Conformally Anosov Flows in Contact Metric Geometry

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Abstract

Mitsumatsu [11], and Eliashberg and Thruston [6] have introduced the notion of a conformally Anosov flow and began a study of its relation to contact structures. In the present paper such flows are studied on contact manifolds via the Riemannian geometry of a metric associated to the contact structure. In particular restrictions on the curvature of an associated metric imply that the characteristic vector field of the contact structure is conformally Anosov. Additional topics are also discussed.

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1 Introduction

In [3] the first author introduced special directions belonging to the contact subbundle of a contact metric manifold with negative sectional curvature for plane sections containing the characteristic vector field ξ of the contact structure, and in the case of 3-dimensional manifolds with ξ an Anosov vector field, compared these with the stable and unstable directions. It is a classical example that the characteristic vector field of the contact structure on the tangent sphere bundle of a negatively curved manifold is (twice) the geodesic flow and therefore generates an Anosov flow. In the case of the tangent sphere bundle of a surface the geodesic flow is closely related to another classical example, namely the standard Anosov flow on the Lie group $SL(2,\mathbf{R})$. These examples are closely related from both the topological and Anosov points of view; however from the Riemannian point of view they are quite different as discussed in [3].

Moreover the classical definition of an Anosov flow is given only for compact manifolds, since in that case, the notion is independent of the metric, but the definition and many properties do not depend on the compactness. For non-compact manifolds the notion of an Anosov flow is metric dependent and in [4] an example of a metric on $\mathbf{R}^3$ was given with respect to which a coordinate vector field is Anosov.

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There has recently been interest in conformally Anosov flows and, in particular, in the case of contact 3-manifolds where the corresponding vector field is not the characteristic vector field but belongs to the contact subbundle (see e.g. [6]). Here we begin to look at the idea of being conformally Anosov from the Riemannian point of view. In the case of an Anosov flow one might expect some negative curvature along the flow as in the case of the Lie group $SL(2, \mathbf{R})$ (see e.g. [3]) or the geodesic flow on the tangent sphere bundle of a negatively curved surface, but the authors know of no result where a curvature hypothesis on a Riemannian manifold with a global vector field implies that the vector field is Anosov. One of the results here, Theorem 4.1, is that on a compact contact metric 3-manifold with negative sectional curvature for plane sections containing the characteristic vector field ξ, the vector field ξ is conformally Anosov. As a corollary, if a contact metric 3-manifold admits Anosov-like special directions as introduced in [15], then ξ is conformally Anosov.

In sectin 5 we give an example of a flow belonging to the contact subbundle which is conformally Anosov with respect to a Sasakian metric and which leaves the characteristic vector field invariant. In section 6 we show that the phenomena of the example in section 5 does not occur in the case of the tangent sphere bundle of a surface. Finally in section 7 we discuss Anosov-like and conformally Anosov flows on 3-dimensional homogeneous contact metric manifolds.

First we review the geometry of contact manifolds and the ideas of Anosov and conformally Anosov flows.

2 Contact manifolds

By a *contact manifold* we mean a C^∞ manifold M^{2n+1} together with a 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. It is well known that given η there exists a unique vector field ξ such that $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$ called the *characteristic vector field* or *Reeb vector field* of the contact structure η. A classical theorem of Darboux states that on a contact manifold there exist local coordinates with respect to which $\eta = dz - \sum_{i=1}^{n} y^i dx^i$. We denote the *contact subbundle* or *contact distribution* defined by the subspaces $\{X \in T_mM : \eta(X) = 0\}$ by $\mathcal{D}$. Roughly speaking the meaning of the contact condition, $\eta \wedge (d\eta)^n \neq 0$, is that the contact subbundle is as far from being integrable as possible. In fact for a contact manifold the maximum dimension of an integral submanifold of $\mathcal{D}$ is only n; whereas a subbundle defined by a 1-form η is integrable if and only if $\eta \wedge d\eta \equiv 0$. A 1-dimensional integral submanifold is called a *Legendre curve*.

A Riemannian metric g is an *associated metric* for a contact form η if there exists a tensor field ϕ of type (1.1) such that

$$\phi^2 = -I + \eta \otimes \xi, \;\; \eta(X) = g(X, \xi), \;\; d\eta(X, Y) = g(X, \phi Y).$$

We refer to (η, g) or (ϕ, ξ, η, g) as a *contact metric structure*. All associated metrics have the same volume element, viz., $\frac{(-1)^n}{2^n n!}\eta \wedge (d\eta)^n$. Since $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$, computing Lie derivatives, we have $\mathcal{L}_\xi \eta = 0$ and $\mathcal{L}_\xi d\eta = 0$. Thus the flow generated by ξ is volume preserving for any associated metric.

In the theory of contact metric manifolds there is another tensor field that plays a fundamental role, viz. $h = \frac{1}{2}\mathcal{L}_\xi \phi$. h is a symmetric operator which anti-commutes with ϕ, $h\xi = 0$ and h vanishes if and only if ξ is Killing in which case the contact metric structure is said to be *K-contact*. We denote by ∇ the Levi-Civita connection of g and by R its curvature tensor defined by

$$R_{X\,Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

On a contact metric manifold we have the following further important relations involving h,

$$\nabla_X \xi = -\phi X - \phi h X, \tag{2.1}$$

$$\frac{1}{2}(R_{\xi\,X}\xi - \phi R_{\xi\,\phi X}\xi) = h^2 X + \phi^2 X, \tag{2.2}$$

$$(\nabla_\xi h)X = \phi(X - h^2 X - R_{X\,\xi}\xi). \tag{2.3}$$

As a corollary we see from equation (2.2) that on a contact metric manifold M^{2n+1} the Ricci curvature in the direction ξ is given by

$$Ric(\xi) = 2n - \mathrm{tr} h^2. \tag{2.4}$$

Since $h\phi + \phi h = 0$, if λ is an eigenvalue of h with eigenvector X, then $-\lambda$ is also an eigenvalue with eigenvector ϕX. Thus, since $h\xi = 0$, in dimension 3 we have only one eigenfunction λ on the manifold to be concerned with.

The sectional curvature of a plane section containing ξ is called a *ξ-sectional curvature*. If $X \in \mathcal{D}$ we denote the ξ-sectional curvature by $K(\xi, X)$. In this paper the notion of a Sasakian manifold arises only occassionally and we refer to [2] as a reference.

In the geometry of contact metric 3-manifolds the condition

$$\nabla_\xi h = 0 \tag{2.5}$$

arises when ξ is Anosov and other conditions are satisfied (see [3], [15] and the next section of the present paper). In [13] the second author showed that the standard contact metric structure of the tangent sphere bundle of a Riemannian manifold satisfies (2.5) if and only if the base manifold is of constant curvature 0 or +1. Also for a compact 3-dimensional contact manifold the second author [12] showed that (2.5) is the critical point condition for the integral of the scalar curvature considered as a functional on the set of all associated metrics. Equation (2.5) is equivalent to $\nabla_\xi \tau = 0$ where $\tau = \mathcal{L}_\xi g$ and is related to h by $\tau(X,Y) = 2g(h\phi X, Y)$. Contact metric 3-manifolds satisfying (2.5) were called *3-τ-manifolds* in [9].

We end this section by briefly stating the results of [3]. One may regard equation (2.1) as indicating how ξ or, by orthogonality, the contact subbundle, rotates as one moves around on the manifold. For example when $h = 0$, as we move in a direction X orthogonal to ξ, ξ is always "turning" or "falling" toward $-\phi X$. If $hX = \lambda X$, then $\nabla_X \xi = -(1+\lambda)\phi X$ and again ξ is turning torward $-\phi X$ if $\lambda > -1$ or toward ϕX if

$\lambda < -1$. Recall, as we noted above, that if λ is an eigenvalue of h with eigenvector X, then $-\lambda$ is also an eigenvalue with eigenvector ϕX.

Now one can ask if there can ever be directions, say Y orthogonal to ξ, along which ξ "falls" forward or backward in the direction of Y itself; this is answered by the following theorem.

Theorem 2.1 *Let M^{2n+1} be a contact metric manifold. If the tensor field h admits an eigenvalue $\lambda > 1$ at a point P, then there exists a vector Y orthogonal to ξ at P such that $\nabla_Y \xi$ is collinear with Y. In particular, if M^{2n+1} has negative ξ-sectional curvature, such directions Y exist.*

Note that when there exists a direction Y along which $\nabla_Y \xi$ is collinear with Y, there is also a second such direction. Let $\{X, \phi X, \xi\}$ be a local eigenvector field basis of h with $hX = \lambda X$; write $Y = aX + b\phi X$ with $a > 0, b > 0, a^2 + b^2 = 1$. The proof of Theorem 2.1 gives $\nabla_Y \xi = \alpha Y$ with $\alpha = -\sqrt{\lambda^2 - 1}$. Then for $Z = aX - b\phi X$ we have $\nabla_Z \xi = -\alpha Z$; thus we think of ξ as falling backward as we move in the direction Y and falling forward as we move in the direction Z. Next note that

$$g(Y, Z) = a^2 - b^2 = -\frac{1}{\lambda}$$

and hence that such directions Y and Z are never orthogonal. We refer to directions such as those determined by Y and Z as *special directions*.

In section 3 we will point out the metric dependency of Anosov flows on non-compact manifolds. In the remainder of this paper we will have occasion to speak of a flow being *Anosov or conformally Anosov with respect to a particular metric*. For 3-dimensional contact metric manifolds if ξ generates an Anosov flow, one can ask what happens if these special directions agree with the stable and unstable bundles (Anosov directions).

Theorem 2.2 *Let (M^3, η, g) be a contact metric manifold with negative ξ-sectional curvature. If the characteristic vector field ξ is an Anosov flow with respect to g and the special directions agree with the Anosov directions, then the contact metric structure is a 3-τ-structure. Moreover if M is compact, it is a compact quotient of $\widetilde{SL}(2, \mathbf{R})$.*

We mentioned in the introductions that the geodesic flow of a surface the geodesic flow is closely related to the standard Anosov flow on the Lie group $SL(2, \mathbf{R})$. That these examples differ from the Riemannian point of view is given by the following result.

Theorem 2.3 *With respect to the standard contact metric structure on the tangent sphere bundle of a negatively curved surface, the characteristic vector field is Anosov, but the special directions never agree with the stable and unstable directions.*

3 Anosov and conformally Anosov flows

Classically an Anosov flow is defined as follows [1, pp.6-7]. Let M be a compact differentiable manifold, ξ a non-vanishing vector field and $\{\psi_t\}$ its 1-parameter group of diffeomorphisms. $\{\psi_t\}$ is said to be an *Anosov flow* (or ξ to be *Anosov*) if there exist subbundles E^s and E^u which are invariant along the flow and such that $TM = E^s \oplus E^u \oplus \{\xi\}$ and there exists a Riemannian metric such that

$$|\psi_{t*}Y| \leq ae^{-ct}|Y| \text{ for } t \geq 0 \text{ and } Y \in E^s_p,$$

$$|\psi_{t*}Y| \leq ae^{ct}|Y| \text{ for } t \leq 0 \text{ and } Y \in E^u_p$$

where a and c are positive constants independent of $p \in M$ and Y in E^s_p or E^u_p. E^s and E^u are called the *stable* and *unstable* subbundles or the *contracting* and *expanding* subbundles.

When M is compact the notion is independent of the Riemannian metric. If M is not compact the notion is metric dependent; the example of a 3-τ-manifold on which the Ricci operator does not commute with ϕ given in [4] is a metric on $\mathbf{R}^3$ with respect to which the coordinate field $\frac{\partial}{\partial z}$ is Anosov, even though $\frac{\partial}{\partial z}$ is clearly not Anosov with respect to the Euclidean metric on $\mathbf{R}^3$.

Mitsumatsu [11] and Eliashberg and Thurston [6] have introduced a generalization of the notion of an Anosov flow. A flow ψ_t or its corresponding vector field is said to be *conformally Anosov* [6] (*projectively Anosov* [11]) if there is a continuous Riemannian metric and a continuous, invariant splitting of $TM = E^s \oplus E^u \oplus \{\xi\}$ as in the Anosov case such that for $Z \in E^u$ and $Y \in E^s$,

$$\frac{|\psi_{t*}Z|}{|\psi_{t*}Y|} \geq e^{ct}\frac{|Z|}{|Y|}$$

for some constant $c > 0$ and all $t \geq 0$.

Now a contact structure η on a 3-dimensional contact manifold M^3 determines an orientation on M^3. This is true in dimension 3 even for a contact structure in the wider sense, since the sign of $\eta \wedge d\eta$ is independent of the choice of local contact form η. In particular we can say that a contact structure is positive or negative with respect to a fixed orientation according as $\eta \wedge d\eta$ induces the same or opposite orientation.

The main result for our purpose from Mitsumatsu [11, p.1418] and Eliashberg and Thurston [6, pp.26-27] is the following.

Theorem 3.1 *If two contact structures on a compact 3-dimensional contact manifold M^3 induce opposition orientations, then the vector field directing the intersection of the two contact subbundles is a conformally Anosov flow. Conversely given a conformally Anosov flow on M^3, there exist two contact structures giving opposite orientations on M^3 whose contact subbundles intersect tangent to the flow.*

Utilizing the special directions Y and Z belonging to the contact subbundle on a contact metric manifold of negative ξ-sectional curvature, the second author [15] introduced another notion. Let $\mathcal{Y}$ and $\mathcal{Z}$ denote the subbundles generated by Y and Z. The special directions are said to be *Anosov-like* if the subbundles $\mathcal{Y} \oplus \{\xi\}$ and $\mathcal{Z} \oplus \{\xi\}$ are integrable and we have the following result from [15].

Theorem 3.2 *A contact metric 3-manifold admits Anosov-like special directions if and only if it is a 3-τ-manifold with negative Ricci curvature in the direction ξ.*

4 Curvature and conformally Anosov flows

In this section we show that certain curvature hypotheses on a compact contact metric 3-manifold will imply that the characteristic vector field ξ is conformally Anosov; in particular negative ξ-sectional curvature is such a hypothesis. A variation of this theorem was given in [5]. In the remainder of this paper λ will always denote a positive eigenvalue of h.

Theorem 4.1 *Let $(M^3, \phi, \xi, \eta, g)$ be a compact 3-dimensional contact metric manifold. If the ξ-sectional curvature satisfies the condition*

$$K(\xi, X) < (1-\lambda)^2, \quad \lambda \geq 0$$

for every $X \in \mathcal{D}$, then ξ is conformally Anosov. In particular, if the ξ-sectional curvature is negative, ξ is conformally Anosov.

Proof. Let $\{e_1, e_2(=\phi e_1), \xi\}$ be an orthonormal eigenvector basis of h with $he_1 = \lambda e_1$, λ the non-negative eigenvalue. Since $Ric(\xi) = 2(1-\lambda^2)$ and $Ric(\xi) = K(\xi, e_1) + K(\xi, e_2)$, the hypothesis first gives us that $(1-\lambda^2) < (1-\lambda)^2$. Thus h is non-vanishing and its positve eigenvalue is $\lambda > 1$. Moreover since the three eigenvalues of h are everywhere distinct the corresponding line fields are global and by the orientability the basis may be taken to be global. Now let ω^1, ω^2 be the dual 1-forms of e_1 and e_2. By staightforward computation we have

$$\omega^1 \wedge d\omega^1(e_1, e_2, \xi) = \frac{\lambda - 1}{6} - \frac{1}{6} g(\nabla_\xi e_1, e_2), \tag{4.1}$$

$$\omega^2 \wedge d\omega^2(e_1, e_2, \xi) = -\frac{\lambda + 1}{6} - \frac{1}{6} g(\nabla_\xi e_1, e_2). \tag{4.2}$$

Applying equation (2.3) to e_1 and e_2 we have

$$K(\xi, e_1) = 1 - \lambda^2 - 2\lambda g(\nabla_\xi e_1, e_2), \tag{4.3}$$

$$K(\xi, e_2) = 1 - \lambda^2 + 2\lambda g(\nabla_\xi e_1, e_2). \tag{4.4}$$

Combining (4.4) with (4.1) and (4.3) with (4.2) we have

$$\omega^1 \wedge d\omega^1(e_1, e_2, \xi) = \frac{1}{12\lambda}((\lambda - 1)^2 - K(\xi, e_2))$$

and

$$\omega^2 \wedge d\omega^2(e_1, e_2, \xi) = \frac{1}{12\lambda}(-(\lambda + 1)^2 + K(\xi, e_1)).$$

Since for $\lambda > 0$, $(1-\lambda)^2 < (1+\lambda)^2$, the hypothesis implies that $\omega^1 \wedge d\omega^1(e_1, e_2, \xi) > 0$ and $\omega^2 \wedge d\omega^2(e_1, e_2, \xi) < 0$. Therefore ξ is conformally Anosov by the result of Mitsumatsu and Eliashberg-Thurston.

We remark that our argument uses two contact structures to study a third, an idea that might be exploited further in view of the result of Gonzalo [8] that a 3-dimensional compact orientable manifold admits three independent contact structures.

Using the idea of Anosov-like special directions we have the following result.

Theorem 4.2 *If a compact 3-dimensional contact metric manifold* $(M^3, \phi, \xi, \eta, g)$ *admits Anosov-like special directions, then* ξ *is conformally Anosov.*

Proof. Theorem 3.2 implies that (M^3, η, g) has negative Ricci curvature in the directions ξ and satisfies $\nabla_\xi h = 0$. The Ricci curvature in the direction ξ being negative implies by (2.4), that the positive eigenvalue $\lambda > 1$. Finally, $\nabla_\xi h = 0$ implies, by virtue of (2.3), that $K(\xi, X) = 1 - \lambda^2 < 0$ for any $X \in \mathcal{D}$ and the result follows from Theorem 4.1.

5 Conformally Anosov flows with Legendre orbits

It is immediate that the characteristic vector field ξ of a contact structure can never be Anosov or conformally Anosov with respect to a Sasakian metric. This is a consequence of the fact that on a Sasakian manifold ξ is a Killing vector field; thus its flow is metric preserving and therefore cannot satisfy the exponential growth behavior in the definition of a classical or conformal Anosov flow. It is possible however for a vector field to belong to the contact subbundle of a Sasakian structure and be conformally Anosov with respect to this metric. We present an example of this as a proposition to emphasize the additional feature of the example that the characteristic vector field is actually invariant along the flow. We begin with the following lemma.

Lemma 5.1 *On a 3-dimensional contact manifold in terms of local Darboux coordinates* (x, y, z) $(\eta = \frac{1}{2}(dz - ydx))$ *any associated metric is of the form*

$$g = \frac{1}{4}\begin{pmatrix} a & b & -y \\ b & c & 0 \\ -y & 0 & 1 \end{pmatrix}$$

with $ac - b^2 - cy^2 = 1$*; the metric is Sasakian if and only if the functions* a, b *and* c *are independent of* z.

Proof. The form of the last row and column follow from the requirement that $\eta(X) = g(X, \xi)$. In dimension 3 the remaining requirements reduce to the determinant of the matrix (without the $\frac{1}{4}$) being 1. Also in dimension 3 the Sasakian condition is equivalent to the contact metric structure being K-contact. Thus evaluating the Lie derivative, $\mathcal{L}_\xi g$, on the coordinate vector fields we see that $\xi = 2\frac{\partial}{\partial z}$ is Killing if and only if the functions a, b and c are independent of z.

Proposition 5.2 *There exists a Sasakian manifold admitting a conformally Anosov flow* ψ_t *whose orbits are Legendre curves and which leaves the characteristic vector field invariant, i.e.* $\psi_{t*}\xi = \xi$.

Proof. Consider $\mathbf{R}^3_+ = \{(x, y, z) | y > 0\}$ with the standard Darboux contact form $\eta = \frac{1}{2}(dz - ydx)$. The characteristic vector field is $2\frac{\partial}{\partial z}$ and the Riemannian metric given by the following matrix is clearly Sasakian by the Lemma.

$$g = \frac{1}{4}\begin{pmatrix} e^{2y} & \sqrt{e^{2y}-y^2-1} & -y \\ \sqrt{e^{2y}-y^2-1} & 1 & 0 \\ -y & 0 & 1 \end{pmatrix}$$

The vector field $\dfrac{\partial}{\partial y}$ is conformally Anosov with respect to this metric. To see this we observe that the subbundles determined by $\dfrac{\partial}{\partial z} = \dfrac{1}{2}\xi$ and $\dfrac{\partial}{\partial x}$ correspond to E^s and E^u respectively. The flow simply maps a point $P_0(x, y, z)$ to the point $P(x, y+t, z)$ and we have easily that

$$\frac{\left|\frac{\partial}{\partial x}(P)\right|}{\left|\frac{\partial}{\partial z}(P)\right|} = e^t \frac{\left|\frac{\partial}{\partial x}(P_0)\right|}{\left|\frac{\partial}{\partial z}(P_0)\right|}.$$

Clearly this flow satisfies $\psi_{t*}\xi = \xi$.

6 Tangent sphere bundles

In contrast to the Proposition 5.2, we show that in the case the tangent sphere bundle of a surface the phenomena of the proposition cannot occur.

We begin with a brief review of the standard contact metric structure on the unit tangent bundle T_1M. Let (M, G) be an $(n+1)$-dimensional Riemannian manifold and $\bar{\pi} : TM \longrightarrow M$ its tangent bundle. If $(x^1, \ldots, x^{n+1})$ are local coordinates on M, identify x^i with $x^i \circ \bar{\pi}$; then $(x^1, \ldots, x^{n+1})$ together with the fibre coordinates $(v^1, \ldots, v^{n+1})$ form local coordinates on TM. If X is a vector field on M, denote by X^V its vertical lift to TM. Using the Levi-civita connection D of G on M, one defines the horizontal lift X^H of X by $X^H\omega = D_X\omega$ where ω is a 1-form on M which on the left is regarded as a function on TM. The *connection map* $K : TTM \longrightarrow TM$ is defined by

$$KX^H = 0, \; K(X_t^V) = X_{\bar{\pi}(t)}, \; t \in TM;$$

TM admits an almost complex structure J defined by $JX^H = X^V$, $JX^V = -X^H$. It is well known that J is integrable if and only if the base manifold is flat.

Define a Riemannian metric $\bar{g}$ on TM called the *Sasaki metric*, by

$$\bar{g}(X, Y) = G(\bar{\pi}_*X, \bar{\pi}_*Y) + G(KX, KY)$$

where X and Y are vector fields on TM. Since $\bar{\pi}_* \circ J = -K$ and $K \circ J = \bar{\pi}_*$, $\bar{g}$ is Hermitian for the almost complex structure J. On TM define a 1-form β by $\beta(X)_t = G(t, \bar{\pi}_*X)$, $t \in TM$ or equivalently by the local expression $\beta = \sum G_{ij}v^i dx^j$. Then $d\beta$ is a symplectic structure on TM and in particular $2d\beta$ is the fundamental 2-form of the almost Hermitian structure $(J, \bar{g})$.

Let $\mathbf{R}$ denote the curvature tensor of G, then the Levi-Civita connection of $\bar{g}$, $\bar{\nabla}$, is given by

$$(\bar{\nabla}_{X^H}Y^H)_t = (D_XY)_t^H - \frac{1}{2}(\mathbf{R}(X, Y)t)_t^V, \quad (\bar{\nabla}_{X^V}Y^H)_t = \frac{1}{2}(\mathbf{R}(t, X)Y)_t^H,$$

$$(\bar{\nabla}_{X^H}Y^V)_t = \frac{1}{2}(\mathbf{R}(t,Y)X)^H_t + (D_XY)^V_t, \quad \bar{\nabla}_{X^V}Y^V = 0.$$

The tangent sphere bundle $\pi : T_1M \longrightarrow M$ is the hypersurface of TM defined by $\sum G_{ij}v^iv^j = 1$. The vector field $N = v^i\frac{\partial}{\partial v^i}$ is a unit normal, as well as the position vector for a point t. The Weingarten map of the hypersurface annihilates horizontal vectors and acts as minus the identity on vertical tangent vectors (cf. [2, p.132]).

Let g' denote the metric on T_1M induced from $\bar{g}$ on TM. Define ϕ', ξ' and η' on T_1M by $\xi' = -JN,\ JX = \phi'X + \eta'(X)N$. η' is the contact form on T_1M induced from the 1-form β on TM as one can easily check. However $g'(X, \phi'Y) = 2d\eta'(X,Y)$, so strictly speaking (ϕ', ξ', η', g') is not a contact metric structure. Of course the difficulty is easily rectified and we shall take

$$\eta = \frac{1}{2}\eta',\ \xi = 2\xi',\ \phi = \phi',\ g = \frac{1}{4}g'$$

as the standard contact metric structure on T_1M. In local coordinates

$$\xi = 2v^i(\frac{\partial}{\partial x^i})^H;$$

the vector field $v^i(\frac{\partial}{\partial x^i})^H$ is the well known geodesic flow.

Theorem 6.1 *The tangent sphere bundle of a complete surface (M, G) cannot admit a flow ψ_t that is conformally Anosov with respect to its standard contact metric structure, whose orbits are Legendre curves and which leaves the characteristic vector field invariant.*

Proof. Reparametrizing the flow if necessary, let A be a unit vector field corresponding to the flow ψ_t. Write A as $\alpha X + \beta U$ where U is a unit vertical vector field and X a unit horizontal vector field orthogonal to ξ. Then using the above formulas of $\bar{\nabla}$ we find that at a point $t \in T_1M$

$$\nabla_\xi U = -(\mathbf{R}(KU,t)t)^H, \quad \nabla_\xi X = (\mathbf{R}(\pi_*X,t)t)^V,$$

$$\nabla_U \xi = 2X - (\mathbf{R}(KU,t)t)^H, \quad \nabla_X\xi = -(\mathbf{R}(\pi_*X,t)t)^V.$$

Using the invariance of ξ by the flow we have

$$0 = [A,\xi] = -2\alpha(\mathbf{R}(\pi_*X,t)t)^V + 2\beta X - (\xi\alpha)X - (\xi\beta)U.$$

Taking horizontal and vertical parts we have

$$\xi\alpha - 2\beta = 0, \quad \xi\beta + 2\alpha g((\mathbf{R}(\pi_*X,t)t)^V, U) = 0$$

from which

$$-2\alpha\beta + 2\alpha\beta g((\mathbf{R}(\pi_*X,t)t)^V, U) = 0.$$

Therefore either $\alpha = 0$, $\beta = 0$ or $G(\mathbf{R}(\pi_*X,t)t), KU) = 4$.

If $\alpha = 0$, U is conformally Anosov and $0 = [U,\xi] = 2X$ a contradiction.

If $\beta = 0$, X is conformally Anosov and $0 = [X,\xi] = -2(\mathbf{R}(\pi_*X,t)t)^V$ giving that (M, G) is flat. Thus locally (M,G) is the Euclidean plane and G is given by

$ds^2 = dx^2 + dy^2$. Instead of using the coordinates (x, y, v^1, v^2), we will use coordinates (x, y, θ) on T_1M given by $v^1 = \cos\theta$, $v^2 = \sin\theta$. Then the vertical space of T_1M is spanned by $U = 2\frac{\partial}{\partial\theta}$ and the geodesic flow is given by $\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} = \frac{1}{2}\xi$. The vector field X is given by $X = 2(-\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y})$. The metric g on T_1M is given by $ds^2 = \frac{1}{4}(dx^2 + dy^2 + d\theta^2)$. The flow ψ_t of X mapping a point $P(x, y, \theta)$ to $\bar{P}(\bar{x}, \bar{y}, \bar{\theta})$ is given by $\bar{x} = x - 2t\sin\theta$, $\bar{y} = y + 2t\cos\theta$, $\bar{\theta} = \theta$ and its differential by

$$\psi_{t*} = \begin{pmatrix} 1 & 0 & -2t\cos\theta \\ 0 & 1 & -2t\sin\theta \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus under the action of ψ_{t*}, the length of vectors belonging to E^u and E^s has less than exponential growth and hence X is not conformally Anosov with respect to the standard contact metric structure, a contradiction.

If $G(\mathbf{R}(\pi_*X, t)t), KU) = 4$, (M, G) is of constant curvature $+1$ and hence by completeness, (M, G) is isometric to the sphere or real projective plane. The tangent sphere bundle T_1M is then the Lie group $SO(3)$ or the lens space $L(4, 1)$ respectively. Lifting the vector field to the universal covering space, the 3-sphere would admit a conformally Anosov vector field which is impossible as pointed out by Mitsumatsu, [11, p.1420].

7 3-dimensional homogeneous contact metric manifolds

A result of Ghys [7] shows that if ξ is Anosov on a compact 3-dimensional contact manifold and the Anosov directions are smooth, then M is a homogeneous space of the form $\widetilde{SL}(2, \mathbf{R})/\Gamma$ (see also [3] for a different proof). Here we give the following extension.

Theorem 7.1 *Let (M, η, g) be a 3-dimensional homogeneous contact metric manifold. If ξ is Anosov-like, then the universal cover $\tilde{M}$ of M is $\widetilde{SL}(2, \mathbf{R})$ and ξ is Anosov.*

Proof. By Theorem 3.2 ξ is Anosov like if and only if $\nabla_\xi\tau = 0$ and $Ric(\xi) < 0$ and by Theorem 3.1 of [14] (M, η, g) is a Lie group G with a left invariant contact metric structure. The result now follows from [14] which we summarize here. If G is unimodular,

$$(\nabla_\xi\tau)(X, Y) = (2 - 4W)\tau(X, \phi Y)$$

where W is the Webster scalar curvature; the positive eigenvalue of h is $\lambda = \frac{|\tau|}{2\sqrt{2}}$. Thus $\nabla_\xi\tau = 0$ and $Ric(\xi) < 0$ is equivalent to $W = \frac{1}{2}$ and $|\tau| > 2\sqrt{2}$ ($\lambda > 1$). Now on the Heisenberg group $|W| = |\tau| = 0$. On $SU(2)$, $|\tau| < 4\sqrt{2}W = 2\sqrt{2}$. On the group $\tilde{E}(2)$ of motions of the Euclidean plane, $|\tau| = 4\sqrt{2}W = 2\sqrt{2}$. On the group

$E(1,1)$ of motions of the Minkowski plane, $4\sqrt{2}W = -|\tau| < 0$. Finally on $\widetilde{SL}(2,\mathbf{R})$, $4\sqrt{2}W < |\tau|$, the only case where $W = \frac{1}{2}$ and $|\tau| > 2\sqrt{2}$ are satisfied. In this case ξ is known to be Anosov (cf. [3]).

In the non-unimodular case, the Lie algebra is given by

$$[e_1, e_2] = \alpha e_2 + 2\xi, \quad [e_1, \xi] = \gamma e_2, \quad [e_2, \xi] = 0$$

with $\alpha \neq 0$ [14]. Here $\nabla_\xi \tau = 0$ implies $\gamma = -2$ and $Ric(\xi) = 0$. Thus we have only the case of $\widetilde{SL}(2,\mathbf{R})$.

Turning to flows belonging to contact subbundle we prove the following theorem.

Theorem 7.2 *Let (M,η,g) be a compact 3-dimensional homogeneous contact metric manifold and $\{e_1, e_2 (= \phi e_1), \xi\}$ an orthonormal eigenvector basis of h. Then M admits a conformally Anosov flow defined by one of the e_i if and only if the universal cover $\tilde{M}$ is either $\widetilde{SL}(2,\mathbf{R})$ or $E(1,1)$. In particular the conformally Anosov flow belongs to the contact subbundle and moreover its integral curves are geodesics.*

Proof. By [14] we know that (M,η,g) is a Lie group G with a left invariant contact metric structure. Milnor shows in [10, Lemma 6.2] that the non-unimodular case has no compact quotients. Thus G is unimodular. The Lie algebra of G is given by

$$[e_1, e_2] = 2\xi, \quad [e_2, \xi] = \lambda_1 e_1, \quad [\xi, e_1] = \lambda_2 e_2 \tag{7.1}$$

where λ_1, λ_2 are constants. Let ω^1, ω^2 the dual 1-forms of e_1 and e_2. Using (7.1) we find

$$d\eta = -2\omega^1 \wedge \omega^2, \quad d\omega^1 = \lambda_1 \eta \wedge \omega^2, \quad d\omega^2 = -\lambda_2 \eta \wedge \omega^1,$$

from which

$$\eta \wedge d\eta = -2\eta \wedge \omega^1 \wedge \omega^2, \quad \omega^1 \wedge d\omega^1 = -\lambda_1 \eta \wedge \omega^1 \wedge \omega^2, \quad \omega^2 \wedge d\omega^2 = -\lambda_2 \eta \wedge \omega^1 \wedge \omega^2. \tag{7.2}$$

From (7.2) and the theorem of Mitsumatsu and Eliashberg-Thurston we see that e_i is conformally Anosov only in the following cases:

a) $\lambda_2 > 0, \lambda_1 < 0$ or $\lambda_2 < 0, \lambda_1 > 0$;

b) $\lambda_2 = 0, \lambda_1 < 0$ or $\lambda_2 < 0, \lambda_1 = 0$.

The case a) corresponds to the Lie group $SL(2,\mathbf{R})$ and case b) to $E(1,1)$ (see Theorem 3.1 in [14]). Moreover from the first equation in (7.1) we have

$$\nabla_{e_1} e_1 = \nabla_{e_2} e_2 = 0$$

showing that the integral curves of e_1 and e_2 are geodesics.

Even though $SL(2,\mathbf{R})$ and $E(1,1)$ are not compact they do admit flows that are Anosov with respect to contact metric structures. Anosov flows on $SL(2,\mathbf{R})$ are well know and discussed in the contact metric context in [3]. We now exhibit an Anosov flow on $E(1,1)$. The group $E(1,1)$ can be represented by matrices of the form

$$\begin{pmatrix} e^y & 0 & z \\ 0 & e^{-y} & x \\ 0 & 0 & 1 \end{pmatrix}.$$

Consider the matrices

$$\begin{pmatrix} 0 & 0 & \dfrac{-1}{\sqrt{\lambda}} \\ 0 & 0 & \dfrac{1}{\sqrt{\lambda}} \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2\sqrt{\lambda} & 0 & 0 \\ 0 & -2\sqrt{\lambda} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda > 0,$$

in the Lie algebra $e(1,1)$ regarded as the tangent space at the identity. Applying the differential of left translation by $\begin{pmatrix} e^y & 0 & z \\ 0 & e^{-y} & x \\ 0 & 0 & 1 \end{pmatrix}$ to these matrices we obtain the vector fields

$$e_1 = \frac{1}{\sqrt{\lambda}}\left(e^{-y}\frac{\partial}{\partial x} - e^y\frac{\partial}{\partial z}\right), \quad e_2 = 2\sqrt{\lambda}\frac{\partial}{\partial y}, \quad \xi = e^{-y}\frac{\partial}{\partial x} + e^y\frac{\partial}{\partial z},$$

whose Lie brackets satisfy

$$[e_1, e_2] = 2\xi, \quad [e_2, \xi] = -2\lambda e_1, \quad [\xi, e_1] = 0.$$

Consider the left invariant metric g such that $\{e_1, e_2, \xi\}$ is an orthonormal basis. The dual 1-form η of ξ is

$$\eta = \frac{1}{2}\left(e^y dx + e^{-y} dz\right)$$

and (η, g) is a contact metric structure on $E(1,1)$

The flow ψ_t generated by the vector field e_2 maps a point $P_0(x,y,z)$ to the point $P(x, y + 2\sqrt{\lambda}t, z)$. Then

$$\left|\psi_{t*}\left(\frac{\partial}{\partial x}(P_0)\right)\right| = \left|\frac{\partial}{\partial x}(P)\right| = \frac{1}{2}\sqrt{1+\lambda}\, e^{y+2\sqrt{\lambda}t} = e^{2\sqrt{\lambda}t}\left|\frac{\partial}{\partial x}(P_0)\right|$$

and

$$\left|\psi_{t*}\left(\frac{\partial}{\partial z}(P_0)\right)\right| = \left|\frac{\partial}{\partial z}(P)\right| = \frac{1}{2}\sqrt{1+\lambda}\, e^{-y-2\sqrt{\lambda}t} = e^{-2\sqrt{\lambda}t}\left|\frac{\partial}{\partial z}(P_0)\right|$$

showing that e_2 is Anosov.

In the non-unimodular case the simply conntected Lie group can also be realized on $\mathbf{R}^3$. As shown in [14] one can find an orthonormal basis $\{e_1, e_2(=\phi e_1), \xi\}$ of the Lie algebra such that

$$[e_1, e_2] = \alpha e_2 + 2\xi, \quad [e_2, \xi] = 0, \quad [e_1, \xi] = \gamma e_2,$$

$\alpha \neq 0$. This gives rise on $\mathbf{R}^3$ to the vector fields

$$e_1 = 2\frac{\partial}{\partial x} - (\alpha y + \gamma z)\frac{\partial}{\partial y} - 2y\frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad \xi = \frac{\partial}{\partial z}.$$

Moreover one has the contact form $\eta = dz + y dx$ and associated metric

$$g = \begin{pmatrix} \frac{1}{4}(1 + 4y^2 + (\alpha y + \gamma z)^2) & \frac{1}{2}(\alpha y + \gamma z) & y \\ \frac{1}{2}(\alpha y + \gamma z) & 1 & 0 \\ y & 0 & 1 \end{pmatrix}.$$

We will show that in the case $\gamma > 0$, the vector field e_1 is Anosov. Let $\mu_1, \mu_2 = \dfrac{-\alpha \pm \sqrt{\alpha^2 + 8\gamma}}{2}$. The flow ψ_t determined by the vector field e_1 is given by

$$P_0(x,y,z) \longrightarrow P(x+2t, \frac{\mu_1 y - \gamma z}{\mu_1 - \mu_2} e^{\mu_1 t} + \frac{\gamma z - \mu_2 y}{\mu_1 - \mu_2} e^{\mu_2 t}, \frac{2}{\mu_1} \frac{\gamma z - \mu_1 y}{\mu_1 - \mu_2} e^{\mu_1 t} \frac{2}{\mu_2} \frac{\gamma z - \mu_2 y}{\mu_1 - \mu_2} e^{\mu_2 t}).$$

Now consider the vector fields $Y = -\mu_2 e_2 + 2\xi$ and $Z = -\mu_1 e_2 + 2\xi$. Applying ψ_{t*} we find that Y and Z are invariant and computing lengths we have

$$|\psi_{t*}(Y(P_0))| = e^{\mu_2 t}|Y(P_0)|, \quad |\psi_{t*}(Z(P_0))| = e^{\mu_1 t}|Z(P_0)|.$$

For $\gamma > 0$, $\mu_1 > 0$ and $\mu_2 < 0$. If $\mu_1 < |\mu_2|$,

$$|\psi_{t*}(Y(P_0))| = e^{\mu_2 t}|Y(P_0)| \leq e^{-\mu_1 t}|Y(P_0)| \text{ for } t \geq 0;$$

if $\mu_1 > |\mu_2|$,

$$|\psi_{t*}(Z(P_0))| = e^{\mu_1 t}|Z(P_0)| \leq e^{|\mu_2| t}|Z(P_0)| \text{ for } t \leq 0.$$

Since $\mu_1 = |\mu_2|$ would imply $\alpha = 0$, we see that for $\gamma > 0$, e_1 is Anosov.

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