Variations on the Theme of Twistor Spaces

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Abstract

The construction of the *twistor space* of an even-dimensional Riemannian manifold was transferred to a similar construction of a *reflector space* of a *neutral manifold* i.e., a manifold with a pseudo-Riemannian metric of signature zero [6], and to the study of *symplectic twistor spaces* [19] and of *Lagrangian-Grassmannian bundles* [21]. In the present paper, we discuss the common basis of all these constructions, the *isosplitting bundles*. These bundles are equipped with a pair of complex distributions which are involutive under the conditions known from twistor theory. We give an elementary proof of the involutivity conditions. In the particular case of a pseudo-Riemannian manifold, we study a subbundle of the isosplitting bundle, called the *twist-reflector space*, which composes twistors and reflectors.

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1 Introduction

The *twistor spaces*, an invention of R. Penrose e.g. [13], were intensively studied in view of their utility in theoretical physics, and their interesting differential geometric properties. The four-dimensional case is both the most interesting and the most important e.g., [1] but, there also are general constructions for any even dimension e.g., [12]. Usually, group representation theory is used in the proof of the integrability conditions but, it is also possible to use a more elementary tool, the moving frame method [3], [4], [5]. The basic fact is that a twistor is a complex structure in a tangent space of the original differentiable manifold.

Geometric objects which are analogous to twistors, and can be studied by analogous methods have also been discussed. Such are the *reflectors* i.e., paracomplex structures in the tangent spaces of a pseudo-Riemannian manifold of signature zero (a neutral manifold) [6], and *pairs of transversal Lagrangian subspaces* tangent to an almost symplectic manifold [21]

In the present work we again investigate the basics of twistor spaces and of the analogous constructions quoted above. We show that the essence of a twistor-like object is that it decomposes the complexified tangent space into a direct sum of

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two isotropic subspaces. In the case of a complex structure, these are the subspaces of complex type $(1,0)$ and $(0,1)$, respectively. This assertion about twistors follows from the fact that, whenever we have a differentiable manifold M^{2n} endowed with either a symmetric or a skew symmetric, 2-covariant, non degenerate tensor field g, the set of the isotropic decompositions mentioned earlier is the total space of a bundle over M which, in a certain sense, behaves like a twistor space. Namely, it has a pair of complex distributions, called the *main structural distributions* with the same involutivity conditions as in the case of the twistor spaces. A second pair of interesting, but never involutive, complex distributions also exists. A bundle of the mentioned kind is called an *isosplitting bundle*.

Section 2 is devoted to the description of the isosplitting bundles for a symmetric tensor g. The method which we use is the moving frame method of [5], [6]. We define the relevant complex distributions, and establish the involutivity conditions of these distributions by using an elementary argument instead of representation theory. In case $n = 2$, and for the Levi-Civita connection of g, we again find the condition of self or anti-self duality of the Weyl curvature tensor. Section 3 gives the corresponding results for the case of a skew symmetric tensor. In Section 4 we show that the structural distributions are conformal invariants.

Of course, one may ask what is the interest of having an involutive complex distribution S on a manifold, since the latter is not related to a foliation, and it doesn't even provide a *partial complex structure* [11]? The answer is that an involutive, complex distribution still allows for the definition of an exterior differential along the distribution S i.e., a differential on the global sections of $\wedge^k S^*$ which makes the latter into an interesting cochain complex. Moreover, the existence of S leads to a spectral sequence of complex differential forms, similar to the spectral sequence of a real foliation [7], which converges to the complex valued de Rham cohomology of the manifold. The study of this spectral sequence was not yet made but, it might lead to interesting results.

However, a more interesting situation is that where the distribution is integrable in the sense of L. Nirenberg [11] since then the manifold has an atlas with a certain number of complex analytic coordinates. We find such a situation in Section 5, where we take (M^{2n}, g) to be a pseudo-Riemannian manifold of signature $2s \geq 0$. Namely, we define a *twist-reflector* as an object which unifies a twistor and a reflector. The twist-reflector space is a subbundle of the isosplitting bundle. For a pseudo-Riemannian manifold of constant sectional curvature, and its Levi-Civita connection, a conveniently chosen distribution is *Nirenberg integrable*. For $s = n$, the twist-reflector space is the classical twistor space, and for $s = 0$ the twist-reflector space is the reflector space of [6].

Finally, let us say that all the objects of the present paper are in the C^∞-category.

2 Riemannian Isosplitting Spaces

The twistor space of an even-dimensional manifold is the space of the complex structures of the tangent spaces of the manifold e.g. [12], and we will use the interpretation of such a complex structure as the decomposition of the complexified tangent space into the $(1,0)$ and the $(0,1)$ components. A formal, complex generalization is available, and it was studied by G. Legrand [8]. We consider this generalization, under an appropriate terminology, and use it for a formal, twistor type construction.

Let M be a $2n$-dimensional differentiable manifold, endowed with a *complex Riemannian metric* $g \in \Gamma \odot^2 T^{*^c}M$ ($T^cM := TM \otimes_{\mathbf{R}} \mathbf{C}$ is the complexified tangent bundle, Γ denotes the space of global cross sections, and $\odot$ is the symmetric tensor product). Then, an *isosplitting* of (M,g) at $x \in M$ is an ordered pair (L_1, L_2) of complementary, maximal (i.e., n-dimensional), g-isotropic subspaces of T^c_xM ($T^c_xM = L_1 \oplus L_2$). Equivalently, an isosplitting may be seen as a complex, $(1,1)$-tensor $F \in End\, T^c_xM$ which satisfies the conditions

$$F^2 = -Id., \qquad g(X, FY) + g(FX, Y) = 0. \tag{2.1}$$

Namely, L_1, L_2 are the $(\pm\sqrt{-1})$-eigenspaces of F.

Now, we will denote by $\mathcal{I}(M,g)$ the space of all the isosplittings, and call it the *isosplitting space* of (M,g). For $n=1$, $\mathcal{I}(M,g)$ is just a double covering space of M. Hence, hereafter, we always assume that $n \geq 2$. We will see that $\mathcal{I}(M,g)$ is the total space of a certain fibration on M, and that it has certain complex distributions with interesting involutivity conditions, the same as in classical twistor space theory. We get the results by using the moving frame method of [5], [6].

In the tangent spaces T^c_xM ($x \in M$), we distinguish *null frames* (e_a, e_{a^*}) $(a, b, \ldots = 1, \ldots n;\ a^* := a + n)$, characterized by

$$g(e_a, e_b) = g(e_{a^*}, e_{b^*}) = 0, \quad g(e_a, e_{b^*}) = \delta_{ab}. \tag{2.2}$$

The set of the null frames is a $O(2n, \mathbf{C})$-principal bundle $\pi : \mathcal{N}(M,g) \to M$, where $O(2n,\mathbf{C})$ is realized by matrices of complex (n,n)-blocks as follows

$$O(2n, \mathbf{C}) = \tag{2.3}$$

$$= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \;/\; {}^tAC + {}^tCA = 0,\ {}^tBD + {}^tDB = 0,\ {}^tAD + {}^tCB = Id \right\}.$$

The corresponding Lie algebra is

$$o(2n, \mathbf{C}) = \left\{ \begin{pmatrix} X & Y \\ Z & U \end{pmatrix} \;/\; U = -{}^tX,\ Y = -{}^tY,\ Z = -{}^tZ \right\}. \tag{2.4}$$

Furthermore, there exists a natural projection $q : \mathcal{N}(M,g) \to \mathcal{I}(M,g)$ defined by

$$q(e_a, e_{a^*}) = \{L_1 = span(e_a),\ L_2 = span(e_{a^*})\}, \tag{2.5}$$

which shows that $\mathcal{I}(M,g)$ is the quotient of $\mathcal{N}(M,g)$ with respect to the right translations by

$$Gl(n, \mathbf{C}) = \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \right\}. \tag{2.6}$$

Hence, q is a $Gl(n,\mathbf{C})$-principal fibration, and $\mathcal{I}(M,g)$ is the total space of an associated fibration $p : \mathcal{I}(M,g) \to M$ of $\mathcal{N}(M,g)$ with the homogeneous fiber $O(2n,\mathbf{C})/Gl(n,\mathbf{C})$. The real dimension of this fiber is $2n(n-1)$, and the dimension of the isosplitting space $\mathcal{I}(M,g)$ is $2n^2$.

2.1 Proposition. *Any connection ∇ on $\mathcal{N}(M,g)$ yields a natural decomposition*

$$T^c\mathcal{I} = \mathcal{H}^c\mathcal{I} \oplus \mathcal{V}^c\mathcal{I}, \tag{2.7}$$

where $\mathcal{V}$ denotes the vertical (i.e., tangent to the fibers) distribution, and $\mathcal{H}$ denotes a complementary, horizontal distribution. Moreover, $\mathcal{I}(M,g)$ also gets endowed with two pairs of complex distributions $(P,\tilde{P})$, $(Q,\tilde{Q})$, where P and $\tilde{P}$ are never involutive, and Q, $\tilde{Q}$ are involutive (both, simultaneously) iff the covariant torsion and curvature tensors of ∇ on M vanish on arguments which span a g-isotropic subspace.

Proof. The existence of the decomposition (2.7) is well known in connection theory. Since $\mathcal{N}(M,g)$ is a bundle of frames, it has a canonical, $\mathbf{C}^{2n}$-valued 1-form θ defined in the usual way e.g., [9] i.e., $\forall\mathcal{X} \in T_{(e_a,e_{a^*})}\mathcal{N}$, $\theta(\mathcal{X})$ is given by the components (θ^a,θ^{a^*}) of $\pi_*\mathcal{X}$ with respect to the basis (e_a,e_{a^*}). This 1-form satisfies

$$R^*_\gamma\theta = \gamma^{-1}\circ\theta \qquad (\gamma\in O(2n,\mathbf{C})), \tag{2.8}$$

where R_γ denotes a right translation, and $\theta = 0$ defines the fibers of π. In (2.8), if we take $\gamma \in Gl(n,\mathbf{C})$ of (2.6), we see that the pull-back of each part of the system $\theta^a = 0$, $\theta^{a^*} = 0$ by local cross sections of q is the annihilator of a distribution which is globally defined on $\mathcal{I}(M,g)$, and the intersection of these two distributions is exactly the vertical distribution $\mathcal{V}^c(\mathcal{I})$.

In this paper, following [5], we will use the same notation for differential forms and their pullbacks, and the context will tell which is what.

Then, we consider the horizontal distribution of the connection ∇ on $\mathcal{N}(M,g)$, and see that it projects to the required $\mathcal{H}^c(\mathcal{I})$. We notice that the Lie algebra $o(2n,\mathbf{C})$ has a symmetric decomposition [9]

$$o(2n,\mathbf{C}) = gl(n,\mathbf{C}) \oplus m \tag{2.9}$$

given by

$$\begin{pmatrix} \omega & \lambda \\ \mu & -{}^t\omega \end{pmatrix} = \begin{pmatrix} \omega & 0 \\ 0 & -{}^t\omega \end{pmatrix} + \begin{pmatrix} 0 & \lambda \\ \mu & 0 \end{pmatrix}, \tag{2.10}$$

where each matrix belongs to the corresponding term of (2.9), and the two parts of the right hand side of (2.9) are characterized by the zeroes of the corresponding terms of (2.10). In view of (2.4), we also have

$$\lambda = -\,{}^t\lambda, \quad \mu = -\,{}^t\mu. \tag{2.11}$$

In particular, we have (2.10) for the case where the total matrix, say ϖ, of the left hand side of (2.10) is the connection form of ∇ on $\mathcal{N}(M,g)$, and then, with the notation of (2.8), we have [9]

$$R^*_\gamma\varpi = (ad\,\gamma^{-1})\varpi. \tag{2.12}$$

It is an immediate consequence of (2.12) that the first term of the right hand side of (2.10) is a connection form of the principal fibration q, which we call the *reduction of* ∇ *to* q, and denote by ∇^r. And, the forms λ, μ are horizontal and tensorial for the fibration q [9] hence, the pullback of each system $\lambda = 0$, $\mu = 0$ by local cross sections of q defines a distribution on $\mathcal{I}(M, g)$. The total system

$$\lambda = 0,\ \bar{\lambda} = 0,\ \mu = 0,\ \bar{\mu} = 0 \tag{2.13}$$

defines the distribution $\mathcal{H}^c(\mathcal{I})$.

At this point, it is essential to notice that the pullback of the forms $(\theta^a, \theta^{a^*}, \lambda, \mu, \bar{\lambda}, \bar{\mu})$ by local cross sections of q defines complex, local, tangent cobases on $\mathcal{I}(M, g)$, and any structure defined by R_γ-invariant combinations of these forms $\forall\gamma \in Gl(n, \mathbf{C})$ descends to a well defined global structure on $\mathcal{I}(M, g)$.

In particular, the following equations define the distributions announced by Proposition 2.1

$$\begin{aligned} &(P) \quad \theta^{a^*} = 0,\ \ \lambda = 0, \qquad (\tilde{P}) \quad \theta^a = 0,\ \ \mu = 0, \\ &(Q) \quad \theta^{a^*} = 0,\ \ \mu = 0,\ \ (\tilde{Q}) \quad \theta^a = 0,\ \ \lambda = 0. \end{aligned} \tag{2.14}$$

The complex dimension of $P, \tilde{P}, Q, \tilde{Q}$ is $n(3n-1)/2$, the dimension of $P \cap \tilde{P}$, $Q \cap \tilde{Q}$ is $n(n-1)$, and $P \oplus \tilde{P} = Q \oplus \tilde{Q} = T^c\mathcal{I}$. The manifold $\mathcal{I}(M, g)$ has a bundle involution $\iota(L_1, L_2) = (L_2, L_1)$ which sends P to $\tilde{P}$ and Q to $\tilde{Q}$. The proof is by lifting ι to $\mathcal{N}(M, g)$, and we will give it in Section 4 because the relevant notation is there.

The most interesting case is that where one of these distributions is involutive. Indeed, while in the complex case involutivity does not provide a foliation, it provides, however, an exterior differential along the distribution which is similar to the exterior differential along the leaves of a foliation.

The structure equations of ∇ [9] are

$$d\theta^a = \sum_{b=1}^{n} (\theta^b \wedge \omega_b^a + \theta^{b^*} \wedge \lambda_b^a) + \frac{1}{2} \sum_{i,j=1}^{2n} T_{ij}^a \theta^i \wedge \theta^j, \tag{2.15}$$

$$d\theta^{a^*} = \sum_{b=1}^{n} (\theta^b \wedge \mu_b^a - \theta^{b^*} \wedge \omega_a^b) + \frac{1}{2} \sum_{i,j=1}^{2n} T_{ij}^{a^*} \theta^i \wedge \theta^j, \tag{2.15*}$$

$$d\lambda_b^a = \sum_{c=1}^{n} (\lambda_b^c \wedge \omega_c^a - \omega_c^b \wedge \lambda_c^a) + \frac{1}{2} \sum_{i,j=1}^{2n} R_{b^* ij}^a \theta^i \wedge \theta^j, \tag{2.16}$$

$$d\mu_b^a = \sum_{c=1}^{n} (\omega_b^c \wedge \mu_c^a - \mu_b^c \wedge \omega_a^c) + \frac{1}{2} \sum_{i,j=1}^{2n} R_{bij}^{a^*} \theta^i \wedge \theta^j, \tag{2.16*}$$

where T and R are the torsion and curvature coefficients, respectively, on $\mathcal{N}(M, g)$. Accordingly, the Frobenius theorem tells us that the distribution P is never involutive, and the distribution Q is involutive iff

$$T_{bc}^{a^*} = 0 \quad \text{(the torsion condition)}, \tag{2.17}$$

$$R^{a^*}_{bcd} = 0 \quad \text{(the curvature condition).} \tag{2.18}$$

If we recall that the covariant torsion and curvature tensors are

$$\tau(X,Y,Z) := g(T(X,Y),Z),\; R(X,Y,Z,U) := g(R(Z,U)Y,X), \tag{2.19}$$

where the arguments are tangent vectors at a point of M, we see that (2.17), (2.18) are equivalent with

$$\tau_{abc} = 0, \qquad R_{abcd} = 0. \tag{2.20}$$

Since (2.20) must hold everywhere on $\mathcal{N}(M,g)$, the involutivity conditions become

$$\tau(X,Y,Z) = 0, \qquad R(X,Y,Z,U) = 0 \tag{2.21}$$

for all the arguments such that *span*$\{X,Y,Z,U\}$ is a g-isotropic, complex, tangent subspace of some tangent space of M. Similarly, we see that $\tilde{P}$ is never involutive, and $\tilde{Q}$ is involutive iff (2.21) holds for arguments with an isotropic *span*. Q.e.d.

The involutivity conditions established above can be made more precise. Namely, one has

2.2 Theorem. *The distributions Q and $\tilde{Q}$ of the isosplitting space $\mathcal{I}(M,g)$ are involutive iff the following two conditions are satisfied: i) the skew symmetric part of the Ricci curvature tensor ρ of ∇ is an exact 2-form*

$$\frac{1}{2}[\rho(X,Y) - \rho(Y,X)] = -(n-1)d\alpha(X,Y), \tag{2.22}$$

and the torsion of ∇ is

$$T(X,Y) = \alpha(X)Y - \alpha(Y)X; \tag{2.23}$$

ii) the curvature of ∇ is given by

$$\begin{aligned} R(X,Y,Z,U) &= g(X,Z)s(Y,U) - g(X,U)s(Y,Z) \\ &-g(Y,Z)s(X,U) + g(Y,U)s(X,Z) - \frac{1}{2}[g(X,Z)d\alpha(Y,U) \\ &-g(X,U)d\alpha(Y,Z) - g(Y,Z)d\alpha(X,U) + g(Y,U)d\alpha(X,Z)], \end{aligned} \tag{2.24}$$

where

$$s(X,Y) = \frac{1}{4(n-1)}[\rho(X,Y) + \rho(Y,X) - \frac{\sigma}{2n-1}g(X,Y)], \tag{2.25}$$

and σ is the scalar curvature of ∇.

Proof. The usual proofs of this kind of results is by representation theory e.g., [12], [1]. Instead, we use only a simple algebraic argument here. Namely, $\tau(X,Y,Z)$ and $R(X,Y,Z,U)$ are polynomials with respect to the components of their vector arguments in a fixed basis, and (2.21) means that these polynomials must belong to the ideal generated by the polynomials which express the isotropy of the subspaces

$span\{X,Y,Z\}$ and $span\{X,Y,Z,U\}$, respectively. That is, the involutivity conditions (2.21) are equivalent with

$$\tau(X,Y,Z) = \alpha(X)g(Y,Z) + \beta(Y)g(X,Z) + \gamma(Z)g(X,Y), \tag{2.26}$$

$$R(X,Y,Z,U) = a(X,Y)g(Z,U) + b(X,Z)g(Y,U) \tag{2.27}$$
$$+c(X,U)g(Y,Z) + d(Y,Z)g(X,U) + e(Y,U)g(X,Z) + f(Z,U)g(X,Y),$$

where X,Y,Z,U are arbitrary vector fields, and $\alpha,\beta,\gamma,a,b,c,d,e,f$ are tensor fields. Terms in $g(X,X)$ and similar do not enter because τ, R are linear in each argument.

Now, since $\tau(X,Y,Z) = -\tau(Y,X,Z)$, (2.26) reduces to (2.23).

Furthermore, since for any metric connection we must have

$$R(X,Y,Z,U) = -R(X,Y,U,Z), \quad R(X,Y,Z,U) = -R(Y,X,Z,U),$$

(2.27) becomes

$$R(X,Y,Z,U) = g(X,Z)b(Y,U) - g(X,U)b(Y,Z) \tag{2.28}$$
$$-g(Y,Z)b(X,U) + g(Y,U)b(X,Z).$$

Finally, we must impose the Bianchi identity [9]

$$\sum_{Cycl(Y,Z,U)} R(X,Y,Z,U) = \sum_{Cycl(Y,Z,U)} \tau(T(Z,U),Y,X) \tag{2.29}$$
$$+ \sum_{Cycl(Y,Z,U)} \nabla_Z \tau(U,Y,X),$$

which, if (2.23) holds, is equivalent to

$$\sum_{Cycl(Y,Z,U)} R(X,Y,Z,U) = \sum_{Cycl(Y,Z,U)} d\alpha(Z,U)g(X,Y). \tag{2.30}$$

Inserting here the R of (2.28) one gets

$$\sum_{Cycl(Y,Z,U)} g(X,Z)d\alpha(U,Y) = \sum_{Cycl(Y,Z,U)} g(X,Z)[b(Y,U) - b(U,Y)], \tag{2.31}$$

and a contraction gives

$$b(Y,U) - b(U,Y) = -d\alpha(Y,U). \tag{2.32}$$

It follows that

$$b(X,Y) = s(X,Y) - \frac{1}{2}d\alpha(X,Y),$$

where

$$s(X,Y) := \frac{1}{2}[b(X,Y) + b(Y,X)],$$

and, if this expression is inserted in (2.28), one gets (2.24), with (2.25) following by contractions. Q.e.d.

From Theorem 2.2, we get the standard result

2.3 Corollary. *If ∇ is the Levi-Civita connection of g, Q and $\tilde{Q}$ are involutive iff g is a locally conformal flat metric.*

Proof. Indeed, the torsion condition (2.23) holds with $\alpha = 0$, and the curvature condition (2.24) reduces to the vanishing of the Weyl curvature tensor. Since we assumed that $dim\, M \geq 4$, the result follows. Q.e.d.

The distributions $P, Q, \tilde{P}, \tilde{Q}$ defined by (2.14) are the main structural ingredient of the isosplitting space. Hence, they deserve a special name. We will say that $Q, \tilde{Q}$ are the *main structural distribution*, and $P, \tilde{P}$ are the *secondary structural distribution* of $(\mathcal{I}(M,g), \nabla)$.

As in classical twistor theory, in real dimension 4 ($n = 2$), if M is oriented, and if *g has a real volume* i.e., $det(g)$ is a positive, real number, a more interesting result is available.

For any n, and if M is orientable and g has a real volume, $\mathcal{I}(M,g)$ has two components as follows. If $\xi \in \mathcal{I}(M,g)_x$, $x \in M$, and we see it as F of (2.1), ξ has an associated 2-form $\omega \in \wedge^2 T_x^{c*}$ defined by $\omega(X,Y) = g(FX,Y)$, and, up to the sign, $\omega^n/n!$ is equal to the volume of g. To see this, use a null frame (e_a, e_{a^*}) such that $q(e_a, e_{a^*}) = \xi$. Then, $Fe_a = \sqrt{-1}e_a$, $Fe_{a^*} = -\sqrt{-1}e_{a^*}$, $\omega = \sqrt{-1}\sum_{a=1}^{n} \theta^a \wedge \theta^{a^*}$, and going over to the g-orthonormal frame

$$f_a := \frac{1}{\sqrt{2}}(e_a + e_{a^*}), \qquad f_{a^*} := \frac{\sqrt{-1}}{\sqrt{2}}(e_a - e_{a^*}), \tag{2.33}$$

and its dual coframe

$$\varphi^a = \frac{1}{\sqrt{2}}(\theta^a + \theta^{a^*}), \qquad \varphi^{a^*} = \frac{\sqrt{-1}}{\sqrt{2}}(\theta^{a^*} - \theta^a), \tag{2.34}$$

we get

$$\omega^n = (-1)^{n(n-1)/2} n! \varphi^1 \wedge \ldots \wedge \varphi^{2n}. \tag{2.35}$$

Accordingly, the two announced components, say $\mathcal{I}_{\pm}(M,g)$ consist of those ξ which have ω^n positive or negative, respectively, with respect to a fixed orientation of M.

Now, we may ask whether it is possible for the distribution Q to be involutive on one of these two components only. The involutivity conditions will still be (2.20) but, only for either positive or negative null frames. If $n \geq 3$ there is no difference between this situation and that of Proposition 2.1 and Theorem 2.2 since any triple of tangent vectors (X,Y,Z) which span an isotropic subspace may be seen as a subset of the set (e_a) of both a positive and a negative null frame. Hence (2.20) still implies Proposition 2.1 and Theorem 2.2, and no weaker condition will ensure the involutivity of Q on one of the manifolds $\mathcal{I}_{\pm}(M,g)$.

The situation is different for $n = 2$. In this case, let $span\{X,Y\}$ be a g-isotropic plane, let (X_*, Y_*) be another pair of tangent vectors at the same point as (X,Y) such that (X, Y, X_*, Y_*) is a null frame, and let (F_1, F_2, F_3, F_4) be the orthonormal frame associated with this null frame by (2.33). Then, (2.35) shows that ω_x^2 is positive (negative) iff (F_1, F_2, F_3, F_4) is negative (positive) with respect to the orientation of M.

It is known that the orientation of this latter frame is positive iff $*(F_1 \wedge F_2) = F_3 \wedge F_4$, and negative iff $*(F_1 \wedge F_2) = -F_3 \wedge F_4$, where $*$ is the Hodge star operator transposed to multivectors. By (2.33), the equivalent form of this condition is

$$(2.36) \qquad *(X \wedge Y) = \pm X \wedge Y,$$

respectively. Thus, the pair (X, Y) is either a part of a frame with positive corresponding orthonormal frame or of a negative one, but not both. This implies that we have separate involutivity conditions of Q on each component $\mathcal{I}_\pm(M, g)$.

Let us look at the following decompositions of the torsion and the curvature of ∇ seen as defined on bivectorial arguments:

$$(2.37) \qquad T = T_1 + T_2, \qquad R = R_1 + R_2,$$

where

$$T_1(X \wedge Y) = \frac{1}{2}(T(X \wedge Y) + T(*(X \wedge Y))),$$

$$T_2(X \wedge Y) = \frac{1}{2}(T(X \wedge Y) - T(*(X \wedge Y))),$$

$$R_1(X \wedge Y, Z \wedge U) = \frac{1}{2}(R(X \wedge Y, Z \wedge U) + R(X \wedge Y, *(Z \wedge U))),$$

$$R_2(X \wedge Y, Z \wedge U) = \frac{1}{2}(R(X \wedge Y, Z \wedge U) - R(X \wedge Y, *(Z \wedge U))).$$

Then, we get

2.4 Proposition. *Let (M, g) be an oriented, complex-Riemannian, four-dimensional manifold, with a real g-volume, the isosplitting space $\mathcal{I}_\pm(M, g)$, and the g-compatible connection ∇. Then, the main structural distributions $Q, \tilde{Q}$ are involutive on $\mathcal{I}_+(M, g)$ iff, for any isotropic plane $s = span\{X, Y\}$, the following two conditions hold: i) $T_2(X \wedge Y) \in s$, and ii) $R_2 = 0$. The same conditions for T_1, R_1 ensure the involutivity of $Q, \tilde{Q}$ on $\mathcal{I}_-(M, g)$.*

Proof. The conditions stated by this proposition are exactly the involutivity conditions (2.21) expressed for isotropic planes which satisfy the second (first) condition (2.36). Notice that the form of the torsion condition is specific for $n = 2$. Q.e.d.

As a consequence of Proposition 2.4, we get the basic result [1]

2.5 Theorem. *Let (M^4, g) be as in Proposition 2.4, and let ∇ be the Levi-Civita connection of g. Then Q and $\tilde{Q}$ are involutive on $\mathcal{I}_+(M, g)$ iff the Weyl curvature tensor of ∇ is self-dual, and $Q, \tilde{Q}$ are involutive on $\mathcal{I}_-(M, g)$ iff the Weyl curvature tensor of ∇ is anti-self-dual.*

Proof. Generally, the Hodge star also acts on tensor-valued forms and, on four-manifolds, $*$ is involutive on the spaces of 2-forms, and decomposes these spaces into the sum of the (± 1)-$*$-eigenspaces called the self-dual and anti-self-dual parts. It follows easily that T_1, R_1, T_2, R_2 are the self-dual and anti-self-dual parts, respectively, of T, R, where T is seen as a TM-valued 2-form, and R is seen as a $\wedge^2 M$-valued 2-form.

By the same polynomial argument as in the proof of Theorem 2.2, the torsion condition of Proposition 2.4 is equivalent to

$$T_a(X, Y) = \alpha(X)Y - \alpha(Y)X \qquad (a = 1, 2),$$

for any arguments X, Y. In particular, this condition holds with $\alpha = 0$ in the case of the Levi-Civita connection ∇.

And, the polynomial argument yields an expression of type (2.28) for $R_a(X \wedge Y, X \wedge Y)$, $a = 1, 2$, respectively, which, in view of the symmetries of R and of the Bianchi identity, turns out to provide the self-duality and anti-self-duality of the Weyl curvature tensor precisely. Q.e.d.

2.6 Remark. *The orientation considerations, Proposition 2.4 and Theorem 2.5 can also be transposed to the case where the metric g has a purely imaginary volume i.e., $det(g)$ is a negative, real number. In order to do so it suffices to apply the mentioned results to the metric $(-1)^{1/2n}g$ which has a real volume, and the same isosplitting space as g.*

The following result is an immediate consequence of (2.12)

2.7 Proposition. *The connection ∇ defines a complex Riemannian metric γ and two global γ-isosplittings on the isosplitting space $\mathcal{I}(M, g)$.*

Proof. The required metric is

$$\gamma = \sum_{a=1}^{n} (\theta^a \otimes \theta^{a^*} + \theta^{a^*} \otimes \theta^a) \tag{2.38}$$

$$+tr(\lambda \otimes \mu + \mu \otimes \lambda + \bar{\lambda} \otimes \bar{\mu} + \bar{\mu} \otimes \bar{\lambda}).$$

The distributions which define the global isosplittings are

$$(I) \quad (P') \quad \theta^{a^*} = 0, \ \lambda = 0, \ \bar{\lambda} = 0, \quad (P'') \quad \theta^a = 0, \ \mu = 0, \ \bar{\mu} = 0,$$

$$(II) \quad (Q') \quad \theta^{a^*} = 0, \ \mu = 0, \ \bar{\mu} = 0, \quad (Q'') \quad \theta^a = 0, \ \lambda = 0, \ \bar{\lambda} = 0.$$

Q.e.d.

We end this section by

2.8 Proposition. *Different metric connections of (M, g) define different distributions P, Q.*

Proof. In order to change the connection, we must add to the matrix (2.10) an $o(2n, \mathbf{C})$-valued, horizontal, tensorial 1-form of type ad, say $\begin{pmatrix} \tau & \sigma \\ \nu & -{}^t\tau \end{pmatrix}$, where σ and ν are skew-symmetric matrices [9]. The equations (2.14) of P, Q show that the change of connection preserves one of these distributions if either σ or ν are combinations of the forms θ^{a^*} alone. But, it is easy to see that, in this case, the 1-form of the connection change cannot be tensorial unless it vanishes. Q.e.d.

3 Symplectic Isosplitting Spaces

In this section, we consider a manifold M^{2n} endowed with a non degenerate, skew symmetric tensor field $\sigma \in \Gamma \wedge^2 T^{*c}M$ i.e., a *complex-almost symplectic manifold*, and define the isosplitting bundle $\mathcal{I}(M, \sigma)$ in exactly the same way as for a Riemannian metric g in Section 2. In particular, an isosplitting, which, now, is a pair of complex, tangent, transversal, Lagrangian subspaces (L_1, L_2), can be seen as F satisfying (2.1) again.

In order to describe the structure of $\mathcal{I}(M,\sigma)$, we look at the principal bundle $\mathcal{S}(M,\sigma)$ of symplectic frames (e_a, e_{a^*}), with dual coframes (θ^a, θ^{a^*}) $(a = 1, ..., n;\ a^* = a + n)$, such that

$$\sigma = \sum_{a=1}^{n} \theta^a \wedge \theta^{a^*}. \tag{3.1}$$

Again, (2.5) defines a projection $q : \mathcal{S}(M,g) \to \mathcal{I}(M,\sigma)$ which is a principal fibration of structure group $Gl(n,\mathbf{C})$ seen as a subgroup of the complex symplectic group $Sp(n,\mathbf{C})$. On the other hand, $\mathcal{I}(M,\sigma)$ is an associated bundle of $\mathcal{S}(M,\sigma)$ over M with the fiber $Sp(n,\mathbf{C})/Gl(n,\mathbf{C})$ of real dimension $2n(n+1)$. Hence, $\mathcal{I}(M,\sigma)$ has the real dimension $2n(n+2)$.

We recall that the Lie algebra of the complex symplectic group is

$$sp(n,\mathbf{C}) = \left\{ \begin{pmatrix} X & Y \\ Z & U \end{pmatrix} \ / \ U = -{}^tX,\ Y = {}^tY,\ Z = {}^tZ \right\}, \tag{3.2}$$

and it has the symmetric decomposition

$$sp(n,\mathbf{C}) = gl(n,\mathbf{C}) \oplus n, \tag{3.3}$$

where n consists of the matrices (3.2) with $X = U = 0$.

Accordingly, if we fix a connection ∇ on $\mathcal{S}(M,\sigma)$, the connection form will be given by (2.10) with the only difference that the matrices λ, μ will be symmetric matrices now. As in Section 2, the connection form satisfies (2.12), we also have the canonical 1-form θ which satisfies (2.8), and the pullback of $(\theta^a, \theta^{a^*}, \lambda, \mu, \bar{\lambda}, \bar{\mu})$ by local cross sections of q is a local cobasis on $\mathcal{I}(M,\sigma)$. Furthermore, the tangent space $T^c\mathcal{I}(M,\sigma)$ again has the decomposition (2.7), and, on the other hand, we may define the structural distributions $P, Q, \tilde{P}, \tilde{Q}$ given by (2.14) in the present case too. The (complex) dimension of these distributions is $n(3n+5)/2$, and, here again, we have the natural involution $\iota : \mathcal{I}(M,\sigma) \to \mathcal{I}(M,\sigma)$ given by $\iota(L_1, L_2) = (L_2, L_1)$ which exchanges P by $\tilde{P}$ and Q by $\tilde{Q}$. (The proof is in the next section.) In fact, the whole Proposition 2.1 is valid and, in particular, we have

3.1 Proposition. *The distributions $P, \tilde{P}$ are never involutive, and the main structural distributions $Q, \tilde{Q}$ are involutive iff the covariant torsion and curvature tensors of ∇, defined by (2.19) vanish on arguments which span a σ-isotropic subspace.*

The same proof as for Proposition 2.1 holds.

The significance of the involutivity condition of $Q, \tilde{Q}$ given by Proposition 3.1 can be revealed by using again the polynomial argument of the proof of Theorem 2.2 namely, that we must have the expressions (2.26), (2.27) with g replaced by σ. But, some of the symmetries of the tensors are now different. In particular, instead of (2.23) we get

$$T(X,Y) = \alpha(X)Y - \alpha(Y)X + \sigma(X,Y)U, \tag{3.4}$$

where U is a (complex) vector field on M, or, equivalently

$$T = \alpha \wedge (Id) + \sigma \otimes U. \tag{3.5}$$

This torsion condition has an interesting significance. Namely, if $d\sigma(X,Y,Z)$ is evaluated via the ∇-covariant derivative, (3.4) implies

$$d\sigma = \beta \wedge \sigma, \tag{3.6}$$

where $\beta = 2\alpha + i(U)\sigma$. It is well known that, if $n \geq 3$, (3.6) implies that $d\beta = 0$, and (M,σ) is a locally conformal symplectic manifold e.g., [17]. If $n = 2$, (3.6) always holds but, β may not be closed.

On the other hand, it is known that there always are connections ∇ on $\mathcal{S}(M,\sigma)$ such that

$$\tau(X,Y,Z) = \frac{1}{3}d\sigma(X,Y,Z) \tag{3.7}$$

e.g., [18], [21]. Thus, if (3.6) holds, connections ∇ which satisfy the torsion condition (3.5) exist. More exactly, a comparison of (3.4) and (3.7) yields

$$T(X,Y) = \beta(X)Y - \beta(Y)X + \sigma(X,Y)\sharp_\sigma\beta, \tag{3.8}$$

where $\sharp_\sigma\beta$ is defined by $i(\sharp_\sigma\beta)\sigma = -\beta$. It is well known that the case $T = 0$ may occur iff $d\sigma = 0$.

In the symplectic case, the symmetries of the curvature are [18]

$$R(X,Y,Z,U) = R(Y,X,Z,U), \;\; R(X,Y,Z,U) = -R(X,Y,U,Z),$$

hence, the curvature involutivity condition becomes

$$R(X,Y,Z,U) = a(X,Y)\sigma(Z,U) + b(X,Z)\sigma(Y,U) \tag{3.9}$$

$$-b(X,U)\sigma(Y,Z) + b(Y,Z)\sigma(X,U) - b(Y,U)\sigma(X,Z),$$

where a is symmetric, and a,b are such that the Bianchi identity (with torsion) is satisfied.

Accordingly, we get

3.2 Theorem. *The distributions $Q,\tilde{Q}$ of the isosplitting space $\mathcal{I}(M,\sigma)$ are involutive iff the torsion and curvature of ∇ are given by the formulas (3.5) and (3.9), respectively. If this happens then, necessarily, either $n = 2$ or (M,σ) is a locally conformal symplectic manifold. In particular, if $d\sigma = 0$, if ∇ is torsionless, and if ∇ has a reducible curvature $Q,\tilde{Q}$ are involutive.*

Proof. Except for the last, all the assertions of this theorem have already been proven. The notion of a *reducible curvature* was defined in [18], and it exactly means that $R(X,Y,Z,U)$ is given by (3.9) with $a = 2b$. Q.e.d.

The basic example of a symplectic connection with a reducible curvature is the Levi-Civita connection of a Kähler manifold of constant, holomorphic, sectional curvature.

Notice that the reducibility of the curvature is not a necessary condition of involutivity even in the torsionless case. Indeed, if $T = 0$, the Bianchi identity without torsion for the curvature (3.9) is

$$(3.10)\qquad \sum_{Cycl(Y,Z,U)} R(X,Y,Z,U) = ((i(X)a)\wedge\sigma)(Y,Z,U)$$

$$-2((i(X)b)\wedge\sigma)(Y,Z,U) + 2((i(X)\sigma)\wedge b_0)(Y,Z,U) = 0,$$

where

$$b_0(X,Y) := \frac{1}{2}[b(X,Y) - b(Y,X)].$$

This relation yields $a = 2b$, and a reducible curvature, if b is symmetric, but not in the general case.

3.3 Remark. *The development of this section is, in fact, the same as that given for a real 2-form σ in [21] but, here, we gave a more detailed analysis of the involutivity conditions. The present development also includes the symplectic twistor spaces of [19]. We notice that the present Theorem 3.2 shows that there was an error in Theorem 2.1 of [19]. Namely, the flatness condition of that theorem was only a sufficient but not a necessary condition for the integrability of the complex structure of a symplectic twistor space. On the other hand, we notice that Proposition 2.8 holds in the symplectic case as well.*

Finally, we should notice that we do not have an analog of Theorem 2.5 in the symplectic case. The reason for this is as follows. If we assume that σ^n is a real form on M, it defines a volume form and a symplectic Hodge star [10] given by

$$*\Phi = i(\sharp_\sigma\Phi)\frac{\sigma^n}{n!},$$

which is such that $*^2 = Id$. Hence, if $n = 2$ it is still true that the spaces of tangent bivectors decomposes into a self-dual and an anti-self-dual component. However, by computing with a symplectic frame, it turns out that, whenever $span(X,Y)$ is isotropic, we must have $*(X\wedge Y) = -X\wedge Y$, unlike in the Riemannian case where both signs could be encountered in (2.36). Thus $\mathcal{I}(M^4,\sigma)$, has only one component.

4 Conformal Invariance Properties

As in the classical twistor theory, the isosplitting spaces of Sections 2, 3 remain unchanged if the metric g or the 2-form σ undergo a *conformal transformation*

$$(4.1)\qquad g' = e^{2h}g, \qquad \sigma' = e^{2h}\sigma,$$

respectively, where h is an arbitrary (complex) function. Moreover, we will see that the behaviour of the distributions P, Q by the transformation (4.1) is the same as that of the almost complex structures of the classical twistor spaces. We show this for the Riemannian and the symplectic case, simultaneously ; the symplectic version is written in parentheses. An accent always denotes objects related to the new metric g (2-form σ).

Following a well known result of Riemannian geometry, let us denote by $\mathcal{C}(g)$ ($\mathcal{C}(\sigma)$) the set of g-metric (σ-almost symplectic) connections on the manifold M^{2n}, and define the *Weyl mapping* $w : \mathcal{C}\to\mathcal{C}'$ by $\nabla' = w(\nabla)$ where

$$\nabla'_X Y = \nabla_X Y + (Xh)Y + (Yh)X - g(X,Y)grad_g h. \tag{4.2}$$

$$(\nabla'_X Y = \nabla_X Y + (Xh)Y + (Yh)X - \sigma(X,Y)grad_\sigma h.)$$

In formulas (4.2), X, Y are vector fields on M, and $grad_\sigma h$ is the *Hamiltonian vector field* of h characterized by $i(grad_\sigma h)\sigma = -dh$. Remember that the Levi-Civita connections of g, g' are related by the Weyl mapping.

What we are going to prove is

4.1 Theorem. *Two connections which are related by the Weyl mapping define the same distributions $Q, \tilde{Q}$ on the isosplitting space but, they do not define the same distributions $P, \tilde{P}$ unless $h = const.$, and the two connections coincide.*

Proof. First, we establish a notation [20] which will allow us to treat the Riemannian and symplectic case simultaneously. Namely, let Greek indices run from 1 to $2n$, and put $\alpha^* := \alpha + n \ (mod. 2n)$ $(\alpha = 1, ..., 2n)$. Then, define the symbol $s_g(\alpha)$ to be the constant 1, and the symbol $s_\sigma(\alpha)$ to be -1 for $\alpha = a$, and 1 for $\alpha = a^*$ $(a = 1, ..., n)$. Hereafter, we do not write the index of the symbol $s(\alpha)$, and we make the convention that g and $\mathcal{N}(M, g)$ mean σ and $\mathcal{S}(M, \sigma)$, respectively, in the symplectic case. This provides the formulas of this section with a common, Riemannian-symplectic character. In particular, a null frame (symplectic frame) $(e_\alpha) = (e_a, e_{a^*})$ is characterized by the following values of the components of g (σ)

$$g_{\alpha\beta} = s(\beta)\delta_{\alpha\beta^*} \tag{4.4}$$

Furthermore, using the local connection equations, one gets the local connection forms ϖ of ∇ with respect to local fields of null-frames (symplectic frames):

$$\varpi_\alpha^{\beta^*} = s(\beta)g(\nabla e_\alpha, e_\beta). \tag{4.5}$$

Accordingly, the global connection form (2.10) on the principal bundle $\mathcal{N}(M, g)$ written as

$$\begin{pmatrix} \omega_a^b & \omega_{a^*}^b \\ \omega_a^{b^*} & \omega_{a^*}^{b^*} \end{pmatrix} := \begin{pmatrix} \omega_a^b & \lambda_a^b \\ \mu_a^b & -\omega_b^a \end{pmatrix} \tag{4.6}$$

is given by [9]

$$\omega_\alpha^\beta = \eta_{\gamma^*}^\beta d\xi_\alpha^{\gamma^*} + \eta_{\gamma^*}^\beta \xi_\alpha^\kappa \pi^* \varpi_\kappa^{\gamma^*}, \tag{4.7}$$

where π is the projection of the bundle space on M, ξ are the natural coordinates on $\mathcal{N}(M, g)$ with respect to the local frames of (4.5), and η is the inverse matrix of ξ. (The Einstein summation convention is used here.)

Now, the conformal change (4.1) provides a bundle equivalence

$$\Phi : \mathcal{N}(M, g') \longrightarrow \mathcal{N}(M, g) \qquad (\pi' = \pi \circ \Phi), \tag{4.8}$$

where $\Phi((e_\alpha)) = (e^h e_\alpha)$, and using (4.7) (4.2) and (4.5) we get

$$\omega'^{\beta}_{\alpha} = \eta'^{\beta}_{\gamma^*} d\xi'^{\gamma^*}_{\alpha} + \eta'^{\beta}_{\gamma^*} \xi'^{\kappa}_{\alpha} \pi'^* \varpi'^{\gamma^*}_{\kappa}$$

$$= \Phi^*[\eta^{\beta}_{\gamma^*} d\xi^{\gamma^*}_{\alpha} + \sum_{\gamma,\kappa} s(\gamma)\eta^{\beta}_{\gamma^*}\xi^{\kappa}_{\alpha}\pi^*(s(\gamma)\varpi^{\gamma^*}_{\kappa} + s(\gamma)h_{\kappa}\theta^{\gamma^*} - s(\kappa)h_{\gamma}\theta^{\kappa^*})],$$

where the computation is at an arbitrary null-frame, θ is the canonical 1-form, and we put $dh = h_\alpha\theta^\alpha$. Furthermore, keeping in mind the coordinate expression of the right translations on $\mathcal{N}(M,g)$, and using the fact that if (e_α) is a null-frame (symplectic frame) so is $(s(\alpha)e_{\alpha^*})$, the result of the previous computation becomes

$$\omega'^{\beta}_{\alpha} = \Phi^*\omega^{\beta}_{\alpha} + h_\alpha\theta'^{\beta} - s(\beta)s(\alpha)h_{\beta^*}\theta'^{\alpha^*}, \tag{4.9}$$

where, in fact, h means $h \circ \pi'$.

From (4.9), and keeping in mind our notation, of course, we get

$$\lambda'^{b}_{a} = \Phi^*\lambda^{b}_{a} + h_{a^*}\theta'^{b} + h_{b^*}\theta'^{a}, \tag{4.10}$$

$$\mu'^{b}_{a} = \Phi^*\mu^{b}_{a} + h_a\theta'^{b^*} + h_b\theta'^{a^*}, \tag{4.11}$$

whence the conclusions of the theorem follow. Q.e.d.

In particular, we get

4.2 Corollary. *The distributions $Q, \tilde{Q}$ defined by the Levi-Civita connection of a metric g on the isosplitting space $\mathcal{I}(M,g)$ are invariant by a conformal transformation of the metric. The distribution $P, \tilde{P}$ are invariant only by homothetical transformations (i.e., $h = const.$ in (4.1)).*

Corollary 4.2 tells us that the triple $(\mathcal{I}(M,g), Q(\nabla), \tilde{Q}(\nabla))$, where ∇ is the Levi-Civita connection of g is, in fact, determined by the conformal structure defined by g i.e., by the maximal "atlas" $\{(U_\sigma, g_\sigma)\}$, where $\{U_\sigma\}$ covers M, each g_σ is a complex-Riemanniann metric on U_σ, any two metrics g_σ are related by (4.1) on their common domain, and (M,g) belongs to the atlas. This explains why the integrability of Q is characterized by a condition on the Weyl curvature tensor which is a conformal invariant of the metric.

Notice that the involutivity conditions of Theorem 2.5 also are conformally invariant. The $*$-operator needed there can be taken with respect to any metric g_σ in the conformal structure since this operator is conformally invariant in the involved situation. The invariance may be seen by expressing $*$ either in an orthonormal or a null frame (as in (2.36)).

Furthermore, if the mentioned conformal structure has a *homothetical substructure* i.e., a similar atlas where the metrics are related by (4.1) with constant functions h, the distributions $P, \tilde{P}$ are invariants of this homothetical structure. A nice example is given by the *locally conformal almost Kähler manifolds* [15] which are almost Hermitian manifolds (M,g) where the conformal structure defined by g has a distinguished substructure $\{(U_\sigma, g_\sigma)\}$ given by local almost Kähler metrics g_σ. This substructure necessarily is homothetical.

The notation used during the proof of Theorem 4.1 allows us to keep a previous promise, and prove

4.3 Proposition. *The involution $\iota(L_1, L_2) = (L_2, L_1)$ defined in Sections 2, 3 sends P to $\tilde{P}$ and Q to $\tilde{Q}$ in both the Riemannian and the symplectic case.*

Proof. The formula $\tilde{\iota}((e_\alpha)) := (s(\alpha)e_{\alpha^*})$ defines a lift of ι to the corresponding bundle of frames, which has the local equations $\tilde{\xi}^\beta_\alpha = s(\alpha)\xi^\beta_{\alpha^*}$ in the coordinates used in (4.7). Now, the definition of $\tilde{\iota}$ and (4.7) yield

$$\tilde{\iota}^*(\theta^\alpha) = s(\alpha)\theta^{\alpha^*}, \qquad \tilde{\iota}^*(\omega^\beta_\alpha) = s(\alpha)s(\beta)\omega^{\beta^*}_{\alpha^*}, \tag{4.12}$$

whence the required results. Q.e.d.

5 Twist-Reflector Spaces

Now, we will apply the method of Section 2 to the geometrically and physically important case where g is a real, pseudo-Riemannian metric of a non-negative, even signature $2s \geq 0$, on an even-dimensional manifold M^{2n}. We will also denote by $p = n + s$ the positive inertial index of g, and by $q = n - s$ its negative index.

In this case the isosplitting space has an interesting subspace which we want to discuss here. (Subspaces of twistor spaces also appeared in a different context namely, the bundles of analytical tangent subspaces on almost Hermitian manifolds [12, 22]. We give the following definition: a *twist-reflector* of (M, g) is an orthogonal decomposition $T_xM = U \oplus V$, where $g/_U$ has signature zero, $g/_V$ is positive definite, $(U, g/_U)$ is endowed with a real isosplitting, $U = U_1 \oplus U_2$, and $(V, g/_V)$ is endowed with a *complex isosplitting*, $V = K \oplus \bar{K}$, where the bar denotes complex conjugation. The name comes from the fact that, if $s = n$, the twist-reflector is a usual twistor, and, if $s = 0$, it is a reflector [6].

For a given twist-reflector, the pair $(U_1 \oplus K, U_2 \oplus \bar{K})$ is an isosplitting. Before characterizing all the twist-reflector isosplittings, we give one more definition. Let L be a subspace of the complexification $V \otimes \mathbf{C}$, where V is a real vector space. M. Then $L \cap \bar{L}$ is the complexification of a subspace of V, and the dimension of this subspace is called the *real index* of L. Coming back to the pseudo-Riemannian manifold (M, g), we notice that the maximal possible, real index of a g-isotropic (complex), tangent subspace L is q. Indeed, if L is isotropic so is $L \cap \bar{L}$, and it is known that the maximal possible dimension of a real, isotropic, tangent subspace of (M, g) is q, e.g., [14].

An isosplitting (L_1, L_2) at $x \in M$ comes from a twist-reflector if the distributions L_1, L_2 have real index q, and are of the form

$$L_1 = (L_1 \cap \bar{L}_1) \oplus A, \qquad L_2 = (L_2 \cap \bar{L}_2) \oplus \bar{A}, \tag{5.1}$$

for some subspace A of L_1. It is easy to understand that, if A exists, it is unique and, in the notation used at the beginning, the twist-reflector has $U_1 = L_1 \cap \bar{L}_1$, $U_2 = L_2 \cap \bar{L}_2$, $V = A \oplus \bar{A}$. Any isotropic, tangent subspace of M which has the real index q belongs to isosplittings which are twist-reflectors.

We will denote by $\mathcal{Z}(M, g)$ the space of all the twist-reflectors, and use the method of Section 2 to study this space, which is the announced subspace of $\mathcal{I}(M, g)$.

A tangent null frame (e_i, e_{i^*}) $(i = 1, \ldots, n)$ of g at $x \in M$ will be called an *adapted null frame* if it puts g into the form

$$g = \sum_{a=1}^{q}(\theta^a \otimes \theta^{a^*} + \theta^{a^*} \otimes \theta^a) + \sum_{u=1}^{s}(\vartheta^u \otimes \bar{\vartheta}^u + \bar{\vartheta}^u \otimes \vartheta^u), \tag{5.2}$$

where $(\theta^a, \theta^{a^*}, \vartheta^u, \bar{\vartheta}^u)$ is the dual coframe of the given frame decomposed as $(e_i, e_{i^*}) = (e_a, f_u, e_{a^*}, \bar{f}_u)$ $(a = 1, ..., q;\ a^* = a + n;\ u = 1, ..., s)$. The set of the adapted null frames is a principal subbundle $\pi : \mathcal{A}(M, g) \to M$ of $\mathcal{N}(M, g)$ of Section 2, with the structure group $O(p, q)$ represented as a subgroup of $O(2n, \mathbf{C})$ given by formula (2.3). The corresponding Lie algebra is the subalgebra of (2.4) which has matrices X, Y, Z of the form

$$(5.3)\qquad X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix},\ Y = \begin{pmatrix} Y_1 & \bar{X}_2 \\ -{}^t\bar{X}_2 & Y_4 \end{pmatrix},\ Z = \begin{pmatrix} Z_1 & -{}^t\bar{X}_3 \\ \bar{X}_3 & \bar{Y}_4 \end{pmatrix},$$

where X_1 is a real, square matrix of order q, Y_1, Z_1 are real, square, skew symmetric matrices of order q, X_2, X_3 are complex (s, q)-matrices, X_4 is an anti-Hermitian matrix of order s, and Y_4 is a complex, square, skew symmetric matrix of order s. This follows either by a straightforward inspection of the action of $O(p, q)$ on adapted null frames or, easier, by writing down the *moving frame equations* i.e., the expression of $(de_a, df_u, de_{a^*}, d\bar{f}_u)$ as combinations of the basic vectors themselves, and by looking at the consequences of the relations $de_a = \overline{de_a}$, $de_{a^*} = \overline{de_{a^*}}$, $d\bar{f}_u = \overline{df_u}$ and at the conditions of (2.4).

In particular, for the connection form (2.10) of any metric connection ∇ of (M, g), we have

$$(5.4)\qquad \omega = \begin{pmatrix} \omega_1 & \omega_2 \\ \omega_3 & \omega_4 \end{pmatrix},\ \lambda = \begin{pmatrix} \lambda_1 & \bar{\omega}_2 \\ -{}^t\bar{\omega}_2 & \lambda_4 \end{pmatrix},\ \mu = \begin{pmatrix} \mu_1 & -{}^t\bar{\omega}_3 \\ \bar{\omega}_3 & \bar{\lambda}_4 \end{pmatrix},$$

with the same reality and symmetry properties as for the corresponding blocks of (5.3).

The projection q of (2.5) will be replaced by $\psi : \mathcal{A}(M, g) \to \mathcal{Z}(M, g)$ given by

$$(5.5)\qquad \psi(e_a, f_u, e_{a^*}, \bar{f}_u) = \{U = span(e_a, e_{a^*}), V = span(f_u, \bar{f}_u)\},$$

with the obvious isosplittings of the two subspaces. And, it follows that ψ is a principal fibration with the structure group $Gl(q, \mathbf{R}) \times U(s)$ seen as the group of the matrices with a diagonal of blocks

$$(5.6)\qquad (A, B, {}^tA^{-1}, \bar{B}) \quad (A \in Gl(q, \mathbf{R}), B \in U(s)),$$

and zeroes on all the other places. Furthermore, the natural projection $p' : \mathcal{Z}(M, g) \to M$ is an associated bundle of $\mathcal{A}$ with a homogeneous fiber of real dimension $n(n - 1) + 2qs$. Therefore, the real dimension of $\mathcal{Z}(M, g)$ is $n(n + 1) + 2qs$.

Now, if we choose a metric connection ∇, we can proceed as in Proposition 2.1, and get a horizontal-vertical decomposition $T\mathcal{Z} = \mathcal{H}\mathcal{Z} \oplus \mathcal{V}\mathcal{Z}$, where $\mathcal{V}$ has the equations

$$(5.7)\qquad \theta^a = 0,\ \vartheta^u = 0,\ \theta^{a^*} = 0,\ \bar{\vartheta}^u = 0,$$

given by the components of the canonical 1-form on $\mathcal{A}(M, g)$ as in the case of Proposition 2.1, and $\mathcal{H}$ has the equations (2.13) where the blocks of the connection form are those shown by (5.4).

Then, the important point to be made is that (2.8) and (2.12) now lead to the fact that each part of the canonical 1-form shown by (5.7), and each block of the connection form (5.4) transform independently of the other parts or blocks by right translations by elements of the form (5.6). Moreover, the symmetric decomposition (2.9) is now replaced by a symmetric decomposition

$$o(p,q) = (gl(q,\mathbf{R}) \oplus u(s)) \oplus v, \tag{5.8}$$

given by

$$\begin{pmatrix} \omega_1 & \omega_2 & \lambda_1 & \bar{\omega}_2 \\ \omega_3 & \omega_4 & -{}^t\bar{\omega}_2 & \lambda_4 \\ \mu_1 & -{}^t\bar{\omega}_3 & -{}^t\omega_1 & -{}^t\omega_3 \\ {}^t\bar{\omega}_3 & \bar{\lambda}_4 & -{}^t\omega_2 & -{}^t\omega_4 \end{pmatrix} = \tag{5.9}$$

$$= \begin{pmatrix} \omega_1 & 0 & 0 & 0 \\ 0 & \omega_4 & 0 & 0 \\ 0 & 0 & -{}^t\omega_1 & 0 \\ 0 & 0 & 0 & \bar{\omega}_4 \end{pmatrix} + \begin{pmatrix} 0 & \omega_2 & \lambda_1 & \bar{\omega}_2 \\ \omega_3 & 0 & -{}^t\bar{\omega}_2 & \lambda_4 \\ \mu_1 & -{}^t\bar{\omega}_3 & 0 & -{}^t\omega_3 \\ {}^t\bar{\omega}_3 & \bar{\lambda}_4 & -{}^t\omega_2 & 0 \end{pmatrix},$$

and it follows that the first term of (5.9) is a connection form of the principal fibration ψ (the *reduction* ∇^r of ∇ to ψ), while the second term is a horizontal, tensorial form and, therefore, the pullback of the blocks of this term by local cross sections of ψ define a vertical cobasis on $\mathcal{Z}(M,g)$. If we add the horizontal cobases given by the pullback of $(\theta^a, \vartheta^u, \theta^{a^*}, \bar{\vartheta}^u)$, we get complete local cobases on $\mathcal{Z}(M,g)$.

Let us recall the following notions which are important for complex distributions L on arbitrary differentiable manifolds M. If: i) L is involutive, and ii) $L+\bar{L}$ is also an involutive distribution of constant dimension, L is *Nirenberg integrable* e.g., [11], [16], and M has a *partial complex structure* i.e., an atlas with some of the coordinates complex, and with transition functions which are complex analytic in the complex coordinates, and C^∞ in the real coordinates [16]. On the other hand, if the real index of the complex distribution L is zero, L is an *almost CR-structure*, and, if L is also Nirenberg integrable, L is a *CR-structure* on M [2].

Now, instead of looking at the distributions P, Q of Section 2 (which, however, can be considered), we define the complex distributions $\mathcal{P}$, $\mathcal{Q}$ by the equations

$$(\mathcal{P}) \qquad \theta^{a^*} = 0,\ \bar{\vartheta}^u = 0,\ \omega_2 = 0,\ \bar{\omega}_2 = 0, \tag{5.10}$$

$$\omega_3 = 0,\ \lambda_1 = 0,\ \lambda_4 = 0$$

$$(\mathcal{Q}) \qquad \theta^{a^*} = 0,\ \bar{\vartheta}^u = 0,\ \omega_2 = 0,\ \omega_3 = 0, \tag{5.11}$$

$$\bar{\omega}_3 = 0,\ \mu_1 = 0,\ \bar{\lambda}_4 = 0.$$

The dimension of these distributions is $n(n+1)/2$, and their real index is $q(q+1)/2$. On the twist-reflector space $\mathcal{Z}(M,g)$, we will give the name *structural distributions* to $\mathcal{P}, \mathcal{Q}$ rather than to P, Q of Section 2. We could also define $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{Q}}$ analogous to $\tilde{P}, \tilde{Q}$ of Section 2 but, this doesn't provide important new information. A change of the metric connection used will lead to different distributions $\mathcal{P}, \mathcal{Q}$ for the same reasons as in Proposition 2.8.

5.1 Theorem. *i). If either $s = 0$ or $s = n$, $\mathcal{P} = P$, $\mathcal{Q} = Q$ on $\mathcal{Z}(M, g)$, and the results of Theorem 2.2 hold.*

ii). If $s \neq 0, n$, the distribution $\mathcal{P}$ is never involutive, and if the distribution $\mathcal{Q}$ is involutive the torsion and curvature of ∇ are those given by (2.23), (2.24). In particular, if ∇ is the Levi-Civita connection, and $\mathcal{Q}$ is involutive g is locally conformal flat. If, moreover, g has constant sectional curvature, Q is Nirenberg integrable.

Proof. We begin by discussing the involutivity of the distributions $\mathcal{P}, \mathcal{Q}$. The structure equation (2.15*) shows that $d\theta^{a^*}$ has terms which are products of forms θ^b and forms in the block μ_1 hence, $\mathcal{P}$ is never involutive. The same equation (2.15*), written for the present forms $\theta^{a^*}, \bar{\vartheta}^u$, and the connection matrix (5.9) shows that the torsion condition for the involutivity of $\mathcal{Q}$ is the annulation of the torsion tensor τ of (2.19) on any triple of arguments among the vectors (e_a, f_u) of any adapted null frame of (M, g).

Then, the structure equations (2.16*) applied to the other equations of $\mathcal{Q}$, with the exception of $\omega_2 = 0$, $\omega_3 = 0$, show that the involutivity of $\mathcal{Q}$ implies a curvature condition which is the vanishing of the covariant curvature tensor R of ∇ on the same kind of arguments on which the torsion vanished.

Let us notice that, if the tensors τ, R vanish on some arguments, they also vanish on linear combinations of these arguments. On the other hand, for any isotropic subspace I of any dimension h, there exists a basis of the subspace which may be embedded into an adapted null frame. It is enough to start with a real basis (i_α) of $I \cap \bar{I}$ ($\alpha = 1, ..., real\ index\ of\ I$) , and to continue with a basis (j_β) of a complement of $I \cap \bar{I}$ in I. Because $span(i_\alpha)$ is a maximal, real subspace of I, the vectors $\bar{j}_\beta$ will be outside I. Therefore, we may embed the (i_α) into the (e_a) part, and the (j_β) into the (f_u) part of an adapted null frame.

Hence, if (X, Y, Z, U) span an isotropic subspace I, then, by expressing them as linear combinations of the partial, adapted null frame of I constructed above, we see that τ, R vanish on X, Y, Z, U. Thus, the involutivity conditions obtained so far are equivalent with the vanishing of τ and R on any arguments which span a g-isotropic subspace of some tangent space $T^c_x M$, $x \in M$.

Now, we are exactly in the same situation as in the proof of Theorem 2.2, and the proof of the latter, as given in Section 2, also provides us with the required expressions of the torsion and curvature tensors, which proves i) and the first two assertions of ii).

For $s \neq 0, n$, these necessary involutivity conditions are not sufficient since they do not ensure that $d\omega_2$, $d\omega_3$ belong to the ideal generated by the equations of $\mathcal{Q}$.

On the other hand, in the case of a metric of constant curvature and its Levi-Civita connection, (2.24) has a classical form, e.g. [9], which shows that the Frobenius involutivity condition of $\mathcal{Q} + \bar{\mathcal{Q}}$, defined by the equations

$$\theta^{a^*} = 0,\ \omega_3 = 0,\ \bar{\omega}_3 = 0,\ \mu_1 = 0,$$

holds. It is also clear that $\mathcal{Q} + \bar{\mathcal{Q}}$ is of a constant dimension. Therefore, $\mathcal{Q} + \bar{\mathcal{Q}}$ is Nirenberg integrable. Q.e.d.

5.2 Remark. *If $\mathcal{Q}$ is Nirenberg integrable, according to Nirenberg's theorem [11], $\mathcal{Z}(M,g)$ has a local atlas with coordinates of the form $(x^\lambda, z^\sigma, \bar{z}^\sigma, y^\mu)$, where x, y are real coordinates, z are complex coordinates, and*

$$\mathcal{Q} = span\left(\frac{\partial}{\partial x^\lambda}, \frac{\partial}{\partial \bar{z}^\sigma}\right) \quad (\lambda, \mu = 1, ..., q(q+1)/2, \sigma = 1, ..., s(2n-s+1)/2).$$

5.3 Remark. *If we come back to the distributions P, Q of Section 2, but on $\mathcal{Z}(M,g)$, the proof of Theorem 5.1 shows that, again, P is never involutive, and Q is involutive iff the conditions of Theorem 2.2 hold.*

The metrics g of the present section are real hence, they have either a real or a purely imaginary volume, and the twist-reflector space $\mathcal{Z}$ has the components $\mathcal{Z}_\pm$ defined as in Section 2. Indeed, a twist-reflector is a particular isosplitting, and we have for it the volume form (2.35) (multiplied by i if $det(g) < 0$). If the twist-reflector has a neutral component U and a positive component V, ω of (2.35) is a sum of a 2-form on U and a 2-form on V, both defined as in Section 2. As in Section 2, we have again Proposition 2.4 and Theorem 2.5 for the distribution Q. For $\mathcal{Q}$, if $s \neq 0, n$, these results only give necessary involutivity conditions. (As in the proof of Theorem 5.1, we first act on those equations of $\mathcal{Q}$ which, in Section 2, defined the distribution Q.)

If $s = n$, the twist-reflector space is the classical twistor space, and, above, we have proofs of the basic integrability results which is *elementary*, in the sense that it does not use representation theory. We also see that this is the only case where we have a CR-structure on $\mathcal{Z}$, a complex structure, in fact. On the other hand, if $s = 0$ we get the reflector space of a neutral manifold, and the integrability conditions of its natural *almost paracomplex structure* [6].

Furthermore, as in Section 2, we have

5.4 Proposition. *If (M^{2n}, g) is a pseudo-Riemannian manifold of signature $2s \geq 0$ and ∇ is a metric connection, ∇ defines a pseudo-Riemannian metric γ on $\mathcal{Z}(M,g)$, and the distributions $\mathcal{P}$, $\mathcal{Q}$ are γ-isotropic distributions.*

Proof. The required metric is

$$\gamma = \sum_{a=1}^{q}(\theta^a \otimes \theta^{a^*} + \theta^{a^*} \otimes \theta^a) + \sum_{u=1}^{s}(\vartheta^u \otimes \bar{\vartheta}^u + \bar{\vartheta}^u \otimes \vartheta^u) \tag{5.12}$$

$$-tr(\lambda_1 \oplus \mu_1 + \mu_1 \otimes \lambda_1 + \omega_2 \otimes \omega_3 + \omega_3 \otimes \omega_2$$

$$+\bar{\omega}_2 \otimes \bar{\omega}_3 + \bar{\omega}_3 \otimes \bar{\omega}_2 + \lambda_4 \otimes \bar{\lambda}_4).$$

Q.e.d.

We end by the remark that, if $s < n$, the main structural distribution $\mathcal{Q}$ of the Levi-Civita connection also is only a homothetical invariant, not a general conformal invariant. But, of course, in the case $s = n$, $\mathcal{Q}$ defines the conformally invariant almost complex structure of the classical twistor space.

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