrix} + \begin{pmatrix} b & \\ & -b \end{pmatrix} + \begin{pmatrix} & d \\ & \end{pmatrix}$$

so that we can write $\mathfrak{so}(2,2) \approx \mathfrak{gl}(2,\mathbb{R}) + \mathfrak{so}(2) + \mathbb{R}$.

3 The reflector spaces

Proposition. *The total spaces and the fibres of the reflector bundles*

$$\begin{array}{ccc} H/K \hookrightarrow & Z_{\pm}(M) \approx G/K & \\ & \downarrow & \\ & M \approx G/H \approx T^*N & \end{array}$$

of either positively or negatively oriented almost para-Hermitian structures over the 4-dimensional Kaneyuki and Kozai's symmetric spaces $M = G/H$ (which are diffeomorphic to the cotangent bundle T^*N of a Riemannian symmetric space N) are diffeomorphic to the spaces G/K and H/K, respectively, given in the two tables:

	H/K	G/K	$M = G/H$	N
1a	$\frac{SL(2,\mathbb{R})}{S(GL_0(1,\mathbb{R})^2)}$	$\frac{SL(3,\mathbb{R})}{S(GL_0(1,\mathbb{R})^3)}$	$\frac{SL(3,\mathbb{R})}{S(GL_0(2,\mathbb{R})\times GL_0(1,\mathbb{R}))}$	S^2
1b	$\frac{SL(2,\mathbb{R})}{S(GL(1,\mathbb{R})^2)} \times \mathbb{Z}_2$	$\frac{SL(3,\mathbb{R})}{S((GL(1,\mathbb{R})^2)_0\times GL_0(1,\mathbb{R}))}$	$\frac{SL(3,\mathbb{R})}{S(GL(2,\mathbb{R})\times GL(1,\mathbb{R}))}$	$P_2(\mathbb{R})$
2	$\frac{GL(1,\mathbb{C})^2}{\triangle GL(1,\mathbb{C})}$	$\frac{GL(2,\mathbb{C})}{\triangle GL(1,\mathbb{C})}$	$\frac{GL(2,\mathbb{C})}{GL(1,\mathbb{C})^2}$	$P_1(\mathbb{C})$
3	$\frac{GL(2,\mathbb{C})}{\rho(SO(3,\mathbb{C}))}$	$\frac{SO(4,\mathbb{C})}{SO(3,\mathbb{C})}$	$\frac{SO(4,\mathbb{C})}{GL(2,\mathbb{C})}$	$\frac{SO(4)}{U(2)}$
4a	$SO(2)\cdot\mathbb{R}^+$	$SO_0(3,1)$	$\frac{SO_0(3,1)}{SO(2)\cdot\mathbb{R}^+}$	$Q_{3,1}(\mathbb{R})$
4b	$\frac{SO_0(1,1)\cdot\mathbb{R}^+}{\mathbb{Z}}$	$\frac{SO_0(2,2)}{\mathbb{Z}}$	$\frac{SO_0(2,2)}{SO_0(1,1)\cdot\mathbb{R}^+}$	$Q_{2,2}(\mathbb{R})$
5	$GL(1,\mathbb{C})$	$SO(3,\mathbb{C})$	$\frac{SO(3,\mathbb{C})}{GL(1,\mathbb{C})}$	$Q_1(\mathbb{C})$
6	$GL(1,\mathbb{C})$	$Sp(1,\mathbb{C})$	$\frac{Sp(1,\mathbb{C})}{GL(1,\mathbb{C})}$	$\frac{Sp(1)}{U(1)}$

Proof. The infinitesimal classification of para-Hermitian symmetric spaces with semisimple group was obtained in [10], from which we have the following table of para-Hermitian symmetric (semi)-simple Lie algebras:

	$(\mathfrak{g},\mathfrak{h})$	M_0^* $(=N)$
1	$(\mathfrak{sl}(m+n,\mathbb{R}),\mathfrak{sl}(m,\mathbb{R})+\mathfrak{sl}(n,\mathbb{R})+\mathbb{R})$	$G_{m,n}(\mathbb{R})$
2	$(\mathfrak{sl}(m+n,\mathbb{C}),\mathfrak{sl}(m,\mathbb{C})+\mathfrak{sl}(n,\mathbb{C})+\mathbb{C})$	$G_{m,n}(\mathbb{C})$
3	$(\mathfrak{su}^*(2m+2n),\mathfrak{su}^*(2m)+\mathfrak{su}^*(2n)+\mathbb{R})$	$G_{m,n}(\mathbb{H})$
4	$(\mathfrak{su}(n,n),\mathfrak{sl}(n,\mathbb{C})+\mathbb{R})$	$U(n)$
5	$(\mathfrak{so}(n,n),\mathfrak{sl}(n,\mathbb{R})+\mathbb{R})$	$SO(n)$
6	$(\mathfrak{so}^*(4n),\mathfrak{su}^*(2n)+\mathbb{R})$	$U(2n)/Sp(n)$
7	$(\mathfrak{so}(2n,\mathbb{C}),\mathfrak{sl}(n,\mathbb{C})+\mathbb{C})$	$SO(2n)/U(n)$
8	$(\mathfrak{so}(m+1,n+1),\mathfrak{so}(m,n)+\mathbb{R})$	$Q_{m+1,n+1}(\mathbb{R})$
9	$(\mathfrak{so}(n+2,\mathbb{C}),\mathfrak{so}(n,\mathbb{C})+\mathbb{C})$	$Q_n(\mathbb{C})$
10	$(\mathfrak{sp}(n,\mathbb{R}),\mathfrak{sl}(n,\mathbb{R})+\mathbb{R})$	$U(n)/O(n)$
11	$(\mathfrak{sp}(n,n),\mathfrak{su}^*(2n)+\mathbb{R})$	$Sp(n)$
12	$(\mathfrak{sp}(n,\mathbb{C}),\mathfrak{sl}(n,\mathbb{C})+\mathbb{C})$	$Sp(n)/U(n)$
13	$(E_6^1,\mathfrak{so}(5,5)+\mathbb{R})$	$G_{2,2}(\mathbb{H})/\mathbb{Z}_2$
14	$(E_6^4,\mathfrak{so}(1,9)+\mathbb{R})$	$P_2(\mathbb{O})$
15	$(E_6^{\mathbb{C}},\mathfrak{so}(10,\mathbb{C})+\mathbb{C})$	$E_6/Spin(10)\times T^1$
16	$(E_7^1,E_6^1+\mathbb{R})$	$SU(8)/Sp(4)\times\mathbb{Z}_2$
17	$(E_7^3,E_6^4+\mathbb{R})$	$T^1\times E_6/F_4$
18	$(E_7^{\mathbb{C}},E_6^{\mathbb{C}}+\mathbb{C})$	$E_7/E_6\times T^1$

In the above list, $G_{m,n}(\mathbb{F})$ denotes the Grassmann manifold of m-planes in $\mathbb{F}^{m+n}$, where $F=\mathbb{R},\mathbb{C}$ or $\mathbb{H}$. $Q_{m,n}(\mathbb{R})$ denotes the real quadric in $P_{m+n-1}(\mathbb{R})$ defined by the quadratic form of signature (m,n). $Q_n(\mathbb{C})$ denotes the complex quadric in $P_{n+1}(\mathbb{C})$. $P_2(\mathbb{O})$ denotes the octonion projective plane. The list on the right of the table contains those Riemannian symmetric spaces M_0^* (which we have denoted in the present paper by N) with the property that if $M=G/H$ is a para-Hermitian symmetric space corresponding to the symmetric pair $(\mathfrak{g},\mathfrak{h})$ associated to the specific M_0^*, then M is diffeomorphic to the cotangent bundle T^*M_0 of a covering manifold M_0 of M_0^*.

It is immediate that the respective (real) dimensions of the spaces G/H are:

1) $2mn$	2) $4mn$	3) $8mn$	4) $2n^2$	5) $n(n-1)$	6) $2n(2n-1)$
7) $2n(n-1)$	8) $2(m+n)$	9) $4n$	10) $n(n+1)$	11) $2n(2n+1)$	12) $2n(n+1)$
13) 32	14) 32	15) 64	16) 54	17) 52	18) 108.

Thus, as it is easily seen, the only 4-dimensional Kaneyuki-Kozai's symmetric spaces, corresponding to symmetric pairs in the table, are the eight spaces $M=G/H$ appearing in the fourth column of the table of the assertion in the Proposition, which we have numbered **1a** to **6**. Each such $M=G/H$ is diffeomorphic to the cotangent bundle of the corresponding 2-dimensional Riemannian symmetric space N appearing in the fifth column of that table. Since they are almost para-Hermitian manifolds, all of them are orientable and we can define their reflector spaces, which:

(i) are in fact bundles because of their structure $H/K\hookrightarrow G/K\to G/H$;

(ii) have as total space a space of real dimension 6;

(iii) have fibre diffeomorphic to S_1^2 (hence diffeomorphic to $SO(2,2)/B_+(2)$ and $SO(2,2)/B_-(2)$), so the fibre is almost paracomplex.

Now, we consider the six cases in the Proposition. The first case has two subcases, with base spaces $G/H \approx GL_0(3,\mathbb{R})/GL_0(2,\mathbb{R}) \times GL_0(1,\mathbb{R})$ (the 0 denoting the identity component) and $G/H \approx SL(3,\mathbb{R})/S(GL(2,\mathbb{R}) \times GL(1,\mathbb{R}))$, which are respectively diffeomorphic to the paracomplex projective model $P_2(\mathbb{B}) \approx T^*S^2$ and to the reduced paracomplex projective model $P_2(\mathbb{B})/\mathbb{Z}_2 \approx T^*P_2(\mathbb{R})$ (see for instance [5, 3]). The respective reflector spaces have the flag spaces in the table as total spaces.

As for the fibre in **1a**, from the Iwasawa decomposition of $SL(2,\mathbb{R}) = KAN$ we have:

$$\frac{SL(2,\mathbb{R})}{S(GL_0(1,\mathbb{R})^2)} \approx (KAN)/A \approx KN \approx SO(2) \times \mathbb{R} \approx S_1^2\,;$$

and for **1b**:

$$\frac{GL(2,\mathbb{R}) \times GL(1,\mathbb{R})}{(GL(1,\mathbb{R})^2)_0 \times GL_0(1,\mathbb{R})} \approx \frac{SL(2,\mathbb{R})}{S((GL(1,\mathbb{R})^2)_0)} \times \mathbb{Z}_2 \approx SO(2) \times \mathbb{R}\,.$$

Note that we have bundles with fibre a Lorentzian space form and with base space a para-Kähler space form [4].

In the case **2**, the base space is $GL(2,\mathbb{C})/GL(1,\mathbb{C}) \times GL(1,\mathbb{C}) \approx T^*P_1(\mathbb{C})$. Considering the usual diffeomorphism in terms of the diagonal $\triangle GL(1,\mathbb{C})$, i.e. $GL(1,\mathbb{C}) \approx (GL(1,\mathbb{C}) \times GL(1,\mathbb{C}))/\triangle GL(1,\mathbb{C})$, we have the reflector space as in the table, with fibre $(GL(1,\mathbb{C}) \times GL(1,\mathbb{C}))/\triangle GL(1,\mathbb{C}) \approx GL(1,\mathbb{C}) \approx U(1) \times \mathbb{R} \approx S_1^2$.

In the third case, the base space is $SO(4,\mathbb{C})/GL(2,\mathbb{C}) \approx T^*(SO(4)/U(2)) \approx T^*S^2$ (the last diffeomorphism by [14, p. 215]). We first consider [11, p. 893]:

$$\mathrm{gl}(2,C) = \left\{ \begin{pmatrix} \mathcal{O} & \mathcal{S} \\ \mathcal{S} & \mathcal{O} \end{pmatrix} : \mathcal{O} \in \ \mathrm{so}\ (\in,\mathcal{C}),\ \mathcal{S} \in \ \mathrm{sym}\ (\in,\mathcal{C}) \right\} =$$

$$= \left\{ \begin{pmatrix} & a+bi & p+qi & r+si \\ -a-bi & & r+si & x+yi \\ p+qi & r+si & & a+bi \\ r+si & x+yi & -a-bi & \end{pmatrix} \right\}.$$

Now, it is immediate that the subset $\mathfrak{h}$ of elements obtained deleting the last row and the last column in the above expression is a Lie subalgebra of $\mathfrak{gl}(2,\mathbb{C})$. We have

$$h \approx \left\{ \begin{pmatrix} & a & b \\ -a & & r \\ p & r & \end{pmatrix} + i \begin{pmatrix} & b & q \\ -b & & s \\ q & s & \end{pmatrix} \right\},$$

that is, $h \approx \mathfrak{so}(2,1) + i\,\mathfrak{so}(2,1)$, but we know (see for instance [1, p. 32]) that $\mathfrak{so}(2,1)$ is a real form of $\mathfrak{so}(3,\mathbb{C})$, so we have $\mathfrak{h} \approx \rho_*(\mathfrak{so}(3,\mathbb{C}))$, where ρ_* is ([12, p. 53]) the differential of an analytic representation ρ of $SO(3,\mathbb{C})$ in $GL(2,\mathbb{C})$, and the reflector space is thus the one given in the table, where ρ denotes the given representation of $SO(3,\mathbb{C})$. This reflector has fibre

$$GL(2,\mathbb{C})/\rho(SO(3,\mathbb{C})) \approx \{x+yi \in \mathbb{C}^*\} \approx GL(1,\mathbb{C}) \approx U(1) \times \mathbb{R} \approx S_1^2\,,$$

as it must be.

The fourth case also has two subcases. In the case **4a**, the base space is $SO_0(3,1)/SO(2)\cdot \mathbb{R}^+ \approx T^*Q_{3,1}(\mathbb{R})$, the cotangent bundle of the subspace of $P_3(\mathbb{R})$ obtained under the identification $(x,y) \sim (-x,-y)$ in $S^2 \times S^0$, that is, S^2. In fact, the space $Q_{3,1}(\mathbb{R})$ in Kaneyuki-Kozai's notation is the space $Q_{n,\nu}(\mathbb{R}) = Q_{4,1}(\mathbb{R})$ in Takeuchi's notation ([15, pp. 144–8]), which is the quadric $x_0^2 - x_1^2 - x_2^2 - x_3^2 = 0$ in $P_3(\mathbb{R})$. As for the sense of the dot in $SO(2) \cdot \mathbb{R}^+$ see [11, p. 893]. The total space of its reflector bundle is the identity component $SO_0(3,1)$ of the homogeneous Lorentz group, and the fibre H/K is thus $SO(2) \cdot \mathbb{R}^+ \approx S_1^2$.

In the case **4b**, the base space is $SO_0(2,2)/SO_0(1,1) \cdot \mathbb{R}^+ \approx T^*Q_{2,2}(\mathbb{R})$, the cotangent bundle of the sphere bundle in $P_3(\mathbb{R})$ obtained under the identification $(x,y) \sim (-x,-y)$ in $S^1 \times S^1$, which is a new $S^1 \times S^1$. In fact, the space $Q_{2,2}$ in Kaneyuki-Kozai's notation is the space $Q_{n,\nu}(\mathbb{R}) = Q_{4,2}(\mathbb{R})$ in Takeuchi's notation ([15, pp. 144–8]), which is the quadric $x_0^2 + x_1^2 - x_2^2 - x_3^2 = 0$ in $P_3(\mathbb{R})$.

In order to obtain its reflector spaces, we consider the Lie algebra $\mathfrak{so}(2,2)'$ isomorphic to $\mathfrak{so}(2,2)$, given by

$$\mathfrak{so}(2,2)' = \mathfrak{so}(2) + \mathbb{R} + \mathbb{R} + \mathfrak{so}(2) + \mathbb{R} + i\mathbb{R}\,,$$

with elements

$$\begin{pmatrix} & -s \\ s & \end{pmatrix} + \begin{pmatrix} a & \\ & -a \end{pmatrix} + \begin{pmatrix} & c \\ & \end{pmatrix} + \begin{pmatrix} & -t \\ t & \end{pmatrix} + \begin{pmatrix} b & \\ & -b \end{pmatrix} + \begin{pmatrix} & -d \\ d & \end{pmatrix}$$

(see the decomposition 1), and its corresponding Lie group, which we shall denote by $SO_0(2,2)/\mathbb{Z}$, and write in terms of the Iwasawa decomposition as

$$SO_0(2,2)/\mathbb{Z} \approx SL(2,\mathbb{R}) \times (SL(2,\mathbb{R})/\mathbb{Z}) \approx KAN \times KAK\,.$$

The space $SO_0(2,2)/\mathbb{Z}$ so defined is diffeomorphic to both total spaces of the reflector bundles over $SO_0(2,2)/SO_0(1,1) \cdot \mathbb{R}^+$, since it has real dimension 6, and the fibre is $(SO_0(1,1) \cdot \mathbb{R}^+)/\mathbb{Z} \approx S_1^2$.

Finally, in the fifth and sixth cases, with $SO(3,\mathbb{C})/GL(1,\mathbb{C}) \approx T^*Q_1(\mathbb{C})$ and $Sp(1,\mathbb{C})/GL(1,\mathbb{C}) \approx T^*(Sp(1)/U(1))$ as respective base spaces, we have the reflector spaces appearing in the table, both with fibre $GL(1,\mathbb{C}) \approx S_1^2$.

Notice that in the 2nd, 3rd, 5th and 6th cases we are considering the cotangent bundles of the Hermitian symmetric spaces $P_1(\mathbb{C})$, $SO(4)/U(2)$, $Q_1(\mathbb{C})$ and $Sp(1)/U(1)$, respectively.

Corollary 2. *The 4-dimensional Kaneyuki-Kozai's para-Hermitian symmetric spaces have only a homotopy class of either positively or negatively oriented almost para-Hermitian structures.*

Proof. The set of positively oriented (resp. negatively oriented) almost para-Hermitian structures on each space $M = G/H$ in the table in the Proposition, coincides with the set of differentiable sections of the corresponding reflector space. The homotopy classes of such sections are a subset of the whole set of homotopy classes $[G/H, G/K]$. To compute the last sets of classes we shall determine the homotopy type (h.t.) of each of the total spaces G/K and base spaces G/H. We have the table

	topology of G/K	h.t. of G/H		topology of G/K	h.t. of G/H
1a	$SO(3) \times \mathbb{R}^3$	S^2	**4a**	$(SU(2)/\mathbb{Z}_2) \times \mathbb{R}^3$	S^2
1b	$(SO(3) \cdot N)/\mathbb{Z}_2$	$P_2(\mathbb{R})$	**4b**	$S^1 \times S^1 \times S^1 \times \mathbb{R}^3$	$S^1 \times S^1$
2	$SU(2) \times \mathbb{R}^3$	S^2	**5**	$SU(2) \times \mathbb{R}^3$	S^2
3	$SU(2) \times \mathbb{R}^3$	S^2	**6**	$SU(2) \times \mathbb{R}^3$	S^2

In fact, the column at the right is obtained from the fact $M \approx T^*N$ as in the table in the Proposition, and the homeomorphisms $SO(4)/U(2) \approx Sp(1)/U(1) \approx S^2$. As for the column at the left, we have:

1a and **1b**) Let $SL(3,\mathbb{R}) = SO(3) \cdot AN$ denote the Iwasawa decomposition of $SL(3,\mathbb{R})$. It is immediate that

$$SL(3,\mathbb{R})/S(GL_0(1,\mathbb{R})^3) \approx (SO(3) \cdot AN)/A \approx SO(3) \cdot N \approx SO(3) \cdot \mathbb{R}^3 \,.$$

Similarly

$$\frac{SL(3,\mathbb{R})}{S((GL(1,\mathbb{R})^2)_0 \times GL_0(1,\mathbb{R}))} \approx (SO(3) \cdot N)/\mathbb{Z}_2 \,,$$

$SO(3) \cdot N$ as in **1a** and $\mathbb{Z}_2$ being the subgroup $\{\mathrm{diag}(1,1,1) \cup \mathrm{diag}(-1,-1,1)\}$ of $SL(3,\mathbb{R})$.

2) We have $G/K \approx (U(2) \cdot \mathbb{R}^4)/(U(1) \cdot \mathbb{R}) \approx SU(2) \cdot \mathbb{R}^3$.

3, **5** and **6**) They follow from the isomorphisms $SO(4,\mathbb{C}) \approx SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ and $SO(3,\mathbb{C}) \approx Sp(1,\mathbb{C}) \approx SL(2,\mathbb{C}) \approx SU(2) \cdot \mathbb{R}^3$.

4a) It follows from $SO_0(3,1) \approx SL(2,\mathbb{C})/\mathbb{Z}_2 \approx (SU(2)/\mathbb{Z}_2) \cdot \mathbb{R}^3$, where $\mathbb{Z}_2$ denotes the center of $SU(2)$ (see [1, pp. 108, 513–515]).

Hence, the homotopy type of G/K is that of $SO(3)$ for **1a**, that of $(SO(3) \cdot N)/\mathbb{Z}_2$ for **1b**, that of $SU(2)$ for **2**, **3**, **5** and **6**, that of $SU(2)/\mathbb{Z}_2$ for **4a** and that of $S^1 \times S^1 \times S^1$ for **4b**. Consequently, we have:

For **1a**: $[G/H, G/K] \approx \pi_2(SO(3)) = 0$;

For **1b**: $[G/H, G/K] \approx [P_2(\mathbb{R}), (SO(3) \cdot N)/\mathbb{Z}_2] \subset \pi_2(SO(3)) = 0$;

For **2**, **3**, **5** and **6**: $[G/H, G/K] \approx \pi_2(SU(2)) = 0$;

For **4a**: $[G/H, G/K] \approx [S^2, SU(2)/\mathbb{Z}_2] = \pi_2(P_3(\mathbb{R})) = 0$ (see [8, p. 105]).

For **4b** one has $[G/H, G/K] \approx [S^1 \times S^1, S^1 \times S^1 \times S^1]$. Now, according to [8, p. 4–5], given the topological spaces X, Y and Z, the C^0 map

$$\theta \colon Map\,(Z \times X, Y) \to Map\,(Z, Map\,(X,Y))\,,$$

which assigns to $f(z,x)$ the C^0 map $Z \to Map\,(X,Y)$, where the image of $z \in Z$ is the map $x \mapsto f(z,x)$, is a homeomorphism onto its image set if the spaces are Hausdorff, and it is a bijective map if X is locally compact. Accordingly, denoting by ΩX the space of loops of a given space X, we have, applying also [8, Th. 3.6, p. 6]: Map $(\mathrm{S}^1 \times S^1, S^1 \times S^1 \times S^1) = (Map\,(S^1 \times S^1, S^1))^3$ $\approx (Map\,(S^1, Map\,(S^1,S^1)))^3 = (Map\,(S^1, \Omega(S^1))^3 \approx (Map\,(S^2,S^1))^3$ so that

$$[S^1 \times S^1, S^1 \times S^1 \times S^1] \approx (\pi_2(S^1))^3 = 0\,.$$

Finally, given two differentiable manifolds M and N, any continuous map from M to N is homotopic to a differentiable map ([6, p. 64, Prop. 4.6]; and, moreover, two differentiable maps which are homotopic by a continuous homotopy are in fact homotopic by a differentiable homotopy ([6, p. 67, Prop. 4.11]. So, we have the above results in this proof for differentiable maps, thus concluding.

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P. M. Gadea
Institute of Mathematics and Fundamental Physics, CSIC,
Serrano 123, 28006-Madrid, Spain.
e-mail: pmgadea@gugu.usal.es

L. Hernández Encinas
Faculty of Education, University of Salamanca,
Paseo de Canalejas 169, 37008-Salamanca, Spain.
e-mail: encinas@gugu.usal.es