Finsler Spaces Admitting a Parallel Vector Field

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Abstract

In this paper we modify the fundamental function of a Finsler space with the help of a parallel Finsler vector field, getting a new Finsler space whose properties are investigated.

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The terminology and notations are referred to Matsumoto's monograph [5].

Let F^n be an n–dimensional Finsler space with a fundamental function $L(x,y)$ $(y=\dot{x})$, and we shall introduce in F^n the Cartan connection $C\Gamma=(F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$.

Let us consider a vector field $X^i(x)$ in F^n: this field is called *parallel*, if it satisfies the partial differential equations

$$X^i{}_{|j} := \partial_j X^i - N^h{}_j \dot{\partial}_h X^i + X^h F_h{}^i{}_j = \partial_j X^i + X^h F_h{}^i{}_j = 0, \tag{1}$$

$$X^i\big|_j := \dot{\partial}_j X^i + X^h C_h{}^i{}_j = X^h C_h{}^i{}_j = 0, \tag{2}$$

where ∂_j and $\dot{\partial}_j$ denote partial differentiations by x^j and y^j, respectively.

From the Ricci identities, the following integrability conditions hold:

$$X^h R_{hijk} = 0, \tag{3}$$

$$X^h P_{hijk} = 0, \tag{4}$$

$$X^h S_{hijk} = 0, \tag{5}$$

where R_{hijk}, P_{hijk} and S_{hijk} are the components of the curvature tensors of $C\Gamma$.

Remark. In particular, the above equations (1) and (2) are satisfied for a stationary vector field $X^i(x)$.

In terms of covariant components $X_i(x)$'s, (1) and (2) are written as

$$X_{i|j} = 0, \tag{1$'$}$$

$$X_i\big|_j = 0. \tag{2$'$}$$

Here we shall consider the modification of a Finsler metric by a parallel vector field $X^i(x)$, as follows. Putting

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$$ {}^*L^2 = L^2 + \beta^2 \quad (\beta = X_i(x)y^i \neq 0), \tag{6} $$

${}^*L$ defines a new Finsler metric of M. It is said that ${}^*L$ is obtained by a β–change of the metric L [6]. The metric tensor derived from ${}^*L$ is written as follows:

$$ {}^*g_{ij} = g_{ij} + X_iX_j, \; {}^*g^{ij} = g^{ij} - X^iX^j/(1+X^2), \tag{7} $$

where X is the length of X^i with respect to the original metric. The coefficients of Cartan's connection are written in the forms

$$ {}^*N^i{}_j = N^i{}_j, \; {}^*F_j{}^i{}_k = F_j{}^i{}_k, \tag{8} $$

and we have ${}^*F_{ijk} = (\delta^h{}_j + X_jX^h)F_{ihk}$.

From (7) we immediately get

$$ {}^*C_{ijk} = C_{ijk}, \; {}^*C_j{}^i{}_k = C_j{}^i{}_k. \tag{9} $$

As a consequence, (8) and (9) yield

Proposition 1. *If a Finsler space with a fundamental function L admits a parallel vector field X^i, then the vector field X^i is parallel with respect to the modified metric (6).*

In view of (9), we can conclude immediately

$$ {}^*S_{hijk} = S_{hijk}. \tag{10} $$

The components of the other two curvature tensors are

$$ {}^*R_{hijk} = R_{hijk}, \quad {}^*R_h{}^i{}_{jk} = R_h{}^i{}_{jk}, \tag{11} $$

$$ {}^*P_{hijk} = P_{hijk}, \quad {}^*P_h{}^i{}_{jk} = P_h{}^i{}_{jk}, \tag{12} $$

For the later use, we shall show here two lemmas.

Lemma 1. ([6]) *The covariant vector $m_i(:= X_i - \beta y_i/L^2)$ is a non–zero vector orthogonal to y^i.*

Proof. Assuming that $m_i = 0$, we have $L^2X_i - \beta y_i = 0$. Differentiating this by y^j and denoting $\dot{\partial}_iL(= y_i/L)$ by ℓ_i, we are led to a contradiction $h_{ij}(:= g_{ij} - \ell_i\ell_j) = 0$.

Lemma 2. *For the angular metric tensor h_{ij} and the covariant vector m_i, we have*

$$ h_{ij}X^j = m_i \quad (\neq 0), \tag{13} $$

$$ m_iX^i = m^2 \quad (\neq 0), \tag{14} $$

where $m^2 = g_{ij}m^im^j$ and $m^i = g^{ij}m_j$.

Proof. Assuming that $m^2 = 0$, we get $L^2X^2 - \beta^2 = 0$. Then $\dot{\partial}_j\dot{\partial}_i(L^2X^2 - \beta^2) = 0$ gives us $g_{ij} = X_iX_j/X^2$,which contradicts $rank(g_{ij}) = n$.

Remark. It follows from $\beta \neq 0$ that the covariant vectors m_i and X_i are non–zero vectors.

Now, we shall consider the T–condition

$$T_{hijk} := LC_{hij}\big|_k + \ell_h C_{ijk} + \ell_i C_{hjk} + \ell_j C_{hik} + \ell_k C_{hij} = 0, \tag{15}$$

where the T–tensor T_{hijk} is completely symmetric.

If we contract (15) by X^h, we have $X^h \ell_h C_{ijk} = 0$ in virtue of (2). But from $X^h \ell_h = \beta/L \neq 0$, we find $C_{ijk} = 0$. Consequently, from (9) we find

Theorem 1. *If a Finsler space F^n satisfying* (15) *admits a parallel vector field, then both the Finsler spaces F^n and ${}^*F^n$ are Riemannian.*

The generalized T–condition is defined by

$$T_{ij} := T_{ijrs} g^{rs} = LC_i\big|_j + \ell_i C_j + \ell_j C_i = 0, \tag{16}$$

where the tensor T_{ij} is called the *contracted T–tensor* and $C_i = C_{ijk} g^{jk}$ being the torsion vector.

Contracting (16) by X^i and using (2), we have $C_j = 0$. According to Deicke's theorem and (9), we have

Theorem 2. *If a Finsler space F^n satisfying* (16) *admits a parallel vector field, then both the Finsler spaces F^n and ${}^*F^n$ are Riemannian.*

We are concerned with a space of scalar curvature in Berwald's sense. It is characterized by the equation

$$R_{i0k} (= R_{ijk} y^j) = L^2 K h_{ik}, \tag{17}$$

or

$$R_{ijk} = h_{ik} K_j - h_{ij} K_k, \quad K_j = L^2 \dot{\partial}_j K/3 + LK\ell_j, \tag{18}$$

where h_{ik} is the angular metric tensor and the scalar curvature K is a Finsler scalar field.

Contracting (17) by X^i and using (3) and (13), we obtain

Proposition 2. *If a Finsler space F^n of scalar curvature K admits a parallel vector field, then the scalar curvature K vanishes.*

From Proposition 2 and (18), we immediately get $R_{ijk} = 0$. In terms of (9) and (11), the Berwald's curvature tensor $H_{hijk} (:= \dot{\partial}_h R_{ijk} - 2C_{ihr} R^r{}_{jk})$ coincides with ${}^*H_{hijk}$, so we find

Theorem 3. *If a Finsler space F^n of scalar curvature K admits a parallel vector field, then both Berwald's curvature tensors H_{hijk} and ${}^*H_{hijk}$ vanish.*

A Finsler space F^n $(n > 2)$ is called quasi–C–reducible, if the torsion tensor C_{ijk} is written as

$$C_{ijk} = A_{ij} C_k + A_{jk} C_i + A_{ki} C_j, \tag{19}$$

where A_{ij} is a symmetric Finsler tensor field satisfying $A_{i0}(:= A_{ij}y^j) = 0$.

Contracting (19) by X^iX^j and using (2), we immediately get $\lambda C_k = 0$, where $\lambda = A_{ij}X^iX^j$. Therefore, taking into account (9) we have

Proposition 3. *If a quasi–C–reducible Finsler space F^n $(n > 2)$ admits a parallel vector field, then both the Finsler spaces F^n and ${}^*F^n$ are Riemannian, provided that $\lambda(= A_{ij}X^iX^j) \neq 0$.*

The condition we consider next is the C–reducibility. A Finsler space F^n $(n > 2)$ which satisfies the equation

$$(n+1)C_{ijk} = h_{ij}C_k + h_{jk}C_i + h_{ki}C_j, \tag{20}$$

is said to be *C–reducible.*

Contracting (20) by X^iX^j and using (2), we obtain $h_{ij}X^iX^jC_k = 0$. Paying attention to Lemma 2, we get $C_k = 0$. Thus, taking into account of (9) we have

Theorem 4. *If a C–reducible Finsler space F^n $(n > 2)$ admits a parallel vector field, then both the Finsler spaces F^n and ${}^*F^n$ are Riemannian.*

A Finsler space F^n $(n > 2)$ with non–zero length C of the torsion vector C^i is called semi–C–reducible, if the torsion tensor C_{ijk} is of the form

$$C_{ijk} = p(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j)/(n+1) + qC_iC_jC_k/C^2, \tag{21}$$

where $C^2 = g^{ij}C_iC_j$ and $p + q = 1$.

As the exceptional case $(p = 0)$ of semi–C–reducibility, we are led to the following definition:

A Finsler space F^n $(n \geq 2)$ with $C^2 \neq 0$ is called *C2–like*, if the torsion tensor C_{ijk} is written in the form

$$C_{ijk} = C_iC_jC_k/C^2. \tag{22}$$

Contracting (21) by X^iX^j and using (2) and Lemma 2, we get $pC_k = 0$. Then we obtain $p = 0$ because of $C^2 \neq 0$. In virtue of $p + q = 1$, we have $q = 1$. Consequently, taking into account of (9), we obtain

Theorem 5. *If a semi–C–reducible Finsler space F^n $(n > 2)$ admits a parallel vector field, then both the Finsler spaces F^n and ${}^*F^n$ are C2–like.*

A Finsler space F^n $(n \geq 2)$ will be called C^h–recurrent, if the torsion tensor C_{ijk} satisfies the equation

$$C_{ijk|\ell} = C_{ijk}K_l, \tag{23}$$

where $K_\ell = K_\ell(x, y)$ is a covariant vector field.

The following expressions are well–known,

$$P_{hijk} = \mathcal{U}_{(hi)}\{C_{ijk|h} + C_{hjr}C_i{}^r{}_{k|0}\}, \tag{24}$$

$$P_{ijk} = C_{ijk|0}, \tag{25}$$

$$S_{hijk} = \mathcal{U}_{(jk)}\{C_{hkr}C_i{}^r{}_j\}, \tag{26}$$

where the index 0 means contraction by y^i and the notation $\mathcal{U}_{(hi)}$ denotes the interchange of indices h, i and subtraction.

Contracting (24) by X^h and using (1),(2) and (4), we have

Lemma 3. *For a torsion tensor C_{ijk} and a parallel vector field X^h, we have*

$$X^h C_{ijk|h} = 0. \tag{27}$$

Contracting (23) by X^ℓ and using (27) and (9), we obtain

Proposition 4. *If a C^h–recurrent Finsler space F^n $(n > 2)$ admits a parallel vector field, then both the Finsler spaces F^n and ${}^*F^n$ are Riemannian, provided that $\mu(= X^\ell K_\ell) \neq 0$.*

Diferentiating (26) h-covariantly we obtain

$$S_{hijk|\ell} = \mathcal{U}_{(jk)}\{C_{hkr|\ell}C_i{}^r{}_j + C_{hkr}C^r_{ij|\ell}\}. \tag{28}$$

Contracting this by X^ℓ and using (27), we get $X^\ell S_{hijk|\ell} = 0$.

A *P2–like Finsler space* F^n $(n > 2)$ is characterized by

$$P_{hijk} = K_h C_{ijk} - K_i C_{hjk}, \tag{29}$$

where $K_h = K_h(x, y)$ is a covariant vector field.

Contracting this by X^h and using (2) and (4) we get $X^h K_h C_{ijk} = 0$. Therefore, taking into acount (9) we have

Theorem 6. *If a P2-like Finsler space F^n $(n > 2)$ admits a parallel vector field, then both the Finsler spaces F^n and $*F^n$ are Riemannian providing that $\nu(= X^h K_h) \neq 0$.*

A Landsberg space is characterized by $P_{ijk}(= C_{ijk|0}) = 0$. Further, a Finsler space is called P–reducible, if the torsion tensor P_{ijk} is written as

$$P_{ijk} = (h_{ij}P_k + h_{jk}P_i + h_{ki}P_j)/(n + 1), \tag{30}$$

where $P_i = P^r{}_{ir} = C_{i|0}$.

Contracting (30) by $X^i X^j$ and using (25),(1) and (2), we obtain $X^i X^j h_{ij} P_k = 0$. From Lemma 2, we get $P_k = 0$. Thus, taking into account (12), we have

Theorem 7. *If a P–reducible Finsler space F^n admits a parallel vector field, then both the Finsler spaces F^n and ${}^*F^n$ are Landsberg spaces.*

A Finsler space F^n $(n > 3)$ is called *S3–like* if the curvature tensor S_{hijk} is written in the form

$$L^2 S_{hijk} = S(h_{hj}h_{ik} - h_{hk}h_{ij}), \tag{31}$$

where the scalar curvature $S(= S_{hijk}g^{hj}g^{ik})$ is a function of position alone.

Contracting (31) by $X^h g^{ik}$ and using (5), (13) and Lemma 1, we get $S = 0$. Therefore, taking into account (10), we have

Theorem 8. *If an $S3$–like Finsler space F^n $(n > 3)$ admits a parallel vector field, then both the curvature tensors S_{hijk} and ${}^*S_{hijk}$ vanish.*

Similar to the case of the $S3$–likeness, we are concerned with the following:

A Finsler space F^n $(n > 4)$ is called *$S4$–like* if the curvature tensor S_{hijk} is written in the form

$$L^2 S_{hijk} = h_{hj} M_{ik} + h_{ik} M_{hj} - h_{hk} M_{ij} - h_{ij} M_{hk}, \tag{32}$$

where M_{ij} is a symmetric and indicatory tensor. Then the tensor M_{ij} of the above definition is given by

$$M_{hi} = L^2\{S_{hi} - Sh_{hi}/2(n-2)\}/(n-3), \tag{33}$$

where $S_{hi} = S_h{}^r{}_{ir}$.

Here we shall prove a lemma.

Lemma 4. *For the angular metric tensors we have*

$$ {}^*h_{ij} = h_{ij} + \tau m_i m_j \qquad (\tau = L^2/{}^*L^2), \tag{34}$$

$$ h_{ij} = M_{ij}/\nu + 2m_i m_j/m^2 \ (\nu = L^2 S/2(n-2)(n-3)). \tag{35}$$

Proof. Contracting (32) and (33) by X^h and using (5) and Lemma 2, we obtain $\nu(h_{ik}m_j - h_{ij}m_k) = M_{ik}m_j - M_{ij}m_k$. Further contracting this by X^j and using Lemma 2, we get (35).

From (5), (7), (10), (33) and (34), we have

$$ {}^*S_{hi} = S_{hi},\ {}^*S = S,\ {}^*M_{hi} = M_{hi}/\tau - \nu m_h m_i.$$

Thus, using (10) and Lemma 4, we obtain

Theorem 9. *If an $S4$–like Finsler space $(n > 4)$ F^n admits a parallel vector field X^i, then the modified metric (6) is also $S4$–like.*

A Finsler space F^n $(n > 2)$ is said to be of *h–isotropic*, if the curvature tensor R_{hijk} is written as

$$R_{hijk} = K(g_{hj}g_{ik} - g_{hk}g_{ij}), \tag{36}$$

where K is a Finsler scalar. In 1961 Akbar–Zadeh proves that K is a constant.

Contracting (36) by $X^h m^j$ and using (3) and (14), we obtain $K(m^2 g_{ik} - m_i X_k) = 0$. Further, contracting this by g^{ik}, we have $K = 0$. Hence, taking into account (11), we get

Theorem 10. *If an h–isotropic Finsler space F^n $(n > 2)$ admits a parallel vector field, then both the curvature tensors R_{hijk} and ${}^*R_{hijk}$ vanish.*

Next, we shall be concerned with the notion of an $R3$–like Finsler space which is defined by the following:

A Finsler space F^n $(n > 3)$ is called *R3–like*, if the curvature tensor R_{hijk} is written in the form

$$R_{hijk} = g_{hj}L_{ik} + g_{ik}L_{hj} - g_{hk}L_{ij} - g_{ij}L_{hk}. \tag{37}$$

It follows from (37) that $R_{ik} = g^{hj}R_{hijk} = (n-2)L_{ik} + Lg_{ik}$, where $L = L_{ij}g^{ij}$ and $R = g^{ik}R_{ik} = 2(n-1)L$. Thus we obtain L_{ik} depending on R_{ik} and R. Since R_{ik} are not symmetric in general, so are L_{ik}. Further, we have to calculate $^*R_{ik}$ and $^*R$. Using (3),(7) and (11) the results are as follows:

$$^*R_{ik} = R_{ik}, \ ^*R = R.$$

Hence, substituting these in the formula giving L_{ik}, we obtain

$$^*L_{ik} = L_{ik} - RX_iX_k/2(n-1)(n-2).$$

Using this and (7), and taking into account (3), (11) and (37), we have

Theorem 11. *If an R3–like Finsler space F^n $(n > 3)$ admits a parallel vector field X^i, then the modified metric* (6) *is also R3–like.*

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