Peano Baker Series Convergence for Matrix Valued Functions of Bounded Variation

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Abstract

The Peano Baker formula is extended for the calculus of generalized fundamental matrix in the case of matrix valued functions of bounded variation. Using the properties of the Perron-Stieltjes integral the convergence of the corresponding matrix series is proved.

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1 Introduction

The theory of generalized differential equations, which uses the concept of the Perron-Stieltjes integral, was initiated by J. Kurzweil [3],[4] and developed by Schwabik, Tvrdý, Vejvoda [9] et al. Boundary value problems for generalized differential equations appeared in Atkinson [1], Halanay and Moro [2]. This theory was employed by Tvrdý [10],[11] and Prepeliţă et al. [5],[6],[7] in the study of generalized dynamical and acausal systems. A basic device in this setting is the fundamental matrix, which gives the possibility to obtain many useful formulae and results.

In this article, the problem of the effective calculus of the (generalized) fundamental matrix is considered. The Peano Baker formula is extended in this framework and the uniform convergence of the matrix series is shown.

2 The Perron-Stieltjes Integral

The definition of the Perron-Stieltjes integral was given in [8],[4] and [9].

Let us consider functions δ , $\delta : [a,b] \to R^+$ for a given interval $[a,b]$ and sequences of numbers

$$S = \{\alpha_0,\ \tau_1,\ \alpha_1, \ldots,\ \tau_k,\ \alpha_k\}$$

such that $a = \alpha_0 < \alpha_1 < \ldots < \alpha_k = b$ and $\alpha_{j-1} \le \tau_j \le \alpha_j, \quad j = \overline{1,k}$.

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For a given function δ, S is called a subdivision of $[a,b]$ subordinate to δ if

$$[\alpha_{j-1},\,\alpha_j] \subseteq (\tau_j - \delta(\tau_j),\, \tau_j + \delta(\tau_j)), \quad j = \overline{1,k}.$$

The set of all subdivisions S subordinate to δ is denoted by $S(\delta)$.

Given two functions $f, g : [a,b] \to R$ and a subdivision S, we associate the integral sum

$$B_{f,g}(S) = \sum_{j=1}^{k} f(\tau_j)\,(g(\alpha_j) - g(\alpha_{j-1}))\ .$$

Definition 2.1. If there is a real number I such that, for any $\varepsilon > 0$ there exists a function $\delta : [a,b] \to R^+$ such that $|B_{f,g}(S) - I| < \varepsilon$, for any $S \in S(\delta)$, then I is denoted by $\int_a^b fdg$ (or $\int_a^b f(t)dg(t)$) and it is called *the Perron-Stieltjes integral of the function f with respect to g from a to b.*

We shall emphasize some properties of this integral that we need in the next paragraph.

Proposition 2.2. *If f is nonnegative and g is nondecreasing in $[a,b]$ and the integral$\int_a^b fdg$ exists, then$\int_a^b fdg \geq 0$.*

Proof. Since $f(\tau_0) \geq 0$ and $g(\alpha_j) - g(\alpha_{j-1}) \geq 0$ we have $B_{f,g}(S) \geq 0$ for any $S \in S(\delta)$; we obtain $I \geq 0$ from $|B_{f,g}(S) - I| < \varepsilon$ for any $\varepsilon > 0$.

By considering $f = f_1 - f_2$ or $g = g_1 - g_2$ we obtain two corollaries:

Corollary 2.3. *If $f_1(t) \geq f_2(t),\ t \in [a,b]$, g is nondecreasing in $[a,b]$ and $\int_a^b f_1dg$ and $\int_a^b f_2dg$ exist, then*

$$\int_a^b f_1dg \geq \int_a^b f_2dg.$$

Corrolary 2.4. *Assume that $f(t) \geq 0, t \in [a,b]$ and $g_1 - g_2$ is a nondecreasing function, where $g_1, g_2 : [a,b] \to R$. If the integrals $\int_a^b fdg_1$ and $\int_a^b fdg_2$ exist, then*

$$\int_a^b fdg_1 \geq \int_a^b fdg_2.$$

Let us denote by BV^n and $BV^{n\times n}$ the Banach spaces

$BV^n = \{f : [a,b] \to R^n, f$ of bounded variation$\}$,

$BV^{n\times n} = \{f : [a,b] \to R^{n\times n}, f$ of bounded variation$\}$

and by BV the space BV^1.

Proposition 2.5. [9, Theorem I.4.19]

If $f, g \in BV$, then the integral $\int_a^b fdg$ exists.

Proposition 2.6. [9, Corollary I.4.27]

If $f, g \in BV$, then $|\int_a^b fdg| \leq \int_a^b |f()t_|d(var_a^t g)$.

Proposition 2.7.[9, Theorem I.4.29.] *If the function $h : [a,b] \to R$ is nonnegative, nondecreasing and continuous from the left in $[a,b]$, then*

$$\int_a^b h^k(t)dh(t) \leq \frac{1}{k+1}[h^{k+1}(b) - h^{k+1}(a)],$$

for any $k = 0, 1, \ldots$.

3 Generalized linear differential equations and the fundamental matrix

Let us consider $A \in BV^{n\times n}$ and $g \in BV^n$. The *generalized linear differential equation* (GLDE) is the symbol:

(1) $$dx = d[A]x + dg.$$

Definition 3.1. A function $x : [a, b] \to R^n$ is said to be a *solution* of the GLDE (1) on the interval [a,b] if for any $t,\ t_0 \in\ [a, b]$ the equality

(2) $$x(t) = x(t_0) + \int -{t_0}^t d\,[A(s)]\,x(s) + g(t) - g(t_0)$$

holds.

One can prove [9, Theorem III.1.3.] that if x is a solution of (2) then $x \in BV^n$.

If $x_0 \in R^n$ and $t_0 \in [a, b]$ are fixed and $x : [a, b] \to R^n$ is a solution of (1) which satisfies the initial condition

(3) $$x(t_0) = x_0,$$

then x is called *the solution of the initial value problem* (1)(3).

Remark 3.2. If the functions $A : [a, b] \to R^{n\times n}$ and $g : [a, b] \to R^n$ have the representations

$$A(t) = \int_a^t \tilde{A}(s)ds, \quad g(s) = \int_a^t \tilde{g}(s)ds, \quad t \in [a, b],$$

where $\tilde{A} : [a, b] \to R^{n\times n}$ and $\tilde{g} : [a, b] \to R^n$ are continuous functions, then A and g era absolutely continuous functions on [a,b] (therefore they are of bounded variation) and the initial value problem (1)(3) is equivalent to the initial value problem for the linear ordinary differential equation

$$\dot{x}(t) = \tilde{A}(t)x(t) + \tilde{g}(t)$$

hence indeed (1) is a generalized equation.

Now let us denote by Δ^- and Δ^+ the operators defined by

$$\Delta^- f(t) = f(t) - f(t-), \quad \Delta^+ f(t) = f(t+) - f(t), \qquad t \in [a, b]$$

(where by definition $f(a-) = f(a)$ and $f(b+) = f(b)$). For a function f of two variables $v(f)$ denotes the two-dimensional Vitali variation of f on $[a, b] \times [a, b]$.

One can associate a (generalized) fundamental matrix:

Theorem 3.3 [9, Theorem III.2.10]
If the matrix $A \in BV^{n\times n}$ has the property

(4) $$\det\left(I-\Delta^{-}A(t)\right)\det\left(I+\Delta^{+}A(t)\right)\neq 0\,, \qquad \forall t\in[a,b],$$

then there exists a unique matrix valued function $U:[a,b]\times[a,b]\to R^{n\times n}$ *such that*

(5) $$U(t,s)=I+\int_s^t d\,[A(r)]\,U(r,s) \qquad .$$

The matrix U has the following properties, for any $r,s,t\in[a,b]$:

$$\begin{aligned}
U(t,s) &= U(t,r)U(r,s)\\
U(t,t) &= I\\
U(t+,s) &= \left[I+\Delta^{+}A(t)\right]U(t,s)\\
U(t-,s) &= \left[I-\Delta^{-}A(t)\right]U(t,s)\\
U(t,s+) &= U(t,s)\left[I+\Delta^{+}A(t)\right]^{-1}\\
U(t,s-) &= U(t,s)\left[I-\Delta^{-}A(t)\right]^{-1}
\end{aligned}$$

$[U(t,s)]^{-1}=U(s,t)$, (hence $U(t,s)$ is nonsingular for any $t,s\in[a,b]$).

There exists a constant $M>0$ such that

$$|U(t,s)|+\mathrm{var}_a^b U(t,\cdot)+\mathrm{var}_a^b U(\cdot,s)+v(U)\leq M\ .$$

The unique solution of the homogeneous initial value problem

$$dx=d[A]x\ ,\quad x(t_0)=x_0$$

is given by the relation

$$x(t)=U(t,t_0)x_0\ ,\forall t,t_0\in[a,b]\ .$$

The fundamental matrix can be used in order to solve the corresponding GLDE:

Theorem 3.4. [9, Theorem III.2.13.]
If (4) holds then the solution of the initial value problem (1)(3) is given by the variation of constants formula:

$$x(t)=U(t,t_0)x_0+g(t)-g(t_0)-\int_{t_0}^t d_s\,[U(t,s)]\,(g(s)-g(t_0)) \qquad .$$

This theorem emphasizes the importance of the calculus of the fundamental matrix for GLDE.

4 Peano-Baker formula for GLDE

In this paragraph the Peano-Baker formula is extended for matrix functions of bounded variation and sufficient conditions for the convergence of the Peano-Baker series are given.

Theorem 4.1. *If the matrix function* $A\in BV^{n\times n}$ *is continuous from the left in* $[a,b]$ *and for any* $t\in[a,b]$ *it verifies the assumption*

$$\det\left(I+\Delta^+A(t)\right)\neq 0\ , \tag{6}$$

then the fundamental matrix $U(t,s)$ which corresponds to A is the sum of the following uniformly convergent series, for any $t,s\in[a,b]$:

$$\begin{aligned}U(t,s)=I&+\int_s^t d[A(s_1)]+\int_s^t d[A(s_1)]\int_s^{s_1} d[A(s_2)]+\dots\\ &+\int_s^t d[A(s_1)]\int_s^{s_1} d[A(s_2)]\dots\int_s^{s_{k-1}} d[A(s_k)]+\dots\qquad .\end{aligned} \tag{7}$$

Proof. If $A=[a_{ij}]_{i,j=\overline{1,n}}$ is continuous from the left it results that $\Delta^-A(t)=A(t)-A(t-)=0,\ \forall t\in[a,b]$, hence $\det(I-\Delta^-A(t))=1\neq 0$, therefore the condition (6) is equivalent to (4) and the existence of the fundamental matrix is a consequence of Theorem 3.3.

Let s be a fixed point, $s\in[a,b]$. Let us consider a function $h:[a,b]\to R$ having the properties:

i) h is nonnegative, nondecreasing and continuous from the left on $[a,b]$
ii) $h(s)=0$
iii) $\forall r,t\in[s,b],\quad r\le t,\quad h(t)-h(r)\ge\max\{\mathrm{var}_r^t a_{ij}\,|\,1\le i,j\le n\}$.

By iii) it results that the function $h(t)-\mathrm{var}_r^t a_{ij}$ is nondecreasing for all $1\le i,j\le n$ and $t\in[s,b]$.

For instance, the function defined by $h(t)=0$ for $t\in[a,s]$ and $h(t)=\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}\mathrm{var}_s^t a_{i,j}$ for $t\in(s,b]$ has the properties i)–iii).

Using iii) and Proposition 2.6 with $f(t)=1,\ t\in[s,b]$ we obtain

$$\left|\int_s^t da_{ij}(s_1)\right|\le\int_s^t d\left(\mathrm{var}_s^{s_1}a_{ij}\right)=\mathrm{var}_s^t a_{ij}-\mathrm{var}_s^s a_{ij}=\mathrm{var}_s^t a_{ij}\le h(t)-h(s)=h(t)\ .$$

Then, by Propositions 2.6 and 2.7 and Corollaries 2.3 and 2.4, it results

$$\left|\int_s^t da_{il}(s_1)\int_s^{s_1} da_{lj}(s_2)\right|\le\int_s^t d\left(\mathrm{var}_s^{s_1}a_{il}\right)\left|\int_s^{s_1} da_{lj}(s_2)\right|$$

$$\le\int_s^t d\left(\mathrm{var}_s^{s_1}a_{il}\right)h(s_1)\le\int_s^t dh(s_1)h(s_1)\le\frac{1}{2}(h^2(t)-h^2(s))=\frac{1}{2}h^2(t)\ .$$

Let us denote by $E_{ij}(M)$ the element of a matrix M situated on the i-th row and j-th column (hence $E_{ij}(A)=a_{ij}$). We obtain:

$$\left|E_{ij}\left(\int_s^t d[A(s_1)]\int_s^{s_1} d[A(s_2)]\right)\right|\le\sum_{l=1}^{n}\left|\int_s^t da_{il}(s_1)\int_s^{s_1} da_{lj}(s_2)\right|\le\frac{n}{2!}h^2(t)$$

Assume that the following inequality holds, for any i,j , $1\le i,j\le n$ and $t\in[s,b]$:

$$\left|E_{ij}\left(\int_s^t d[A(s_1)]\int_s^{s_1} d[A(s_2)]\dots\int_s^{s_{k-1}} d[A(s_k)]\right)\right|\le\frac{n^{k-1}}{k!}h^k(t). \tag{8}$$

Then using again Propositions 2.6, 2,7, Corollaries 2.3, 2.4 and the inequality (8), we obtain:

$$\left|E_{ij}\left(\int_s^t d[A(s_1)]\int_s^{s_1} d[A(s_2)]\dots\int_s^{s_{k-1}} d[A(s_k)]\int_s^{s_k} d[A(s_{k+1})]\right)\right| \leq$$

$$\sum_{l=1}^{n}\left|\int_s^t da_{il}(s_1)E_{lj}\left(\int_s^{s_1} dA(s_2)\dots\int_s^{s_{k-1}} dA(s_k)\int_s^{s_k} dA(s_{k+1})\right)\right| \leq$$

$$\leq \sum_{l=1}^{n}\int_s^t d\left(\mathrm{var}_s^{s_1} a_{il}\right)\left|E_{lj}\left(\int_s^{s_1} dA(s_2)\dots\int_s^{s_{k-1}} dA(s_k)\int_s^{s_k} dA(s_{k+1})\right)\right| \leq$$

$$\leq \sum_{l=1}^{n}\int_s^t d\left(\mathrm{var}_s^{s_1} a_{il}\right)\frac{n^{k-1}}{k!}h^k(s_1) \leq n\frac{n^{k-1}}{k!}\int_s^t d(h(s_1))h^k(s_1) \leq$$

$$\leq \frac{n^k}{k!}\frac{1}{k+1}\left(h^{k+1}(t)-h^{k+1}(s)\right) = \frac{n^k}{(k+1)!}h^{k+1}(t)\ .$$

We proved by induction that (8) holds for any $k \geq 1$.

Since h is nondecreasing, it results that each element of the matrix series (7) verifies

$$\left|E_{ij}\left(\int_s^t d[A(s_1)]\int_s^{s_1} d[A(s_2)]\dots\int_s^{s_{k-1}} d[A(s_k)]\right)\right| \leq \frac{n^{k-1}}{k!}h^k(b)$$

for any $t \in [s,b]$.

But the series of positive numbers $1+\sum\limits_{k=1}^{\infty}\frac{n^{k-1}}{k!}h^k(b)$ converges to $1-\frac{1}{n}+\frac{1}{n}e^{nh(b)}$, hence by the Weierstrass criterion it results that the matrix series in the right hand member of (7) converges uniformly on [s,b].

Obviously, the sum of the matrix series in the right hand member of (7) verifies the equality (5); by unicity (see Theorem 3.3) it results that the sum of this matrix series is the fundamental matrix $U(t,s)$, hence the Peano-Baker formula (7) holds.

5 Conclusion

In the particular case described in Remark 3.2 formula (7) gives the "classical" fundamental matrix corresponding to the continuous matrix function $\tilde{A}$, hence indeed (7) is a generalization of the Peano-Baker formula in the framework of matrix functions of bounded variation.

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