

Applications of Brézis–Browder Principle to the Existence of Fixed Points and Endpoints for Multifunctions

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Abstract

In this paper we study the existence of fixed points and endpoints for certain class of multifunctions. Applying the Brézis–Browder principle, we obtain some generalizations of Clarke's theorem concerning the directional contractions and of Caristi's fixed point theorem. Also we generalize the Edelstein fixed point theorem to the d - and δ -directional contractive functions and multifunctions.

Mathematics Subject Classification: 54H25

Key words: fixed point, endpoint, d - and δ -directional contraction, d - and δ -directional contractive multifunctions.

1 Introduction

The objective of this paper is to apply the Brézis–Browder principle to the fixed point theory. We shall obtain new results concerning the existence of fixed points and endpoints for multifunctions and, in the same time, we shall see that the utilization of this principle leads us to an unitary presentation of these results.

The mentioned principle has been published in 1976 by H. Brézis and F. Browder in [1] and it is remarkable by simplicity and diversity of its applications in the nonlinear functional analysis. We state this principle in the following form:

Theorem 1.1. *Let (X, \leq) be a partial ordered set and $\theta : X \rightarrow \mathbf{R} \cup \{+\infty\}$ be a function. Suppose that*

1° *every increasing sequence in X is bounded, and*

2° *θ is an increasing function.*

Then, for each $x_0 \in X$ there exists an element $x^ \in X$, $x_0 \leq x^*$, such that $x^* \leq x \implies \theta(x) = \theta(x^*)$.*

We point out the fact that this principle can be used to prove the Ekeland's variational principle [4], [5]. Consequently, many results can be obtained in an elementary way if, instead of Ekeland's principle, one utilizes that of Brézis and Browder. Thus, F.H. Clarke extends the contraction principle of Banach to directional contractions with the aid of Ekeland's variational principle [3]. Obviously, an elementary proof of his result can be based on the Brézis–Browder principle.

2 Preliminary definitions and notations

Let X be a nonempty set and $F : X \rightsquigarrow X$ be a multifunction. A point $x \in X$ is said to be a *fixed point (endpoint)* for this multifunction if $x \in Fx$ ($Fx = \{x\}$).

Now, let X be a topological space. The multifunction $F : X \rightsquigarrow X$ (with non-empty values) is said to be *l.s.c. (lower semicontinuous) at x_0* if for any open set U with $Fx_0 \cap U \neq \emptyset$ there is a neighbourhood V of x_0 such that for any $x \in V$ we have $Fx \cap U \neq \emptyset$. F is said to be *u.s.c. (upper semicontinuous) at x_0* if for any open set $U \supset Fx_0$ there exists a neighbourhood V of x_0 such that $U \supset Fx$ for any $x \in V$. The multifunction F is *l.s.c. (u.s.c.)* if it is l.s.c. (u.s.c.) in each point of the space X .

Also, we recall that a real-valued function $f : X \rightarrow \mathbb{R}$ is l.s.c. at $x_0 \in X$ if $x_n \rightarrow x_0 \implies \liminf_{n \rightarrow \infty} f x_n \geq f x_0$.

Let (X, d) be a metric space. We adopt the following notations:

$$\begin{aligned} d(A, B) &= \inf\{d(a, b); a \in A, b \in B\} & \text{and} & & d(x, A) &= d(\{x\}, A); \\ \delta(A, B) &= \sup\{d(a, b); a \in A, b \in B\} & \text{and} & & \delta(x, A) &= \delta(\{x\}, A); \\ e(A, B) &= \sup\{d(a, B); a \in A\}; \\ D(A, B) &= \max\{e(A, B), e(B, A)\}. \end{aligned}$$

We observe that the application $(A, B) \rightarrow D(A, B)$ is a distance on the set of all nonempty bounded closed subsets of X , so-called *Pompeiu–Hausdorff distance*. Moreover, if we consider D on the set of nonempty bounded subsets of X , then the implication $D(A, B) = 0 \implies A = B$ is not satisfied, hence D is only a pseudo-metric on this set.

3 Directional contractions

Let (X, d) be a metric space.

Definition 3.1. A *directional contraction* is a function $f : X \rightarrow X$ for which there exists a number $k \in (0, 1)$ with the following property: for each $x \in X$, $x \neq fx$, there is $y \in X$, $y \neq x$, verifying the *convexity condition*

$$(3.1) \quad d(x, y) + d(y, fx) = d(x, fx),$$

and the *contraction condition*

$$(3.2) \quad d(fx, fy) \leq kd(x, y).$$

Concerning this kind of contraction it is well-known the following

Theorem 3.2. (F.H. Clarke [3, p.269]). *If (X, d) is a complete metric space and $f : X \rightarrow X$ is a continuous directional contraction on X , then this function admits a fixed point.*

In the case of multifunctions, the notion of directional contraction can be generalized in two different manners which correspond with two convexity conditions extending (3.1).

Definition 3.3. A multifunction $F : X \rightsquigarrow X$ (with nonempty bounded values) is called a d -directional contraction if there exists a number $k \in (0, 1)$ having the property: for each $x \in X$, $x \notin Fx$, there is $y \in X$, $y \neq x$, verifying the conditions

- (i) $d(x, y) + d(y, Fx) = d(x, Fx)$, and
- (ii) $D(Fx, Fy) \leq kd(x, y)$.

Definition 3.4. A multifunction $F : X \rightsquigarrow X$ (with nonempty bounded values) is a δ -directional contraction if there exists $k \in (0, 1)$ such that for each $x \in X$ for which $Fx \neq \{x\}$ there is $y \in X$, $y \neq x$, verifying the conditions

- (j) $d(x, y) + \delta(y, Fx) = \delta(x, Fx)$, and
- (jj) $D(Fx, Fy) \leq kd(x, y)$.

We need two lemmas which point out some known properties of the distances d and δ .

Lemma 3.5. *For any $A, B \subset X$ (nonempty and bounded) we have:*

- a) $d(x, A) \leq d(x, B) + D(A, B)$,
- b) $\delta(x, A) \leq \delta(x, B) + D(A, B)$.

Lemma 3.6. *Let $F : X \rightsquigarrow X$ a multifunction with nonempty values. Then:*

- a) *If F has compact values and it is u.s.c. at x_0 , then the function $x \rightarrow d(x, Fx)$ is l.s.c. at x_0 .*
- b) *If F has bounded values and it is l.s.c. at x_0 , then the function $x \rightarrow \delta(x, Fx)$ is l.s.c. at x_0 .*

We are now ready to state and prove the next theorems:

Theorem 3.7. *Let (X, d) be a complete metric space. Suppose that $F : X \rightsquigarrow X$ satisfies the following conditions:*

- 1° *the values of F are nonempty and compact;*
- 2° *F is u.s.c.;*
- 3° *F is a d -directional contraction.*

Then F admits a fixed point.

Theorem 3.8. Let (X, d) be a complete metric space. Suppose that $F : X \rightsquigarrow X$ verifies the conditions:

- 1° the values of F are nonempty and bounded;
- 2° F is l.s.c.;
- 3° F is a δ -directional contraction.

Then F has an endpoint.

The proofs of these results follow the same path, which consists in the application of Brézis–Browder principle. For this reason, we shall prove only the second theorem.

Proof of Theorem 3.8. On X we introduce the next relation:

$$(3.3) \quad x \leq y \text{ if } \delta(y, Fy) + (1 - k)d(x, y) \leq \delta(x, Fx).$$

It is easy to verify that \leq is a partial order on X .

Consider the real-valued function θ defined by $\theta(x) = -\delta(x, Fx)$, $x \in X$. In view of (3.3), we have

$$(3.4) \quad x \leq y \implies \delta(y, Fy) \leq \delta(x, Fx),$$

hence θ is increasing with respect to \leq .

Now, let (x_n) be an increasing sequence in X . Taking into account (3.4), we deduce that the sequence $(\delta(x_n, Fx_n))$ is convergent. Since for $n \leq m$ we have

$$(3.5) \quad \delta(x_m, Fx_m) + (1 - k)d(x_n, x_m) \leq \delta(x_n, Fx_n),$$

it follows that

$$(1 - k)d(x_n, x_m) \leq \delta(x_n, Fx_n) - \delta(x_m, Fx_m), \quad n \leq m$$

and, consequently, (x_n) is a Cauchy sequence. Hence (x_n) is convergent, i.e. there exists $x \in X$ such that $x_n \rightarrow x$. We shall prove that $x_n \leq x$, $n \in \mathbb{N}$, i.e. (x_n) is bounded in X . In order to do this, we take $m \rightarrow \infty$ in (3.5) and obtain

$$\liminf_m \delta(x_m, Fx_m) + (1 - k)d(x_n, x) \leq \delta(x_n, Fx_n), \quad n \in \mathbb{N}.$$

Using Lemma 3.6b, we can write

$$\delta(x, Fx) + (1 - k)d(x_n, x) \leq \delta(x_n, Fx_n), \quad n \in \mathbb{N},$$

that is $x_n \leq x$, $n \in \mathbb{N}$.

Since the hypotheses of Brézis–Browder principle are satisfied, for any $x_0 \in X$ there is $x^* \in X$ such that $x_0 \leq x^*$ and

$$(3.6) \quad x^* \leq x \implies \delta(x, Fx) = \delta(x^*, Fx^*).$$

We shall prove that x^* is an endpoint for F . Indeed, if, by contrary, $Fx^* \neq \{x^*\}$ then there exists $y \neq x^*$ such that

$$(3.7) \quad d(x^*, y) + \delta(y, Fx^*) = \delta(x^*, Fx^*) \quad \text{and}$$

$$(3.8) \quad D(Fx^*, Fy) \leq kd(x^*, y).$$

By means of Lemma 3.5b, (3.7) and (3.8), it follows:

$$\begin{aligned} \delta(y, Fy) &\leq \delta(y, Fx^*) + D(Fx^*, Fy) \leq \delta(y, Fx^*) + kd(x^*, y) \\ &= [\delta(x^*, Fx^*) - d(x^*, y)] + kd(x^*, y) \\ &\leq \delta(x^*, Fx^*) - (1 - k)d(x^*, y), \end{aligned}$$

and, thus,

$$(3.9) \quad \delta(y, Fy) + (1 - k)d(x^*, y) \leq \delta(x^*, Fx^*),$$

i.e. $x^* \leq y$. By (3.6), we conclude

$$(3.10) \quad \delta(y, Fy) = \delta(x^*, Fx^*).$$

Combining (3.9) and (3.10), we deduce $d(x^*, y) = 0$, i.e. $y = x^*$, which is absurd. Consequently, $Fx^* = \{x^*\}$, q.e.d.

The relation between the convexity conditions (i) and (j) as well as between Theorem 3.7 and Theorem 3.8 are clarified by the next

Example 3.9. Let $X = [0, 1]$ and d be the Euclidean distance on X . We define $F : X \rightsquigarrow X$ by $Fx = [\frac{1}{2}(1 - x), 1]$, $x \in X$. It is easy to see that: 1° F has nonempty and compact values, 2° F is continuous (i.e. u.s.c. and l.s.c.), 3° F is a d -directional contraction, 4° F is not a δ -directional contraction, 5° any point $x \in [\frac{1}{2}, 1]$ is a fixed point of F , and 6° F has not endpoints.

Remark 3.10. The previous example shows that we cannot replace in Theorem 3.8 the hypothesis 3° with the condition " F is a d -directional contraction".

Remark 3.11. I. Ekeland states in [5,p.448] that the proof of Clarke's Theorem, based on his variational principle, can be extended to multifunctions "with ad hoc assumptions". Theorems 3.7 and 3.8 indicate sufficient conditions for the existence of fixed points and endpoints, respectively. Moreover, the proofs based on Brézis-Browder principle are simple and accessible.

4 Generalizations of Caristi's fixed point theorem

In 1976, J. Caristi [2] established the following interesting result:

Theorem 4.1. *Let (X, d) be a complete metric space and $\varphi : X \rightarrow \mathbf{R}$ be a l.s.c. function which is bounded below. Assume that $f : X \rightarrow X$ is a function satisfying the condition*

$$(4.1) \quad d(x, fx) \leq \varphi(x) - \varphi(fx), \quad x \in X.$$

Then f has a fixed point.

Also, one knows the next two versions of this result for the case of multifunctions:

Theorem 4.2. *Suppose that the metric space (X, d) and the function φ are as in Theorem 4.1. Let $F : X \rightsquigarrow X$ be a multifunction with nonempty values which satisfies*

$$(4.2) \quad d(x, y) \leq \varphi(x) - \varphi(y) \text{ for any } x \in X \text{ and some } y \in Fx.$$

Then F has a fixed point.

Theorem 4.3. *If the hypotheses of Theorem 4.2 are fulfilled and the multifunction F satisfies the more restrictive condition*

$$(4.2)' \quad d(x, y) \leq \varphi(x) - \varphi(y) \text{ for any } x \in X \text{ and all } y \in Fx,$$

then F admits an endpoint.

In order to generalize the precedent theorems, one can imagine various modalities. We indicate a way of generalization in the following three theorems.

Theorem 4.4. *Let (X, d) be a complete metric space and $\varphi : X \rightarrow \mathbf{R}$ be a l.s.c. function which is bounded below. Suppose $f : X \rightarrow X$ having the property that for each $x \in X$ there exists $\bar{x} \in X$ such that*

$$(4.3) \quad d(x, \bar{x}) \leq \varphi(x) - \varphi(\bar{x}), \quad \text{and}$$

$$(4.4) \quad d(\bar{x}, f\bar{x}) \leq \varphi(\bar{x}) - \varphi(f\bar{x}).$$

Then, the function f has a fixed point.

Theorem 4.5. *Suppose (X, d) and φ as in the precedent theorems. Let $F : X \rightsquigarrow X$ be a multifunction with nonempty values and having the following property: for each $x \in X$ there exists $\bar{x} \in X$ such that the conditions (4.3) and*

$$(4.5) \quad d(\bar{x}, y) \leq \varphi(\bar{x}) - \varphi(y), \text{ for some } y \in F\bar{x},$$

hold. Then F admits a fixed point.

Theorem 4.6. *Suppose that the hypotheses of Theorem 4.5 are fulfilled and, moreover, F verifies the strong condition*

$$(4.5)' \quad d(\bar{x}, y) \leq \varphi(\bar{x}) - \varphi(y), \text{ for all } y \in F\bar{x}.$$

Then F admits an endpoint.

We shall apply again the Brézis–Browder principle to establish Theorems 4.4–4.6. Because the proofs are analogous, we shall prove only one of them.

Proof of Theorem 4.5. Obviously, the relation \leq defined by

$$(4.6) \quad x \leq y \text{ if } \varphi(y) + d(x, y) \leq \varphi(x)$$

is a partial order on X . Further, the function θ given by $\theta(x) = -\varphi(x)$, $x \in X$, is increasing.

Now, let (x_n) be an increasing sequence in the space X . In our hypotheses, the sequence $(\varphi(x_n))$ is convergent. Since

$$(4.7) \quad \varphi(x_m) + d(x_n, x_m) \leq \varphi(x_n), \quad n \leq m,$$

or, equivalently,

$$d(x_n, x_m) \leq \varphi(x_n) - \varphi(x_m), \quad n \leq m,$$

it follows that (x_n) is a Cauchy sequence, hence there exists $x \in X$ such that $x_n \rightarrow x$. Passing to the limit in (4.7) as $m \rightarrow \infty$, and taking into account that φ is l.s.c., we obtain

$$\varphi(x) + d(x_n, x) \leq \varphi(x_n), \quad n \in \mathbf{N},$$

i.e. $x_n \leq x$, $n \in \mathbf{N}$. Thus, (x_n) is bounded with respect to \leq .

Using Brézis–Browder principle, for any $x_0 \in X$ there exists $x^* \in X$, $x_0 \leq x^*$ such that

$$(4.8) \quad x^* \leq x \implies \varphi(x) = \varphi(x^*).$$

In view of our hypotheses, there exists \bar{x}^* such that

$$(4.9) \quad d(x^*, \bar{x}^*) \leq \varphi(x^*) - \varphi(\bar{x}^*), \quad \text{and}$$

$$(4.10) \quad d(\bar{x}^*, y) \leq \varphi(\bar{x}^*) - \varphi(y), \quad \text{for some } y \in F\bar{x}^*.$$

By (4.6), these inequalities can be written in the form $x^* \leq \bar{x}^* \leq y$ for some $y \in F\bar{x}^*$. Then, by (4.8), we deduce

$$(4.11) \quad \varphi(x^*) = \varphi(\bar{x}^*) = \varphi(y).$$

Taking into account (4.9), (4.10) and (4.11) it follows that $x^* = \bar{x}^* = y$. Because $y \in F\bar{x}^*$, we obtain $x^* \in Fx^*$, i.e. x^* is a fixed point of F , q.e.d.

Remark 4.7. The next example shows that Theorems 4.4–4.6 are effective generalizations of the corresponding theorems from the beginning of this section.

Example 4.8. Let $X = [0, +\infty)$, d be the Euclidean distance on X , and φ be defined by $\varphi(t) = e^t$, $t \geq 0$. Let $f : X \rightarrow X$ be given by

$$fx = \begin{cases} \ln(1+x), & x \in X \cap \mathbf{Q}, \\ \sqrt{2}, & x \in X \setminus \mathbf{Q}. \end{cases}$$

(\mathbf{Q} denotes the set of rational numbers.)

Caristi's fixed point theorem can't be applied, since the condition (4.1) is not verified for $x \in [0, \sqrt{2}) \setminus \mathbf{Q}$. On the other hand, it is routine to show that the hypotheses of Theorem 4.4 are fulfilled. Consequently, the function f has a fixed point. As a matter of fact, f has two fixed points: 0 and $\sqrt{2}$.

5 Directional contractive functions and multifunctions

The theorems in Section 3 generalize the well-known Banach's fixed point theorem. Our purpose now is to generalize in the same manner another remarkable fixed point theorem, that of Edelstein concerning contractive functions on compact metric spaces.

Let us start with some definitions.

Definition 5.1. A function $f : X \rightarrow X$ is called *directional contractive* if for any $x \in X$ with $x \neq fx$ there exists a point $y \neq x$ such that

$$(5.1) \quad d(x, y) + d(y, fx) = d(x, fx), \quad \text{and}$$

$$(5.2) \quad d(fx, fy) < d(x, y).$$

Remark 5.2. Unlike the contractive functions, the directional contractive functions are not necessarily continuous and, moreover, they can admit an infinity of fixed points. In order to see this, we give the following

Example 5.3. Let $X = [1, \infty)$ be endowed with the usual distance, and $f : X \rightarrow X$ be defined by

$$fx = \begin{cases} x + \frac{1}{x}, & x \in [1, \infty) \cap \mathbf{Q} \\ x, & x \in [1, \infty) \setminus \mathbf{Q}. \end{cases}$$

It is easy to verify the following assertions: 1° f isn't contractive, 2° f is directional contractive, 3° f isn't continuous, and 4° each $x \in X \setminus \mathbf{Q}$ is a fixed point of f .

Definition 5.4. A multifunction $F : X \rightsquigarrow X$ (with nonempty and bounded values) is called *d-directional contractive* if for any $x \in X$ with $x \notin Fx$ there exists $y \neq x$ such that

$$(5.3) \quad d(x, y) + d(y, Fx) = d(x, Fx), \quad \text{and}$$

$$(5.4) \quad D(Fx, Fy) < d(x, y).$$

Definition 5.5. A multifunction $F : X \rightsquigarrow X$ (with nonempty and bounded values) is called *δ -directional contractive* if for any $x \in X$ with $Fx \neq \{x\}$ there exists $y \neq x$ such that

$$(5.5) \quad d(x, y) + \delta(y, Fx) = \delta(x, Fx), \quad \text{and}$$

$$(5.6) \quad D(Fx, Fy) < d(x, y).$$

Theorem 5.6. Let (X, d) be a compact metric space and $f : X \rightarrow X$ be continuous and directional contractive. Then f has a fixed point.

Proof. Because X is compact and $x \rightarrow d(x, fx)$ is a continuous real-valued function, we can apply a theorem of Weierstrass and point out an element $x^* \in X$ such that

$$(5.7) \quad d(x^*, fx^*) = \inf_{x \in X} d(x, fx).$$

We shall prove that $x^* = fx^*$. Indeed, if $x^* \neq fx^*$ there exists $y \neq x^*$ such that

$$d(x^*, y) + d(y, fx^*) = d(x^*, fx^*) \quad \text{and} \quad d(fx^*, fy) < d(x^*, y).$$

Then

$$d(y, fy) \leq d(y, fx^*) + d(fx^*, fy) < d(y, fx^*) + d(x^*, y) = d(x^*, fx^*),$$

hence $d(y, fy) < d(x^*, fx^*)$, which contradicts (5.7). This completes the proof.

Remark 5.7. Since the space X is compact, it is not necessary the utilization of Brézis–Browder principle. In this case, applying a theorem of Weierstrass we obtain a point which minimizes the function $x \rightarrow d(x, fx)$.

The proof of the next theorem utilizes Lemmas 3.5 and 3.6 and is analogous to the proof of Theorem 5.6.

Theorem 5.8. *Let (X, d) be a compact metric space. A multifunctions $F : X \rightsquigarrow X$ with nonempty and compact values, u.s.c., and d -directional contractive admits a fixed point.*

Remark 5.9. R.E. Smithson observes in [7] that a contractive multifunction F may not be u.s.c., but if, in addition, the values of F are compact, then it is u.s.c. The next example shows us that a d -directional contractive multifunction with compact values is not necessarily u.s.c.

Example 5.10. In the complex plane endowed with the usual metric we consider the compact crown $X = \{z = \rho e^{it}; \rho \in [1, 2], t \in [0, 2\pi)\}$. Let $F : X \rightsquigarrow X$ be given by

$$F(\rho e^{it}) = \begin{cases} \left\{ r e^{it}; r \in \left[1, 2 - \frac{1}{\rho}\right] \right\}, & t \in [0, 2\pi) \cap \mathbb{Q} \\ \left\{ r e^{it}; r \in \left[1 + \frac{1}{\rho}, 2\right] \right\}, & t \in [0, 2\pi) \cap \mathbb{Q}^c. \end{cases}$$

The values of F are compact. It is easy to see that 1° F is not contractive, 2° F is d -directional contractive, 3° F is not u.s.c., and 4° F has an infinity of fixed points.

The existence of endpoints is given by

Theorem 5.11. *Let (X, d) be a compact metric space. A multifunction $F : X \rightsquigarrow X$ with nonempty and bounded values, l.s.c., and δ -directional contractive admits an endpoint.*

The proof is analogous to the proof of Theorem 5.6.

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