On Finite Type Closed Curves on the Pseudo-Hyperbolic Space $H^3(-c^2)$

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Abstract

We obtain some nonexistence theorems of certain finite type closed curves on the pseudo-hyperbolic space $H^3(-c^2)$ in the Minkowski spacetime E_1^4.

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1 Introduction

First, we will survey briefly the fundamental concepts and properties in the pseudo-Riemannian geometry. We refer mainly to O'Neill([9]) and Chen([3],[4]). For the general concepts in the Riemannian geometry, refer to the book of Kobayashi and Nomizu([8]).

Let M be a C^∞-class differentiable manifold of dimension n and g a C^∞-class differentiable symmetric nondegenerate tensor field of type $(0,2)$ on M. The pseudo-Riemannian metric g_p at every point p of M defines the scalar product on the tangent space $T_p(M)$ of M at p. The index of g_p is not necessarily constant in general. If the index of g_p is constant $t(0 \leq t \leq n)$ on M, then we call g a *pseudo-Riemannian metric of signature* $(t, n-t)$. And a C^∞-class differentiable manifold (M,g) furnished with a pseudo-Riemannian metric g is called a *pseudo-Riemannian manifold.* A pseudo-Riemannian manifold of signature $(0,n)$ means a Riemannian manifold. Let v be a tangent vector to a pseudo-Riemannian manifold M with a pseudo-Riemannian metric g. Then v is said to be

spacelike if $g(v,v) > 0$ or $v = 0$; *lightlike* if $g(v,v) = 0$ and $v \neq 0$, *timelike* if $g(v,v) < 0$.

The simplest example of pseudo-Riemannian manifold is a pseudo-Euclidean space.

Let $(x^1, x^2, \cdots, x^m)$ be a point in the set R^m of all ordered m-tuples of real numbers. For each $t(0 \leq t \leq m)$, we define a scalar product g_0 on $T_p(R^m)$ at the point p of R^m by

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$$g_0(v_p, w_p) = -\sum_{i=1}^{t} v^i w^i + \sum_{i=t+1}^{m} v^i w^i,$$

where $v_p = \sum_{i=1}^{m} v^i \partial/\partial x^i$ and $w_p = \sum_{i=1}^{m} w^i \partial/\partial x^i$. E_t^m denotes a R^m with a canonical pseudo-Riemannian metric g_0. In this case, g_0 is called a *pseudo-Euclidean metric* of signature $(t, m-t)$ and E_t^m is called a *pseudo-Euclidean space* of signature $(t, m-t)$. In particular, E_1^m is called a *Minkowski spacetime.*

From now on, we will use $<,>$ instead of a pseudo-Euclidean metric g_0. And we denote by $H_t^m(-c^2) = \{p \in E_{t+1}^{m+1} | <p,p>= -c^2\}$. In this case, it is called the *pseudo-hyperbolic space* of radius $c > 0$ and center 0 in E_{t+1}^{m+1}. For a vector $a_0 = (a_1, a_2, \cdots, a_t, \cdots, a_m)$ in E_t^m,

$$\bar{a}_0 = (-a_1, -a_2, \cdots, -a_t, a_{t+1}, a_{t+2}, \cdots, a_m)$$

is called the *conjugate vector* of a_0. In [7] and [10], the authors proved the following

Theorem A. *Only 1-type closed curve $\gamma(s)$ on $H_t^m(-c^2)$ is an intersection of $H_t^m(-c^2)$ and a 2-plane P lying in Π_{a_0}, where P is determined by two spacelike vectors and Π_{a_0} denotes a hyperplane through a_0 which is orthogonal to the conjugate vector $\bar{a}_0$ in the sense of Euclidean scalar product.*

Ishikawa([7]), and Shin and Pyo([11]) also proved some nonexistence theorems concerning finite type closed curves on pseudo-hyperbolic spaces $H^2(-c^2)$ and $H^4(-c^2)$. For instance,

Theorem B. *There exists neither 2-type closed curves nor 3-type closed curves on $H^2(-c^2)$.*

Remark. Finite type curves in a Euclidean space were investigated in [1], [2], [5], [6] *etc.*

The purpose of this article is to prove some theorems on nonexistence of certain finite type closed curves on the pseudo-hyperbolic space $H^3(-c^2)$ in the Minkowski spacetime E_1^4.

2 Preliminaries

Every closed curve $\gamma : [0, 2\pi r] \to E_t^m$ of the length $2\pi r$ in E_t^m may be regarded as an isometric immersion of a circle of radius r into E_t^m. We use the arc length s as a parameter of γ. Then the Laplacian Δ on the circle is given by $\Delta = -d^2/ds^2$ and the eigenvalues are $\{(l/r)^2; l = 1, 2, \cdots\}$. The corresponding eigenspace V_l is constructed by using $\cos(ls/r)$ and $\sin(ls/r)$. Hence, every closed curve $\gamma : [0, 2\pi r] \to E_t^m$ has the spectral decomposition

$$\gamma(s) = a_0 + \sum_{l=1}^{\infty} \{a_l \cos(ls/r) + b_l \sin(ls/r)\},$$

where a_l, b_l are some vectors in E_t^m (see [2],[5]). In particular, if γ is a k-type closed curve of the length 2π on $H_t^m(-c^2)$, then γ can be expressed as

$$\gamma(s) = a_0 + \sum_{i=1}^{k} \{a_i \cos(p_i s) + b_i \sin(p_i s)\}, \tag{2.1}$$

where a_i or b_i is nonzero vector in E_t^m for each $i = 1, 2, \cdots, k$, p_i are the positive integers with $p_1 < p_2 < \cdots < p_k$ and s is the arc length parameter of γ. Because of $\gamma(s)$ being on $H_t^m(-c^2)$ and a_0 the center of mass of γ, a_0 is a timelike vector in E_{t+1}^{m+1} (see [7]). Furthermore, from $< \gamma(s),\ \gamma(s) >= -c^2$, we have the following

$$2 < a_0, a_0 > +2c^2 + \sum_{i=1}^{k} D_{ii} = 0, \tag{2.2}$$

$$\sum_{p_i = l} M_i + \sum_{2p_i = l} A_{ii} + 2 \sum_{\substack{p_i + p_j = l \\ i > j}} A_{ij} + 2 \sum_{\substack{p_i - p_j = l \\ i > j}} D_{ij} = 0, \tag{2.3}$$

$$\sum_{p_i = l} \bar{M}_i + \sum_{2p_i = l} \bar{A}_{ii} + 2 \sum_{\substack{p_i + p_j = l \\ i > j}} \bar{A}_{ij} - 2 \sum_{\substack{p_i - p_j = l \\ i > j}} \bar{D}_{ij} = 0, \tag{2.4}$$

for each $l \in \{p_i,\ 2p_i,\ p_i + p_j,\ p_i - p_j\ ;\ 1 \le j < i \le k\}$, where

$$\begin{aligned} M_i &= 4 < a_0, a_i >, & \bar{M}_i &= 4 < a_0, b_i >, \\ A_{ij} &=< a_i, a_j > - < b_i, b_j >, & \bar{A}_{ij} &=< a_i, b_j > + < b_i, a_j >, \\ D_{ij} &=< a_i, a_j > + < b_i, b_j >, & \bar{D}_{ij} &=< a_i, b_j > - < b_i, a_j > . \end{aligned}$$

From now on, we call the real numbers M_i and $\bar{M}_i$ (resp. A_{ii} and $\bar{A}_{ii}$, A_{ij} and $\bar{A}_{ij}$, or D_{ij} and $\bar{D}_{ij}$) to be *corresponding to the integer* p_i(resp. $2p_i,\ p_i + p_j$, or $p_i - p_j$). Since s is the arc length parameter of $\gamma(s)$, we have

$$2 = \sum_{i=1}^{k} p_i^2 D_{ii}, \tag{2.5}$$

$$\sum_{2p_i = l} p_i^2 A_{ii} + 2 \sum_{\substack{p_i + p_j = l \\ i > j}} p_i p_j A_{ij} - 2 \sum_{\substack{p_i - p_j = l \\ i > j}} p_i p_j D_{ij} = 0, \tag{2.6}$$

$$\sum_{2p_i = l} p_i^2 \bar{A}_{ii} + 2 \sum_{\substack{p_i + p_j = l \\ i > j}} p_i p_j \bar{A}_{ij} + 2 \sum_{\substack{p_i - p_j = l \\ i > j}} p_i p_j \bar{D}_{ij} = 0. \tag{2.7}$$

Moreover, if $< \gamma^{(r)}(s),\ \gamma^{(r)}(s) >$ is constant $(r = 1, 2, \cdots)$, then we have

$$\sum_{2p_i = l} p_i^{2r} A_{ii} + 2 \sum_{\substack{p_i + p_j = l \\ i > j}} (p_i p_j)^r A_{ij} + (-1)^r 2 \sum_{\substack{p_i - p_j = l \\ i > j}} (p_i p_j)^r D_{ij} = 0, \tag{2.8}$$

$$(2.9) \qquad \sum_{2p_i=l} p_i^{2r}\bar{A}_{ii} + 2\sum_{\substack{p_i+p_j=l\\ i>j}} (p_ip_j)^r\bar{A}_{ij} - (-1)^r 2\sum_{\substack{p_i-p_j=l\\ i>j}} (p_ip_j)^r\bar{D}_{ij} = 0.$$

Next, let γ be a k-type closed curve on $H_t^m(-c^2)$ given in (2.1). Divide the set $\mathcal{A} = \{_{\sqrt{}}\rangle,\ \in_{\sqrt{}}\rangle,\ _{\sqrt{}}\rangle + _{\sqrt{}}|,\ _{\sqrt{}}\rangle - _{\sqrt{}}|\ ;\ \infty \leq | < \rangle \leq \|\}$ as the union of the subsets as follows:

$$(2.10) \qquad \mathcal{A} = \mathcal{A}_\infty \cup \mathcal{A}_\in \cup \cdots \cup \mathcal{A}_\mathcal{N},$$

where all elements in each subset $\mathcal{A}_\backslash(\backslash = \infty, \in, \cdots, \mathcal{N})$ are equal to each other and if $n_1 \neq n_2$, then every element in $\mathcal{A}_{\backslash_\infty}$ is not equal to any element in $\mathcal{A}_{\backslash_\in}$.

3 Main Results

Let γ be a closed k-type curve on $H_t^m(-c^2)$ in E_{t+1}^{m+1}. Then γ is expressed as $\gamma(s) = a_0 + \sum_{i=1}^{k}\{a_i\cos(p_is/r) + b_i\sin(p_is/r)\}$, where a_i or b_i is nonzero vector in E_{t+1}^{m+1} for each $i = 1, 2, \cdots, k$ and p_i are the positive integers satisfying $p_1 < p_2 < \cdots < p_k$. Here s is the arc length parameter of γ and the length of γ is $2\pi r$. Therefore every k-type closed curve $\gamma(s)$ of the length 2π may be described as

$$(3.1) \qquad \gamma(s) = a_0 + \sum_{i=1}^{k}\{a_i\cos(p_is) + b_i\sin(p_is)\},$$

where $a_i \neq 0$ or $b_i \neq 0$ for each i. We prove our results for $r = 1$, because the proof for case $r \neq 1$ is the same as one for case of $r = 1$.

Lemma 3.1([7]). *(1) If* $< \gamma^{(r)}(s),\ \gamma^{(r)}(s) >$ *is constant* $(r = 1, 2, \cdots, l)$ *and the number of members in* $\mathcal{A}_\backslash$ *is less than or equal to* $l+1$*, then* M_i *and* $\bar{M}_i$ *(resp.* A_{ii} *and* $\bar{A}_{ii}$*,* A_{ij} *and* $\bar{A}_{ij}$*, or* D_{ij} *and* $\bar{D}_{ij}$*) of corresponding to the integer* p_i *(resp.* $2p_i$*,* $p_i + p_j$*, or* $p_i - p_j$*) in* $\mathcal{A}_\backslash$ *vanish.*

(2) In particular, for every k*-type closed curve* $\gamma(s)$ *on* $H_t^m(-c^2)$ *in* E_{t+1}^{m+1}*, we have*

$$A_{kk} = \bar{A}_{kk} = 0,$$
$$A_{k(k-1)} = \bar{A}_{k(k-1)} = 0,$$
$$A_{(k-1)(k-1)} = \bar{A}_{(k-1)(k-1)} = 0.$$

Now, let $\gamma(s)$ be a k-type closed curve on $H^3(-c^2)$ in a Minkowski spacetime E_1^4 as (3.1). Then we can obtain the following lemmas.

Lemma 3.2. *If* $\gamma(s)$ *satisfies the following conditions*

$$(3.2) \qquad M_k = \bar{M}_k = M_{k-1} = 0 \ \text{and} \ D_{k(k-1)} = \bar{D}_{k(k-1)} = 0,$$

then either (1) $\{a_0, a_{k-1}, a_k, b_k\}$ *forms a basis for* E_1^4*, or (2)* b_{k-1} *is a lightlike vector and* $\{a_0, b_{k-1}, a_k, b_k\}$ *is a basis of* E_1^4*.*

Proof. Since a_0 is a timelike vector in E_1^4, from the first equation of (3.2), we know that a_{k-1}, a_k and b_k are spacelike vectors. Hence a_k and b_k are nonzero vectors because

$< a_k, a_k >=< b_k, b_k >$ and $\gamma(s)$ is of k-type. From Lemma 3.1(2) and the second equation of (3.2), we have

$$\begin{aligned} < a_{k-1}, a_{k-1} > &= < b_{k-1}, b_{k-1} >, \\ < a_k, b_k > &= < a_{k-1}, b_{k-1} >= 0, \\ < a_k, a_{k-1} > &= < b_k, b_{k-1} >= 0, \end{aligned}$$

and

$$< a_k, b_{k-1} >=< b_k, a_{k-1} >= 0.$$

If $a_{k-1} \neq 0$, then a_0, a_{k-1}, a_k, b_k are linearly independent vectors in E_1^4 and hence $\{a_0, a_{k-1}, a_k, b_k\}$ is a basis of E_1^4.

Suppose $a_{k-1} = 0$. Since $< a_{k-1}, a_{k-1} >=< b_{k-1}, b_{k-1} >= 0$ and $\gamma(s)$ is of k-type, b_{k-1} is a lightlike vector. If we put $Aa_0 + Bb_{k-1} + Ca_k + Db_k = 0$, then we can obtain $A = B = C = D = 0$ because $< a_0, b_{k-1} >\neq 0$(see [11]). Therefore we complete the proof.

Remark. If the k-type closed curve γ on $H^3(-c^2)$ satisfies $M_{k-1} = 0$ and b_{k-1} is a lightlike vector in E_1^4, then $a_{k-1} = 0$ because $< a_{k-1}, a_{k-1} >=< b_{k-1}, b_{k-1} >= 0$.

Lemma 3.3. *Suppose that* $\{a_0, a_{k-1}, a_k, b_k\}$ *is a basis of* E_1^4 *satisfying (3.2). Then* b_{k-1} *is a parallel nonzero vector to* a_0.

Proof. Put $b_{k-1} = Aa_0 + Ba_{k-1} + Ca_k + Db_k$. Combining Lemma 3.1(2) and (3.2), we have $B = C = D = 0$ because a_{k-1}, a_k and b_k are nonzero spacelike vectors. Since $< a_0, b_{k-1} >\neq 0$, $b_{k-1} = Aa_0 \neq 0$ for a constant A.

Next, we can obtain the following

Lemma 3.4. *Suppose that* $\{a_0, a_{k-1}, a_k, b_k\}$ *is a basis of* E_1^4 *satisfying (3.2). If a pair* $\{a_i, b_i\}(i = 1, 2, \cdots, k-2)$ *satisfies*

$$A_{ki} = \bar{A}_{ki} = 0$$

and

$$< a_{k-1}, a_i >=< a_{k-1}, b_i >= 0,$$

then $A_{ii} = \bar{A}_{ii} = 0$ *if and only if* $M_i = \bar{M}_i = 0$.

Proof. Put $a_i = Aa_0 + Ba_{k-1} + Ca_k + Db_k$ and $b_i = Ea_0 + Fa_{k-1} + Ga_k + Hb_k$. Combining Lemma 3.1(2), (3.2) and our assumptions, we have $B = F = 0,\ \ C = H$ and $D = -G$. Hence $a_i = Aa_0 + Ca_k + Db_k$ and $b_i = Ea_0 - Da_k + Cb_k$ for some constants A, C, D and E.

Suppose $A_{ii} = \bar{A}_{ii} = 0$. Then we get $AE = 0$ and $A^2 - E^2 = 0$ because $< a_k, a_k >=< b_k, b_k >$ and $< a_k, b_k >= 0$, and hence $A = E = 0$. Therefore $M_i = 4 < a_0, a_i >= 0$ and $\bar{M}_i = 4 < a_0, b_i >= 0$.

Conversely, if $M_i = \bar{M}_i = 0$, then we have $a_i = Ca_k + Db_k$ and $b_i = -Da_k + Cb_k$ for some constants C and D. Hence $< a_i, a_i >=< b_i, b_i >$ and $< a_i, b_i >= 0$.

Lemma 3.5. *Suppose that* $\{a_0, a_{k-1}, a_k, b_k\}$ *is a basis of* E_1^4 *satisfying (3.2). If a pair* $\{a_i, b_i\}(i = 1, 2, \cdots, k-2)$ *is satisfying*

$$A_{ki} = \bar{A}_{ki} = 0, \qquad D_{ki} = \bar{D}_{ki} = 0$$

and

$$< a_{k-1}, a_i >=< a_{k-1}, b_i >= 0,$$

then a_i *and* b_i *are parallel to* a_0.

Proof. If we put $a_i = Aa_0 + Ba_{k-1} + Ca_k + Db_k$ and $b_i = Ea_0 + Fa_{k-1} + Ga_k + Hb_k$, then we have, from Lemma 3.1(2), (3.2) and our assumptions, $B = C = D = 0$, $F = G = H = 0$. It follows that $a_i = Aa_0$ and $b_i = Ea_0$ for some constants A and B. Finally, we have the following lemma.

Lemma 3.6. *Suppose that $\{a_0, b_{k-1}, a_k, b_k\}$ is a basis of E_1^4 satisfying (3.2). If a pair $\{a_i, b_i\}(i = 1, 2, \cdots, k-2)$ is satisfying*

$$A_{ki} = \bar{A}_{ki} = 0$$

and

$$< b_{k-1}, a_i >=< b_{k-1}, b_i >= 0,$$

then $A_{ii} = \bar{A}_{ii} = 0$.

Proof. If we put $a_i = Aa_0 + Bb_{k-1} + Ca_k + Db_k$ and $b_i = Ea_0 + Fb_{k-1} + Ga_k + Hb_k$, Combining Lemma 3.1(2), (3.2) and our assumptions, we have $C = H$ and $D = -G$. Since b_{k-1} is lightlike and $< a_0, b_{k-1} >\neq 0$, $A = E = 0$. Hence $a_i = Bb_{k-1} + Ca_k + Db_k$ and $b_i = Fb_{k-1} - Da_k + Cb_k$ for some constants B, C, D and F. Therefore $< a_i, a_i >=< b_i, b_i >$ and $< a_i, b_i >= 0$.

From now on, we prove the following nonexistence theorems for a k-type($k \geq 2$) closed curve $\gamma(s)$ on $H^3(-c^2)$.

Theorem 3.1. *There exists no 2-type closed curve $\gamma(s)$ on $H^3(-c^2)$.*

Proof. We assume the existence of the 2-type closed curve

$$\gamma(s) = a_0 + a_1 \cos(p_1 s) + b_1 \sin(p_1 s) + a_2 \cos(p_2 s) + b_2 \sin(p_2 s)$$

on $H^3(-c^2)$. From Lemma 3.1, we see

$$M_1 = \bar{M}_1 = 0, \qquad M_2 = \bar{M}_2 = 0.$$

Hence a_1, b_1, a_2 and b_2 are spacelike vectors in E_1^4. Furthermore a_1, b_1, a_2 and b_2 are nonzero vectors because $A_{11} = A_{22} = 0$ and $\gamma(s)$ is of 2-type. We also have

$$\bar{A}_{11} = \bar{A}_{22} = 0, \;\; A_{21} = \bar{A}_{21} = 0, \;\; D_{21} = \bar{D}_{21} = 0.$$

Therefore a_0, a_1, b_1, a_2, b_2 are linearly independent vectors in E_1^4. It contradicts.

Theorem 3.2. *There exists no 3-type closed curve $\gamma(s)$ on $H^3(-c^2)$ satisfying $M_2 = 0$ and $D_{32} = \bar{D}_{32} = 0$.*

Proof. We assume the existence of the 3-type closed curve

$$\begin{aligned} \gamma(s) = a_0 \;\; &+ \;\; a_1 \cos(p_1 s) + b_1 \sin(p_1 s) + a_2 \cos(p_2 s) \\ &+ \;\; b_2 \sin(p_2 s) + a_3 \cos(p_3 s) + b_3 \sin(p_3 s) \end{aligned}$$

on $H^3(-c^2)$ satisfying the assumptions $M_2 = 0$ and $D_{32} = \bar{D}_{32} = 0$.

First, if we assume that $a_2 \neq 0$, then $\{a_0, a_2, a_3, b_3\}$ is a basis of E_1^4 satisfying (3.2) by Lemmas 3.1 and 3.2.

Case 1. In case of $\{p_1,\ p_2,\ p_3\} = \{p_1,\ 2p_1,\ 3p_1\}$, it follows that $\mathcal{A} = \{\sqrt{\infty}, \sqrt{\in} - \sqrt{\infty}, \sqrt{\ni} - \sqrt{\in}\} \cup \{\sqrt{\in}, \in\sqrt{\infty}, \sqrt{\ni} - \sqrt{\infty}\} \cup \{\sqrt{\infty} + \sqrt{\in}, \sqrt{\ni}\} \cup \{\sqrt{\infty} + \sqrt{\ni}, \in\sqrt{\in}\} \cup \{\sqrt{\in} + \sqrt{\ni}\} \cup \{\in\sqrt{\ni}\}$. Applying (2.3), (2.4), (2.6) and (2.7) for the subclasses $\{p_1,\ p_2 -$

$p_1,\ p_3-p_2\}$ and $\{2p_1,\ p_2,\ p_3-p_1\}$ of $\mathcal{A}$, and combining Lemmas 3.1 and 3.4, we obtain

$$\begin{aligned} A_{21}=\bar{A}_{21}=0, &\quad D_{21}=\bar{D}_{21}=0,\\ A_{31}=\bar{A}_{31}=0, &\quad M_1=\bar{M}_1=0,\\ A_{11}=\bar{A}_{11}=0, &\quad D_{31}=\bar{D}_{31}=0 \end{aligned}$$

by our assumptions. Furthermore, we have $\bar{M}_2=0$. Hence b_2 is a spacelike vector in E^4_1. It is a contradiction to Lemma 3.3.

Case 2. In case of $\{p_1,\ p_2,\ p_3\}=\{p_1,\ 2p_1,\ 4p_1\}$, it follows that $\mathcal{A}=\{\sqrt{\infty},\ \sqrt{\in}-\sqrt{\infty}\}\cup\{\sqrt{\in},\ \in\sqrt{\infty},\ \sqrt{\ni}-\sqrt{\in}\}\cup\{\sqrt{\infty}+\sqrt{\in},\ \sqrt{\ni}-\sqrt{\infty}\}\cup\{\sqrt{\ni},\ \in\sqrt{\in}\}\cup\{\sqrt{\infty}+\sqrt{\ni}\}\cup\{\sqrt{\in}+\sqrt{\ni}\}\cup\{\in\sqrt{\ni}\}$. Applying (2.3), (2.4), (2.6) and (2.7) for the subclass $\{2p_1,\ p_2,\ p_3-p_2\}$ of $\mathcal{A}$, we get $\bar{M}_2=0$ by the assumption $D_{32}=\bar{D}_{32}=0$. Hence Lemma 3.3 leads a contradiction.

Case 3. In case of $\{p_1,\ p_2,\ p_3\}=\{p_1,\ 3p_1,\ 5p_1\}$, $\mathcal{A}=\{\sqrt{\infty}\}\cup\{\in\sqrt{\infty},\ \sqrt{\in}-\sqrt{\infty},\ \sqrt{\ni}-\sqrt{\in}\}\cup\{\sqrt{\in}\}\cup\{\sqrt{\infty}+\sqrt{\in},\ \sqrt{\ni}-\sqrt{\infty}\}\cup\{\sqrt{\ni}\}\cup\{\sqrt{\infty}+\sqrt{\ni},\ \in\sqrt{\in}\}\cup\{\sqrt{\in}+\sqrt{\ni}\}\cup\{\in\sqrt{\ni}\}$. From Lemma 3.1(1), we obtain $\bar{M}_2=0$. This is a contradiction.

Case 4. Let $\{p_1,\ p_2,\ p_3\}\neq\{p_1,\ 2p_1,\ 3p_1\}$, $\{p_1,\ 2p_1,\ 4p_1\}$ or $\{p_1,\ 3p_1,\ 5p_1\}$. In this case, each subset $\mathcal{A}_\backslash$ of $\mathcal{A}$ consists of at most two elements. Hence, we have $\bar{M}_2=0$ by Lemma 3.1(1). It contradicts.

Summarizing all cases, we complete the proof of this theorem in the case of $a_2\neq 0$.

Now, let $a_2=0$. Then, by Lemmas 3.1(1) and 3.2, $\{a_0,b_2,a_3,b_3\}$ forms a basis for E^4_1 satisfying (3.2). In Case 1, applying (2.3), (2.4), (2.6) and (2.7) for the subclass $\{p_1,\ p_2-p_1,\ p_3-p_2\}$ of $\mathcal{A}$, and combining the condition $D_{32}=\bar{D}_{32}=0$ and Lemma 3.1(1), we have

$$A_{31}=\bar{A}_{31}=0,\ \ A_{21}=\bar{A}_{21}=0,\ \ D_{21}=\bar{D}_{21}=0.$$

Hence $A_{11}=\bar{A}_{11}=0$ by Lemma 3.6. Applying (2.3), (2.4), (2.6), (2.7) and the above equation for the subclass $\{2p_1,\ p_2,\ p_3-p_1\}$ of $\mathcal{A}$, we get $\bar{M}_2=0$. Since b_2 is a lightlike vector by Lemma 3.2, it contradicts.

The other cases are also impossible.

Therefore we complete the proof of this theorem.

For a 3-type closed curve $\gamma(s)=a_0+\sum_{t=1}^{3}\{a_t\cos(p_ts)+b_t\sin(p_ts)\}$ on $H^3(-c^2)$, if $a_2=0$, then we have $M_2=0$ and $<b_2,a_3>=<b_2,b_3>=0$ by Lemma 3.1(2). Hence $D_{32}=\bar{D}_{32}=0$. Therefore, from Theorem 3.2, we have the following corollary.

Corollary 3.1. *There exists no 3-type closed curve*

$$\gamma(s)=a_0+\sum_{t=1}^{3}\{a_t\cos(p_ts)+b_t\sin(p_ts)\}$$

on $H^3(-c^2)$ satisfying $a_2=0$.

Corollary 3.2. *There exists no 3-type closed curve with constant curvature on $H^3(-c^2)$.*

Proof. Let

$$\begin{aligned}\gamma(s) = a_0 \quad &+ \quad a_1 \cos(p_1 s) + b_1 \sin(p_1 s) + a_2 \cos(p_2 s) \\ &+ \quad b_2 \sin(p_2 s) + a_3 \cos(p_3 s) + b_3 \sin(p_3 s)\end{aligned}$$

be a 3-type closed curve with constant curvature on $H^3(-c^2)$. Then each subclass of $\mathcal{A}$ consists of at most three elements. From Lemma 3.1(1), we get

$$M_2 = \bar{M}_2 = 0, \qquad M_3 = \bar{M}_3 = 0.$$

Hence a_2, b_2, a_3 and b_3 are spacelike vectors. Furthermore, they are nonzero vector because $A_{22} = A_{33} = 0$ and $\gamma(s)$ is of 3-type. Therefore a_0, a_2, b_2, a_3, b_3 are linearly independent vectors in E_1^4 by Lemma 3.1. This implies a contradiction.

Next, we get the following

Theorem 3.3. *There exists no 4-type closed curve with constant curvature on $H^3(-c^2)$ satisfying $D_{43} = \bar{D}_{43} = 0$.*

Proof. Assume the existence of the 4-type closed curve

$$\gamma(s) = a_0 + \sum_{t=1}^{4} \{a_t \cos(p_t s) + b_t \sin(p_t s)\}$$

satisfying our assumptions. If $\{p_1,\ p_2,\ p_3,\ p_4\} = \{p_1,\ 2p_1,\ 3p_1,\ 4p_1\}$, then $\mathcal{A} = \{\surd^{\infty},\ \surd^{\in} - \surd^{\infty},\ \surd^{\ni} - \surd^{\in},\ \surd^{\triangle} - \surd^{\ni}\} \cup \{\surd^{\in},\ \in\surd^{\infty},\ \surd^{\ni} - \surd^{\infty},\ \surd^{\triangle} - \surd^{\in}\} \cup \{\surd^{\ni},\ \surd^{\infty} + \surd^{\in},\ \surd^{\triangle} - \surd^{\infty}\} \cup \{\surd^{\triangle},\ \in\surd^{\in},\ \surd^{\infty} + \surd^{\ni}\} \cup \{\surd^{\infty} + \surd^{\triangle},\ \surd^{\in} + \surd^{\ni}\} \cup \{\in\surd^{\ni},\ \surd^{\in} + \surd^{\triangle}\} \cup \{\surd^{\triangle} + \surd^{\ni}\} \cup \{\in\surd^{\triangle}\}$. Let $\mathcal{A}_{\rangle}$ be the subclass consisting of all elements in $\mathcal{A}$ to be equal to p_i. Then the number of elements in $\mathcal{A}_{\ni}$ (and $\mathcal{A}_{\triangle}$) is less than or equal to three in this case. Hence, from Lemma 3.1, we have

$$\begin{aligned} M_3 = \bar{M}_3 = 0, &\quad M_4 = \bar{M}_4 = 0, \\ A_{33} = A_{44} = 0, &\quad A_{43} = \bar{A}_{43} = 0. \end{aligned}$$

Since $D_{43} = \bar{D}_{43} = 0$, we get a_0, a_3, b_3, a_4, b_4 are linearly independent vectors in E_1^4 by the same way as the proof Corollary 3.2. It contradicts.

In case of $\{p_1,\ p_2,\ p_3,\ p_4\} \neq \{p_1,\ 2p_1,\ 3p_1,\ 4p_1\}$, we can also imply a contradiction by the same way.

From Theorem 3.3, we can obtain the following corollary.

Corollary 3.3. *There exists no 4-type closed curve*

$$\gamma(s) = a_0 + \sum_{t=1}^{4} \{a_t \cos(p_t s) + b_t \sin(p_t s)\}$$

on $H^3(-c^2)$ satisfying $a_3 = 0$.

Theorem 3.4. *There exists no 5-type closed curve $\gamma(s)$ on $H^3(-c^2)$ with $D_{54} = \bar{D}_{54} = 0$ satisfying $< \gamma^{(l)}(s),\ \gamma^{(l)}(s) >$ is constant $(l = 2, 3)$.*

Proof. Assume the existence of the 5-type closed curve

$$\gamma(s) = a_0 + \sum_{t=1}^{5} \{a_t \cos(p_t s) + b_t \sin(p_t s)\}$$

satisfying our conditions. Let $\{p_1,\ p_2,\ p_3,\ p_4,\ p_5\} = \{p_1,\ 2p_1,\ 3p_1,\ 4p_1,\ 5p_1\}$, it follows that $\mathcal{A} =$ { √∞, √∈ − √∞, √∋ − √∈, √△ − √∋, √▽ − √△ } ∪ { √∈, ∈ √∞, √∋ − √∞, √△ − √∈, √▽ − √∋ } ∪ { √∋, √∞ + √∈, √△ − √∞, √▽ − √∈ } ∪ { √△, ∈ √∈, √∞ + √∋, √▽ − √∞ } ∪ { √▽, √∞ + √△, √∈ + √∋ } ∪ { ∈ √∋, √∞ + √▽, √∈ + √△ } ∪ { √∈ + √▽, √∋ + √△ } ∪ { ∈ √△, √∋ + √▽ } ∪ { √△ + √▽ } ∪ { ∈ √▽ }. Applying Lemma 3.1(1) for the subclasses $\{p_4,\ 2p_2,\ p_1+p_3,\ p_5-p_1\}$ and $\{p_5,\ p_1+p_4,\ p_2+p_3\}$ of $\mathcal{A}$, we obtain

$$M_4 = \bar{M}_4 = 0, \qquad M_5 = \bar{M}_5 = 0.$$

Hence a_4, b_4, a_5 and b_5 are spacelike vectors in E^4_1. Furthermore, from Lemma 3.1(2), we have

$$A_{44} = A_{55} = 0, \qquad A_{54} = \bar{A}_{54} = 0.$$

Therefore a_0, a_4, b_4, a_5, b_5 are linearly independent vectors in E^4_1 because $D_{54} = \bar{D}_{54} = 0$ and $\gamma(s)$ is of 5-type. It contradicts.

By the same way, in case of $\{p_1,\ p_2,\ p_3,\ p_4,\ p_5\} \neq \{p_1,\ 2p_1,\ 3p_1,\ 4p_1,\ 5p_1\}$, we can also imply a contradiction.

Finally, we get the following theorem.

Theorem 3.5. *There exists no 6-type closed curve $\gamma(s)$ on $H^3(-c^2)$ with $D_{65} = \bar{D}_{65} = 0$ satisfying $< \gamma^{(l)}(s),\ \gamma^{(l)}(s) >$ is constant $(l = 2, 3)$.*

Proof. Assume the existence of the 6-type closed curve

$$\gamma(s) = a_0 + \sum_{t=1}^{6} \{a_t \cos(p_t s) + b_t \sin(p_t s)\}$$

satisfying our conditions.

Case 1. Let $\{p_1,\ p_2,\ p_3,\ p_4,\ p_5,\ p_6\} = \{p_1,\ 2p_1,\ 3p_1,\ 4p_1,\ 5p_1,\ 6p_1\}$, it follows that $\mathcal{A} =$ { √∞, √∈ − √∞, √∋ − √∈, √△ − √∋, √▽ − √△, √⌐ √▽ } ∪ { √∈, ∈ √∞, √∋ − √∞, √△ − √∈, √▽ − √∋, √⌐ √△ } ∪ { √∋, √∞ + √∈, √△ − √∞, √▽ − √∈, √⌐ √∋ } ∪ { √△, ∈ √∈, √∞ + √∋, √▽ − √∞, √⌐ √∈ } ∪ { √▽, √∞ + √△, √∈ + √∋, √⌐ √∞ } ∪ { ∈ √∋, √∞ + √▽, √∈ + √△, √ } ∪ { √∈ + √▽, √∋ + √△, √∞ + √ } ∪ { ∈ √△, √∋ + √▽, √∈ + √ } ∪ { √△ + √▽, √∋ + √ } ∪ { ∈ √▽, √△ + √ } ∪ { √▽ + √ } ∪ { ∈ √ }. Applying Lemma 3.1(1) for the subclasses $\{p_5,\ p_1+p_4,\ p_2+p_3,\ p_6-p_1\}$ and $\{p_6,\ 2p_3,\ p_1+p_5,\ p_2+p_4\}$ of $\mathcal{A}$, we obtain

$$M_5 = \bar{M}_5 = 0, \qquad M_6 = \bar{M}_6 = 0.$$

And, from Lemma 3.1(2), we have

$$A_{55} = A_{66} = 0, \qquad A_{65} = \bar{A}_{65} = 0.$$

Hence a_0, a_5, b_5, a_6, b_6 are linearly independent vectors in E^4_1 by our assumptions. It contradicts.

In case of $\{p_1,\ p_2,\ p_3,\ p_4,\ p_5,\ p_6\} \neq \{p_1,\ 2p_1,\ 3p_1,\ 4p_1,\ 5p_1,\ 6p_1\}$, we can also imply a contradiction.

References

[1] B.Y. Chen, *On submanifolds of finite type,* Soochow J. Math. 9 (1983), 65-81.

[2] B.Y. Chen, *Total Mean Curvature and Submanifolds of Finite type,* World Scientific, 1984.

[3] B.Y. Chen, *Finite Type Submanifolds in pseudo-Euclidean spaces and applications,* Kodai Math. J. 8 (1985), 358-374.

[4] B.Y. Chen, *Finite type pseudo-Riemannian submanifolds,* Tamkang J. Math. 17 (1986), 137-151.

[5] B.Y. Chen, J. Deprez, F. Dillen, L. Verstraelen and L. Vrancken, *Finite type curves, Geometry and Topology of Submanifolds II,* World Scientific (1990), 76-110.

[6] B.Y. Chen, F. Dillen and L. Verstraelen, *Finite type space curves,* Soochow J. Math. 12 (1986), 1-10.

[7] S. Ishikawa, *On Biharmonic Submanifolds and Finite Type Submanifolds in a Euclidean Space or a Pseudo-Euclidean Space,* Doctal thesis, Kyushu Univ.

[8] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry,* Vol. I and II, Wiley (Interscience), 1963 and 1969.

[9] B. O'Neill, *Semi-Riemannian Geometry,* Academic press, 1983.

[10] Y.S. Pyo and Y.J. Kim, *Finite type closed curves on pseudo-hyperbolic spaces,* Far East J. Math. Sci. 4(2) (1996), 149-162.

[11] K.H. Shin and Y.S. Pyo, *Some nonexistence theorems of finite type closed curves on the pseudo-hyperbolic space* $H^4(-c^2)$, East Asian Math. J. 14(1) (1988), 205-217.

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