Classification of Locally Symmetric Contact Metric Manifolds

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Abstract

We complete the classification of 5-dimensional locally symmetric contact metric manifolds stated by D. Blair and J.M. Sierra. Furthermore, in general dimension we prove the existence of a foliation with totally geodesic leaves locally isometric to a Riemannian product $E^{m+1} \times S^m(4)$.

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Introduction

In [6], Z. Olszak proved that for dimensions $2n+1 \geq 5$ there are not contact metric manifolds of constant curvature unless the constant is 1 and in this case the structure is Sasakian. On the other hand, in [7], S. Tanno proved that a locally symmetric K-contact manifold is of constant curvature. Motivated by these results, D. Blair and J.M. Sierra proposed the question of classifying locally symmetric contact manifolds, and in [5] they studied the 5-dimensional case, proving the following theorem.

Theorem. *Let M be a complete 5-dimensional locally symmetric contact metric manifold. Then the simply-connected covering space is either $S^5(1)$ or $E^3 \times S^2(4)$ or $H^2(k_1) \times H^2(k_2) \times R$, where $H^2(k_i)$ $\quad i = 1, 2$ is the hyperbolic plane with constant negative curvature k_i.*

However, whereas $S^5(1)$ and $E^3 \times S^2(4)$ admit such a structure ([2], [3]), the problem of the existence in the third case remained still open. We recall also that the 3-dimensional case has been studied in [4] by Blair and Sharma who proved that a 3-dimensional locally symmetric contact metric manifold is of constant curvature $+1$ or 0 .

In this paper we prove that the third possibility in the theorem of Blair and Sierra has to be removed . Moreover, in the general case, we prove that a locally symmetric contact metric manifold M^{2n+1}, $2n+1 > 5$, admits a foliation whose leaves are totally geodesic and locally isometric to the Riemannian product $E^{m+1} \times S^m(4)$, for a suitable m.

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1 Preliminaries

We recall some results on contact metric manifolds and for more details we refer to [1],[3],[5].

A contact metric manifold M^{2n+1} is a C^∞-manifold with a global 1-form η such that $\eta\wedge(d\eta)^n\neq 0$. It is well known that there exists a unique vector field ξ on M^{2n+1} satisfying $\eta(\xi)=1$ and $d\eta(\xi,X)=0$. A manifold M^{2n+1} is said to be a *contact metric* manifold if it admits a contact metric structure (φ,ξ,η,g), where φ is a tensor field of type $(1,1)$ and g is an associated metric such that

$$\varphi^2=-I+\eta\otimes\xi,\quad g(X,\xi)=\eta(X),\quad d\eta(X,Y)=g(X,\varphi(Y)).$$

Denoting by L the Lie-derivation operator, the tensor field $h=\frac{1}{2}L_\xi\varphi$ is a symmetric operator which anticommutes with φ. Obviously, $h(\xi)=0$ and if λ is an eigenvalue of h with eigenvector X, then $-\lambda$ is an eigenvalue with eigenvector $\varphi(X)$. Moreover, we have $h=0$ if and only if ξ is a Killing vector field and in this case M^{2n+1} is called a *K-manifold.*

We have the following formulas, for any vector field X on M^{2n+1}:

$$\nabla_X\xi=-\varphi(X)-\varphi h(X) \tag{1}$$

$$\frac{1}{2}(R_{\xi X}\xi-\varphi R_{\xi\varphi(X)}\xi)=h^2(X)+\varphi^2(X) \tag{2}$$

$$(\nabla_\xi h)(X)=\varphi(X)-h^2\varphi(X)-\varphi R_{X\xi}\xi, \tag{3}$$

where ∇ is the Levi-Civita connection and R its curvature tensor field, [5]. Furthermore, in [2], the following theorem is proved.

Theorem B. *Let M^{2n+1} be a contact metric manifold and suppose that $R(X,Y)\xi=0$ for all vector fields X and Y. Then M^{2n+1} is locally the product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of positive constant curvature* 4.

Finally, supposing that M^{2n+1} is a locally symmetric contact metric manifold we have $\nabla_\xi h=0$, [3]. Consequently, (3) gives

$$R_{\xi X}\xi=-X+\eta(X)\xi+h^2(X) \tag{4}$$

and the following formulas hold for all orthogonal to ξ unit eigenvectors X,Y of h with eigenvalues λ, μ respectively, ([5] lemma 3.3):

$$\begin{aligned}(\lambda^2-\mu^2)g(\nabla_{\varphi X}X,Y)\ &=(1-\lambda)[(1-\lambda)g(\nabla_X\varphi X,Y)\\ &\quad -2\lambda g(\nabla_Y X,\varphi X)+(1+\mu)g(\nabla_X X,\varphi Y)]\end{aligned} \tag{5}$$

and

$$\begin{aligned}(\varphi Y)(\lambda^2)\ &=2(X\mu)(1-\mu)g(\varphi Y,X)+2(1-\mu^2)g(\nabla_X\varphi Y,X)\\ &+2(1-\lambda-\mu+\lambda\mu)g(\nabla_X Y,\varphi X)\\ &+4\lambda(1-\mu)g(\nabla_Y X,\varphi X)\end{aligned} \tag{6}$$

2 The five-dimensional case

Let M^5 be a locally symmetric contact metric manifold. If the tensor field h vanishes, then M^5 is a K-manifold of constant curvature $+1$ and it is realized by $S^5(1)$ with the standard Sasakian structure, [6], [7].

Now, suppose that $h \neq 0$. As discussed in section 4 of [5] , for any $p \in M^5$ there exists a unit vector $X \in T_p(M^5)$ such that $g(X, \xi) = 0$ and $R_{X\xi}\xi = 0$. Using (4), we have

$$h^2(X) - X = 0 \tag{7}$$

and since $h(\xi) = 0$, the spectrum of the operator h is given by $\{0, \lambda, -\lambda, \mu, -\mu\}$. We suppose $\lambda \geq 0$, $\mu \geq 0$ and we denote by $\{\xi, e_1, e_2, e_3, e_4\}$ the set of the corresponding eigenvectors. Writing $X = \sum_{i=1}^{4} X^i e_i$, and applying (7) we obtain that at least one of λ or μ must be 1, say μ. Moreover, $\varphi(e_1) = e_2$, $\varphi(e_3) = e_4$ and the eigenvalues are constant along their eigenvectors.

Blair and Sierra distingueshed three cases:

i) $\lambda = 1$;

ii) $\lambda = 0$;

iii) $\lambda \neq 0, 1$.

In their proof the first case implies that M^5 is locally isometric to the Riemannian product $E^3 \times S^2(4)$ via theorem B, the second one leeds to an empty class and the third one implies the local isometry of M^5 with $H^2(k_1) \times H^2(k_2) \times R$.

Now, we shall prove that the third possibility has to be excluded, obtaining the following classification theorem.

Theorem 1 *Let M be a complete 5-dimensional locally symmetric contact metric manifold. Then the simply-connected covering space is either $S^5(1)$ or $E^3 \times S^2(4)$.*

Proof. Let us suppose $\lambda \neq 0, 1$. In this hypothesis, Blair and Sierra proved the following results:

a) The distribution $[+1] \bigoplus [-1] \bigoplus [\xi]$ is integrable with flat totally geodesic leaves. Here, $[+1]$ and $[-1]$ denote respectively the eigenspaces related to the eigenvalues $+1$ and -1 and $[\xi]$ is the distribution spanned by ξ.

b) The Levi-Civita connection satisfies the following relations:

$$\nabla_\xi = 0 \qquad \nabla_{e_4} = 0$$

$$\nabla_{e_1} e_1 = -\beta_1' e_3 \qquad \nabla_{e_1} e_2 = -\gamma_1' e_3 - \gamma_1 e_4 + (1+\lambda)\xi$$

$$\nabla_{e_1} e_3 = \beta_1' e_1 + \gamma_1' e_2 \qquad \nabla_{e_1} e_4 = \gamma_1 e_2$$

$$\nabla_{e_1} \xi = (-1-\lambda) e_2 \qquad \nabla_{e_2} e_1 = -\beta_2' e_3 - \beta_2 e_4 - (1-\lambda)\xi$$

$$\nabla_{e_2} e_2 = -\gamma_2' e_3 \qquad \nabla_{e_2} e_3 = \beta_2' e_1 + \gamma_2' e_2$$

$$\nabla_{e_2} e_4 = \beta_2 e_1 \qquad \nabla_{e_2} \xi = (1-\lambda) e_1$$

$$\nabla_{e_3} e_1 = \alpha_3 e_2 \qquad \nabla_{e_3} e_2 = -\alpha_3 e_1$$

$$\nabla_{e_3} e_3 = 0 \qquad \nabla_{e_3} e_4 = 2\xi$$

$$\nabla_{e_3} \xi = -2e_4,$$

where $\beta_2 = -\dfrac{1-\lambda}{1+\lambda}\gamma_1$, $\lambda\alpha_3 = -\gamma_1'$, $\xi(\alpha_3) = 0$.

c) $R_{e_1 e_2}\xi = -((1+\lambda)\gamma_2' + (1-\lambda)\beta_1')e_3$.

d) The eigenvalue λ must be a non costant function, and $\xi(\lambda) = 0$, $e_4(\lambda) = 0$.

First at all, we deduce some other formulas. Taking $Y = e_4$ and $X = e_i, i = 1, 2$ in (6) we get

$$-e_3(\lambda^2) = 4(1-\lambda) g(\nabla_{e_1} e_4, e_2) + 8\lambda g(\nabla_{e_4} e_1, e_2)$$

Then, using b), we obtain $-e_3(\lambda^2) = 4(1-\lambda)\gamma_1$ and

$$e_3(\lambda) = -2\frac{1-\lambda}{\lambda}\gamma_1. \tag{8}$$

Now, condition d) implies $\gamma_1 \neq 0$ and applying the first Bianchi identity to e_1, e_3, ξ and using $R_{e_3\xi} = 0$ we obtain:

$$-2\gamma_1 + e_3(\lambda) + (1+\lambda)(\beta_1' - \gamma_2') = 0 \tag{9}$$

Again, using $R_{e_3\xi} = 0$ and c) we find:

$$\gamma_2' = -\frac{1-\lambda}{1+\lambda}\beta_1' \tag{10}$$

and substituing (8) , (10) in (9) , we get

$$\beta_1' = \frac{1}{\lambda}\gamma_1.$$

Finally, by direct computation, we have

$$g(R_{e_1 e_2} e_1, e_2) = -(\gamma_1')^2 - \frac{(1-\lambda)^2}{\lambda^2}(\gamma_1)^2 + 1 - \lambda^2. \tag{11}$$

Now, we suppose that $M^5 = H^2(k_1) \times H^2(k_2) \times R$ and recall that a) holds. Obviously, ξ has non zero component tangent to $H^2(k_1) \times H^2(k_2)$, otherwise we have $R_{XY}\xi = 0$ for all X, Y and $\lambda = 1$. Moreover, since the foliation spanned by $\{e_3, e_4, \xi\}$ induces foliations by geodesics on each $H^2(k_i)$, we can consider (f_1, f_2) orthonormal vectors tangent to $H^2(k_1)$, and (f_3, f_4) orthonormal vectors tangent to $H^2(k_2)$ such that $\{f_2, f_4, f_5\}$ span the distribution $[+1] \bigoplus [-1] \bigoplus [\xi]$. It follows that e_1 and e_2 belong to the $span\{f_1, f_3\}$ and, since the sectional curvature $K(\{f_1, f_3\}) = 0$, we have $K(\{e_1, e_2\}) = 0$ and (11) implies

$$1 - \lambda^2 = (\gamma_1')^2 + \frac{(1-\lambda)^2}{\lambda^2}(\gamma_1)^2 > 0. \tag{12}$$

On the other hand, writing $\xi = af_2 + bf_4 + cf_5$ and using (4) we obtain $R_{f_1\xi}\xi = (1 - \lambda^2)f_1$ whereas using the sectional curvature, we get $R_{f_1\xi}\xi = a^2 k_1$ so that $1 - \lambda^2 = a^2 k_1$.

We conclude that $1 - \lambda^2 < 0$, contradicting (12).

3 Some results in the higher dimensional case

Let M^{2n+1} be a locally symmetric contact metric manifold and suppose that $h \neq 0$. Arguing as at the beginning of section 2, we consider the set

$$\{0, +1, -1, \lambda_1, -\lambda_1, \ldots, \lambda_r, -\lambda_r\}$$

of the distinct eigenvalues of h such that $dim[0] = p + 1, dim[+1] = m, dim[\lambda_1] = m_1, \ldots, dim[\lambda_r] = m_r$ and $2n + 1 = p + 1 + 2m + 2m_1 + \ldots + 2m_r$.
Here $[\lambda]$ denotes the eigenspace corresponding to the eigenvalue λ.
Theorem 2 *Let M^{2n+1}, $2n + 1 > 5$, be a locally symmetric contact metric manifold and suppose that the spectrum of h is given by the set $\{0, +1, -1\}$ with $+1$ and -1 as eigenvalues of multiplicity n and 0 as simple eigenvalue. Then M^{2n+1} is locally isometric to the Riemannian product $E^{n+1} \times S^n(4)$.*
Proof. By means of (4), we get $R_{X\xi}\xi = 0$ for any eigenvector $X \in [\pm 1]$. Consequently, the sectional curvatures of the tangent 2-planes containing ξ vanish.

If M^{2n+1} is irreducible, it is Einstein with $Ric(\xi, \xi) = 2n - tr(h^2) = 0$ and consequently it is Ricci-flat and then flat, contradicting the result of Olszak in [6]. Hence, M^{2n+1} is reducible and the vanishing of the ξ-curvatures implies that ξ has to be tangent to a flat factor. It follows that $R_{XY}\xi = 0$ for all tangent vectors X, Y and theorem B applies.

Now,we suppose $m < n$, we put $[0] = [\xi] \oplus V_0$ (orthogonal sum), and $H = [\xi] \oplus [\pm 1]$. To prove that the distribution H is integrable we need some lemma.

Lemma 1. *For any $X \in H$ we have $[\xi, X] \in H$.*

Proof. Clearly, for $X \in H$ we have:

$$X \in [+1] \Rightarrow (\nabla_X \xi = -2\varphi X \in [-1], \nabla_\xi X \in [+1])$$
$$X \in [-1] \Rightarrow (\nabla_X \xi = 0, \nabla_\xi X \in [-1])$$

Finally, $\nabla_\xi \xi = 0$ and $[X, \xi] \in [\pm 1] \subset H$ follows.

Lemma 2. *For any X, Y belonging to $[+1]$ we have $\nabla_{\varphi Y} X \in [\pm 1] \subset H$.*

Proof. We use the following formula stated as formula (5) in [3]

$$\begin{aligned} R_{YX}\xi + R_{\xi X}Y - R_{hYX}\xi - R_{\xi X}hY &= g(X,Y)\xi - 2\eta(Y)X + \eta(X)Y \\ &\quad -g(X,hY)\xi + 2\eta(Y)h^2 X \\ &\quad -\eta(X)hY + (\nabla_{\varphi Y} h^2)(X). \end{aligned} \tag{13}$$

obtaining $(\nabla_{\varphi Y} h^2)(X) = 0$, i.e.,

$$\nabla_{\varphi Y} X - h^2(\nabla_{\varphi Y} X) = 0 \tag{14}$$

and this implies $\nabla_{\varphi Y} X \in [\pm 1]$. Namely, we decompose $\nabla_{\varphi Y} X$ with respect to the direct sum of the eigenspaces:

$$\nabla_{\varphi Y} X = A_0 + A_{+1} + A_{-1} + A_{\lambda_1} + A_{-\lambda_1} + \ldots + A_{\lambda_r} + A_{-\lambda_r} \tag{15}$$

Then, we have

$$h^2(\nabla_{\varphi Y} X) = A_{+1} + A_{-1} + \lambda_1^2 A_{\lambda_1} + \lambda_1^2 A_{-\lambda_1} + \ldots + \lambda_r^2 A_{\lambda_r} + \lambda_r^2 A_{-\lambda_r}.$$

Using (14) and (15), we get $A_0 = 0, A_{\lambda_1} = 0, A_{-\lambda_1} = 0, \ldots, A_{\lambda_r} = 0$, $A_{-\lambda_r} = 0$, $\lambda_1, \ldots, \lambda_r$ being different from $+1, -1$. Finally, from (15) we obtain $\nabla_{\varphi Y} X = A_{+1} + A_{-1} \in [\pm 1] \subset H$.

Corollary 1. *For any $X \in [-1]$ and $Y \in [+1]$ we have $\nabla_X Y \in [\pm 1]$.*

Proof. Apply Lemma 2 to φX and Y.

Lemma 3. *For any $Y \in [+1]$ and $X \in [-1]$, we have $\nabla_{\varphi Y} X \in [\pm 1]$.*

Proof. From (13), since $g(X, Y) = 0$, we obtain $(\nabla_{\varphi Y} h^2)(X) = 0$ and we continue as in the proof of Lemma 2.

Corollary 2. *We have:* a) $(X \in [-1], Y \in [-1]) \Rightarrow \nabla_X Y \in [\pm 1]$
b) $X, Y \in [-1] \Rightarrow [X, Y] \in [\pm 1]$

Lemma 4. *For any $X \in [-1]$ and $Y \in [-1]$, we have $\nabla_{\varphi Y} X \in H$.*

Proof. Using (13) we have:

$$2R_{XY}\xi + 2R_{\xi X}Y = 2g(X,Y)\xi + (\nabla_{\varphi Y} h^2)(X).$$

Lemma 1 and Corollary 2 easily imply that $R_{YX}\xi \in [\pm 1]$ and $R_{\xi X}Y \in [\pm 1]$.
It follows

$$B = 2g(X,Y)\xi + \nabla_{\varphi Y} X - h^2(\nabla_{\varphi Y} X) \in [\pm 1] \tag{16}$$

On the other hand, decomposing $\nabla_{\varphi Y} X$ as in (15) and computing $h^2(\nabla_{\varphi Y} X)$, we get

(17)
$$B = 2g(X,Y)\xi + A_0 + (1-\lambda_1^2)A_{\lambda_1} + (1-\lambda_1^2)A_{-\lambda_1} + \dots + (1-\lambda_r^2)A_{\lambda_r} + (1-\lambda_r^2)A_{-\lambda_r}$$

Comparing (16) and (17) we conclude

$$A_0 = -2g(X,Y)\xi, A_{\lambda_1} = 0, A_{-\lambda_1} = 0, \dots, A_{\lambda_r} = 0, A_{-\lambda_r} = 0$$

so that

$$\nabla_{\varphi Y}X = -2g(X,Y)\xi + A_{+1} + A_{-1} \in H.$$

Corollary 3. $(X \in [+1], Y \in [-1]) \Rightarrow (\nabla_X Y \in H, [X,Y] \in H)$.

Lemma 5. *For any $Y \in [-1]$ and $X \in [+1]$ we have $\nabla_{\varphi Y}X \in [\pm 1]$.*

Proof. Using (13), since $g(X,Y) = 0$, we get

$$2R_{YX}\xi + 2R_{\xi X}Y = (\nabla_{\varphi Y}h^2)(X)$$

Now, Lemma 1 and the previous corollaries easily imply that $R_{YX}\xi \in [\pm 1]$ and $R_{\xi X}Y \in H$, so that

(18)
$$\nabla_{\varphi Y}X = h^2(\nabla_{\varphi Y}X) \in H.$$

Again, decomposing $\nabla_{\varphi Y}X$ with respect to the direct sum of eigenspaces, (18) implies $A_0 = a\xi, A_{\lambda_1} = 0 \dots, A_{-\lambda_r} = 0$,so that we have

$$\nabla_{\varphi Y}X = a\xi + A_{+1} + A_{-1}$$

Now, since $\varphi Y \in [+1]$, we get $g(\nabla_{\varphi Y}X, \xi) = -g(X, \nabla_{\varphi Y}\xi) = -2g(x, \varphi^2 Y) = 2g(X,Y) = 0$ and $\nabla_{\varphi Y}X \in [\pm 1]$.

Corollary 4. $(X \in [+1], Y \in [+1]) \Rightarrow (\nabla_X Y \in [\pm 1].[X,Y] \in [\pm 1]$.

Proposition 4.1. *The distribution $H = [\xi] \oplus [\pm 1]$ is integrable with totally geodesic leaves.*

Proof. The previous lemma and corollaries imply that $[X,Y] \in H$ for any $X \in H$ and $Y \in H$. Thus the distribution H is involutive and integrable.
Let N be a maximal integral submanifold. Since $\nabla_X Y$ is tangent to N for any vector fields X, Y tangent to N , the second fundamental form vanishes and N is totally geodesic.

Proposition 4.2. *The integral manifolds of the distribution H are locally isometric to the Riemannian product $E^{m+1} \times S^m(4)$.*

Proof. Let N be an integral manifold of H, A local frame for TN is given by ξ and the eigenvectors $\{e_i, \varphi e_i\}$, $\quad i \in \{1, \dots, m\}$ corresponding to the eigenvalues $+1, -1$, and N has a canonically induced contact metric structure (ξ, φ', g) where φ' is the restriction of φ to N. Moreover, N turns out to be locally symmetric since it is totally geodesic in the locally symmetric manifold M^{2n+1}. It is easy to verify that $h' = \frac{1}{2}L_\xi \varphi'$ is the restriction of h to N. Now, h' has eigenvalues $+1, -1$ with multiplicity m and 0 as a simple eigenvalue. Theorem 2 insures that N is locally isometric to $E^{m+1} \times S^m(4)$. Hence, we can conclude with the following theorem:

Theorem 3 *Let M^{2n+1} be a locally symmetric contact metric manifold. Then M^{2n+1} admits a foliation whose leaves are totally geodesic and locally isometric to the Riemannian product $E^{m+1} \times S^m(4)$. The integer m is the multiplicity of the eigenvalue $+1$ of the operator $\frac{1}{2}L_\xi\varphi$.*

References

[1] D.E.Blair, *Contact manifolds in Riemannian geometry*, Lecture Notes in Math., 509 , 1976.

[2] D.E.Blair, *Two remarks on contact metric structures*, Tôhoku Math. J. 29 (1977), 319-324.

[3] D.E.Blair, *When is the tangent sphere bundle locally symmetric?*, in Geometry and Topology, World Scientific, Singapore (1989), 15-30.

[4] D.E.Blair and R.Sharma, *Three dimensional locally symmetric contact metric manifolds*, B.U.M.I (7), 4-A, (1990) , 385-390.

[5] D.E.Blair and J.M.Sierra, *5-dimensional locally symmetric contact metric manifolds*, B.U.M.I. (7), 7-A, (1993), 299-311.

[6] Z.Olszak, *On contact metric manifolds*, Tôhoku Math.J., 31, (1979), 247-253.

[7] S.Tanno, *Locally symmetric K-contact Riemannian manifolds*, Proc. Japan Acad. 43, (1967), 581-583.

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