Geometrical Interpretation of Solutions of Certain PDEs

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Abstract

In §1 the authors mention some open problems and define the notion of harmonic map between two generalized Lagrange spaces. In §2 one proves that for certain systems of differential or partial differential equations, the solutions belong to a class of harmonic maps between two generalized Lagrange spaces. §3 describes the main properties of the generalized Lagrange spaces constructed in §2. These spaces, being convenient relativistic models, allow us to write the Maxwell and Einstein equations.

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1 Introduction

Looking for solving a generalized Poincaré problem, Sasaki tried to find a Riemannian metric on a manifold M such that the orbits of an arbitrary vector field X should be geodesics. This attempt was a failure, but Sasaki discovered the well known almost contact metric structures on a manifold of odd dimension [8]. After the introduction of generalized Lagrange structures [4], the problem of Poincaré-Sasaki was reconsidered by the first author [9, 10, 11] using a suitable prolongation of the flow to a second-order conservative dynamical system. He succeded to discover a Lagrange structure on M, depending of the given vector field X, a $(1,1)$-tensor field built using X, using a metric g, and the covariant derivative induced by g, such that the C^2 orbits belong to a class of pregeodesics. Moreover, replacing the system of ODEs of the orbits of X by a system of PDEs and the notion of geodesic by the notion of harmonic map, same open general problems appear [12], namely

1) There exist Lagrange type structures such that the solutions of certain PDEs should be *harmonic maps*?

2) What is a *harmonic map* between two generalized Lagrange spaces?

3) What Lagrange structure solve the inverse problem in the variational calculus associated to an even-order prolongation obtained by differentiations and mathematical artifice from a given PDE system?

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Using the notion of *direction dependent harmonic map* between a Riemannian manifold and a generalized Lagrange space, a partial answer to the Udrişte questions was offered by the second author [6].

In this paper we will attempt to carry on the development of certain ideas that realize a closely linked between PDEs, Differential Geometry, and Variational Calculus.

Let us introduce, in a natural way, the notion of harmonic map between two Lagrange spaces $(M,\ g_{\alpha\beta}(a,b))$ and $(N,\ h_{ij}(x,y))$, where M (resp. N) has the dimensions m (resp. n) and $(a,b)=(a^\mu,b^\mu)$ (resp. $(x,y)=(x^k,y^k)$) are local coordinates on TM (resp. TN).

Definition. On $M\times N$, a tensor field P of type $(1,2)$ with all components null excepting $P^{\beta}_{\alpha i}(a,x)$ and $P^{j}_{\alpha i}(a,x)$, where $\alpha,\beta=\overline{1,m},\ i,j=\overline{1,n}$, is called *tensor of connection.*

Assume that the manifold M is connected, compact, orientable and endowed also with a Riemannian metric $\varphi_{\alpha\beta}$. This fact ensures the existence of a volume element on M. In these conditions, we can define the $\begin{pmatrix} & P & \\ g & \varphi & h\end{pmatrix}$ *-energy functional*,

$$E\overset{\text{not}}{=}E^P_{g\varphi h}:C^\infty(M,N)\to R,$$

$$E^P_{g\varphi h}(f)=\frac{1}{2}\int_M g^{\alpha\beta}(a,b)h_{ij}(f(a),y)f^i_\alpha f^j_\beta\sqrt{\varphi}da,$$

where
$$\begin{cases} f^i=x^i(f),\ f^i_\alpha=\dfrac{\partial f^i}{\partial a^\alpha},\ \varphi=det(\varphi_{\alpha\beta})\\ b(a)=b^\gamma(a)\ \left.\dfrac{\partial}{\partial a^\gamma}\right|_a\overset{\text{def}}{=}\varphi^{\alpha\beta}(a)f^i_\alpha(a)P^\gamma_{\beta i}(a,f(a))\ \left.\dfrac{\partial}{\partial a^\gamma}\right|_a\\ y(f(a))=y^k(a)\ \left.\dfrac{\partial}{\partial x^k}\right|_{f(a)}\overset{\text{def}}{=}\varphi^{\alpha\beta}(a)f^i_\alpha(a)P^k_{\beta i}(a,f(a))\ \left.\dfrac{\partial}{\partial x^k}\right|_{f(a)}.\end{cases}$$

Definition. A map $f\in C^\infty(M,N)$ is called $\begin{pmatrix} & P & \\ g & \varphi & h\end{pmatrix}$ *-harmonic* if f is a critical point (extremal) for the functional $E^P_{g\varphi h}$.

Let $L(a^\alpha,f^i,f^i_\alpha)=\frac{1}{2}g^{\gamma\mu}(a^\nu,b^\nu)h_{kl}(x^p,y^p)f^k_\gamma f^l_\mu$. The $\begin{pmatrix} & P & \\ g & \varphi & h\end{pmatrix}$-harmonic maps are described by Euler-Lagrange PDEs

$$(H)\qquad \sqrt{\varphi}\frac{\partial L}{\partial f^i}+\frac{\partial}{\partial a^\alpha}\left(\sqrt{\varphi}\frac{\partial L}{\partial f^i_\alpha}\right)=0,\ \forall i=\overline{1,n}.$$

The naturalness of the preceding definitions comes from the following particular cases:

i) If $g_{\alpha\beta}(a,b)=\varphi_{\alpha\beta}(a)$ and $h_{ij}(x,y)=h_{ij}(x)$ are Riemannian metrics, then it recovers the classical definition of a harmonic map between two Riemannian manifolds [2, 3]. In this case, the definition of harmonic maps is independent of the connection tensor field P.

ii) If $M=[a,b]\subset R,\ \varphi_{11}=g_{11}=1$ and $P=(P^1_{1i},\delta^k_i)$, then we find $C^\infty(M,N)=\{c:[a,b]\to N|\ c-C^\infty\text{differentiable}\}\overset{\text{not}}{=}\Omega_{a,b}(N)$ and the energy functional is

$$E^P_{11h}(c) = \frac{1}{2}\int_a^b h_{ij}(c(t),\dot{c}(t))\frac{dc^i}{dt}\frac{dc^j}{dt}dt,\ \forall c \in \Omega_{a,b}(N).$$

In other words, the $\begin{pmatrix} & P & \\ 1 & 1 & h \end{pmatrix}$-harmonic curves are exactly the geodesics of the generalized Lagrange space $(N, h_{ij}(x,y))$ [4].

iii) If we take $N = R,\ h_{11} = 1$ and $P = (\delta^\alpha_\beta, P^1_{\beta 1})$, we obtain $C^\infty(M,N) = \mathcal{F}(M)$ and the energy functional becomes

$$E^P_{g\varphi 1}(f) = \frac{1}{2}\int_M g^{\alpha\beta}(a, grad_\varphi f) f_\alpha f_\beta \sqrt{\varphi} da,\ \forall f \in \mathcal{F}(M).$$

The extremals are *harmonic maps* between the Lagrange space $(M, g_{\alpha\beta}(a,b))$ and the classical Riemannian space $(R,1)$.

iv) In the particular cases when the metric tensors are of the form $g_{\alpha\beta}(a,b) = e^{-2\sigma(a,b)}\varphi_{\alpha\beta}(a)$ and $h_{ij}(x,y) = e^{2\tau(x,y)}\psi_{ij}(x)$, where $\sigma : TN \to R,\ \tau : TN \to R$ are smooth functions and ψ_{ij} is a pseudo-Riemannian metric on N, we have

$$\begin{cases} \dfrac{\partial L}{\partial x^i} = e^{2\sigma+2\tau}\varphi^{\gamma\mu}\varphi^{\delta\varepsilon}\psi_{kl}\left[\dfrac{\partial P^\nu_{\varepsilon p}}{\partial x^i}\dfrac{\partial\sigma}{\partial b^\nu} + \dfrac{\partial P^j_{\varepsilon p}}{\partial x^i}\dfrac{\partial\tau}{\partial y^j}\right] x^p_\delta x^k_\gamma x^l_\mu + \dfrac{1}{2}g^{\gamma\mu}\dfrac{\partial h_{kl}}{\partial x^i}x^k_\gamma x^l_\mu \\ \dfrac{\partial L}{\partial x^i_\alpha} = e^{2\sigma+2\tau}\left\{\varphi^{\gamma\mu}\varphi^{\alpha\varepsilon}\psi_{kl}\left[P^\nu_{\varepsilon i}\dfrac{\partial\sigma}{\partial b^\nu} + P^j_{\varepsilon i}\dfrac{\partial\tau}{\partial y^j}\right] x^k_\gamma x^l_\mu + \varphi^{\gamma\alpha}\psi_{ik}x^k_\gamma\right\}. \end{cases}$$

These expressions can be simplified if we consider the following more particular cases:

1) $\sigma = \sigma(a)$ and $P = (P^\alpha_{\beta i}, A_\beta(a)\delta^i_j)$, where $\{A_\beta\}$ are the components of a covector A on M. In this situation, we obtain

$$(*) \qquad \begin{cases} \dfrac{\partial L}{\partial x^i} = \dfrac{1}{2}g^{\gamma\mu}\dfrac{\partial h_{kl}}{\partial x^i}x^k_\gamma x^l_\mu \\ \dfrac{\partial L}{\partial x^i_\alpha} = e^{2\sigma+2\tau}\left\{\varphi^{\gamma\mu}\varphi^{\alpha\varepsilon}\psi_{kl}A_\varepsilon\dfrac{\partial\tau}{\partial y^i}x^k_\gamma x^l_\mu + \varphi^{\gamma\alpha}\psi_{ik}x^k_\gamma\right\}. \end{cases}$$

2) $\tau = \tau(x)$ and $P = (\delta^\alpha_\beta\xi_i(x), P^\alpha_{\beta i})$, where $\{\xi_i\}$ are the components of an 1-form ξ on N. Now, we find

$$(**) \qquad \begin{cases} \dfrac{\partial L}{\partial x^i} = e^{2\sigma+2\tau}\varphi^{\gamma\mu}\varphi^{\delta\varepsilon}\psi_{kl}\dfrac{\partial\xi_p}{\partial x^i}\dfrac{\partial\sigma}{\partial b^\varepsilon}x^p_\delta x^k_\gamma x^l_\mu + \dfrac{1}{2}g^{\gamma\mu}\dfrac{\partial h_{kl}}{\partial x^i}x^k_\gamma x^l_\mu \\ \dfrac{\partial L}{\partial x^i_\alpha} = e^{2\sigma+2\tau}\left\{\varphi^{\gamma\mu}\varphi^{\alpha\varepsilon}\psi_{kl}\dfrac{\partial\sigma}{\partial b^\varepsilon}\xi_i x^k_\gamma x^l_\mu + \varphi^{\gamma\alpha}\psi_{ik}x^k_\gamma\right\}. \end{cases}$$

2 Geometrical interpretation

By the preceding ideas, we shall offer some beautiful geometrical interpretations for the C^2 solutions of certain PDEs of order one.

We start with a smooth map $f \in C^\infty(M,N)$. This map induces the tensor field $\delta f \overset{\text{not}}{=} f^i_\alpha da^\alpha \otimes \left.\dfrac{\partial}{\partial y^i}\right|_{f(x)} \in \Gamma(T^*M \otimes f^{-1}(TN))$. On $M \times N$, let T be a tensor field of type $(1,1)$ with all components null excepting $(T^i_\alpha)_{\substack{i=\overline{1,n}\\ \alpha=\overline{1,m}}}$. These geometrical objects determine the system of PDEs,

$$(E) \qquad \delta f = T, \quad \frac{\partial f^i}{\partial a^\alpha} = T^i_\alpha(a,f).$$

If $(M,\varphi_{\alpha\beta})$ and (N,ψ_{ij}) are Riemannian manifolds, then we can build a scalar product on $\Gamma(T^*M\otimes f^{-1}(TN))$, namely $<T,S>=\varphi^{\alpha\beta}\psi_{ij}T^i_\alpha S^j_\beta$, where $T=T^i_\alpha da^\alpha\otimes\dfrac{\partial}{\partial y^i}$ and $S=S^j_\beta da^\beta\otimes\dfrac{\partial}{\partial y^j}$. Obviously, the Cauchy-Schwartz inequality

$$<T,S>\leq\|S\|^2\|T\|^2,\ \forall S,T\in\Gamma(T^*M\times f^{-1}(TN)),$$

is an equality iff there exists $\mathcal{K}\in\mathcal{F}(M)$ such that $T=\mathcal{K}S$.

In these conditions, we prove the following

Theorem. *If $(M,\varphi),(N,\psi)$ are Riemannian manifolds and the map $f\in C^\infty(M,N)$ is solution of the PDE system (E), then f is an extremal of the functional*

$$\mathcal{L}_T : C^\infty(M,N)\backslash\{\exists a\in M \textit{ such that } <\delta f,T>(a)=0\}\to R_+,$$

$$\mathcal{L}_T(f)=\frac{1}{2}\int_M\frac{\|\delta f\|^2\|T\|^2}{<\delta f,T>^2}\sqrt{\varphi}da=\frac{1}{2}\int_M\frac{\|T\|^2}{<\delta f,T>^2}\varphi^{\alpha\beta}\psi_{ij}f^i_\alpha f^j_\beta\sqrt{\varphi}da.$$

Proof. Let f be an arbitrary map from the definition domain of $\mathcal{L}_T$. Applying the preceding Cauchy-Schwarz inequality, we obtain

$$\mathcal{L}_T(f)=\frac{1}{2}\int_M\frac{\|\delta f\|^2\|T\|^2}{<\delta f,T>^2}\sqrt{\varphi}da\geq\frac{1}{2}\int_M\sqrt{\varphi}da=\frac{1}{2}Vol_\varphi(M).$$

Obviously, if f is solution of the system (E), it follows $\mathcal{L}_T(f)=\dfrac{1}{2}Vol_\varphi(M)$, that is, f is a global minimum point of the functional $\mathcal{L}_T$. In conclusion, the map f verifies the Euler-Lagrange equations of $\mathcal{L}_T$ □.

Generally, the global minimum points of the functional $\mathcal{L}_T$ are solutions of the PDE system $\delta f=\mathcal{K}T$, where $\mathcal{K}\in\mathcal{F}(M)$. They are not necessarily solutions of the initial PDE system (E).

Now, we remark that, in certain particular cases, the functional $\mathcal{L}_T$ becomes exactly a functional of type $\begin{pmatrix} & P & \\ g & \varphi & h\end{pmatrix}$-energy and, consequently, the Euler-Lagrange equations are equations of harmonic maps. This idea can be applied to the following important cases:

1. Orbits

Taking $M=([a,b],1)$ and $T=\xi\in\Gamma(c^{-1}(TN))$, the PDE system (E) reduces to the differential system of orbits

$$\frac{dc^i}{dt}=\xi^i(c(t)),\ c:[a,b]\to N,$$

and the functional $\mathcal{L}_\xi$ comes to

$$\mathcal{L}_\xi(c) = \frac{1}{2}\int_a^b \frac{\|\xi\|^2_\psi}{[\xi^b(\dot{c})]^2}\psi_{ij}\frac{dc^i}{dt}\frac{dc^j}{dt}dt,$$

where $\xi^b = \xi_i dx^i = \psi_{ij}\xi^j dx^i$. Hence the functional $\mathcal{L}_\xi$ is a $\begin{pmatrix} & P & \\ 1 & 1 & h \end{pmatrix}$-energy, where the Lagrange metric tensor

$$h_{ij}(x,y) = \frac{\|\xi\|^2_\psi}{[\xi^b(y)]^2}\psi_{ij}(x) = \psi_{ij}(x)\exp\left[2\ln\frac{\|\xi\|_\psi}{|\xi^b(y)|}\right]$$

is defined on $TN\backslash\{y|\xi^b(y) = 0\}$. This case was studied, in other way, by Udrişte [9]-[12].

Replacing $\sigma = 0$, $\tau(x,y) = \ln(\|\xi\|_\psi/|\xi^b(y)|)$, $\varphi_{11} = 1$ and $A_1 = 1$ in the equations $(*)$, we obtain $\frac{\partial L}{\partial c^i} = \frac{1}{2}\frac{h_{kl}}{\partial c^i}\dot{c}^k\dot{c}^l$ and $\frac{\partial L}{\partial \dot{c}^i} = e^{2\tau}\left\{\psi_{kl}\frac{\partial \tau}{\partial \dot{c}^i}\dot{c}^k\dot{c}^l + \psi_{ik}\dot{c}^k\right\}$, and $\frac{\partial L}{\partial c^i} + \frac{d}{dt}\frac{\partial L}{\partial \dot{c}^i} = 0$, $\forall i = \overline{1,n}$, are the equations of these harmonic curves.

2. Pfaff systems

If we put $N = (R,1)$ and $T = A \in \Lambda^1(T^*M)$, then the PDE system (E) becomes the Pfaff system

$$df = A,\ f \in \mathcal{F}(M)$$

and the functional $\mathcal{L}_T$ is

$$\mathcal{L}_A(f) = \frac{1}{2}\int_M \frac{\|A\|^2_\varphi}{[A(grad_\varphi f)]^2}\varphi^{\alpha\beta}f_\alpha f_\beta\sqrt{\varphi}da.$$

Consequently, the functional $\mathcal{L}_A$ is a $\begin{pmatrix} & P & \\ g & \varphi & 1 \end{pmatrix}$-energy, where the components $g_{\alpha\beta} : TM\backslash\{b|A(b) = 0\} \to R$ are defined by

$$g_{\alpha\beta}(a,b) = \frac{[A(b)]^2}{\|A\|^2_\varphi}\varphi_{\alpha\beta}(a) = \varphi_{\alpha\beta}(a)\exp\left[2\ln\frac{|A(b)|}{\|A\|_\varphi}\right].$$

In this case, replacing $\tau = 0$, $\sigma(a,b) = \ln(\|A\|_\varphi/|A(b)|)$, $\psi_{11} = h_{11} = 1$ and $\xi_1 = 1$ in $(**)$, we find the form of harmonic maps equations. These are the equations (H) of harmonic maps corresponding to $n = 1$, where

$$\frac{\partial L}{\partial f_\alpha} = e^{2\tau}\left\{\varphi^{\gamma\mu}\varphi^{\alpha\varepsilon}\frac{\partial\sigma}{\partial b^\varepsilon}f_\gamma f_\mu + \varphi^{\gamma\alpha}f_\gamma\right\},\ \frac{\partial L}{\partial f} = 0.$$

3. Pseudolinear functions

We suppose that $T^k_\beta(a,x) = \xi^k(x)A_\beta(a)$, where ξ^k is a vector field on N and A_β is an 1-form on M. In this case the functional $\mathcal{L}_T$ is expressed by

$$\mathcal{L}_T(f) = \frac{1}{2}\int_M \frac{\|\xi\|^2_\varphi\|A\|^2_\varphi}{[A(b)]}\varphi^{\alpha\beta}\psi_{ij}f^i_\alpha f^j_\beta\sqrt{\varphi}da =$$

$$= \frac{1}{2}\int_M g^{\alpha\beta}(a,b)h_{ij}(f(a))f^i_\alpha f^j_\beta \sqrt{\varphi}da,$$

where $P^\gamma_{i\beta}(x) = \delta^\gamma_\beta \xi_i(x)$, $b^\gamma = \varphi^{\alpha\beta} f^i_\alpha P^\gamma_{i\beta}$, $h_{ij}(x) = \|\xi\|^2_\psi \psi_{ij}(x)$ and the Lagrange metric tensor has the components $g_{\alpha\beta} : TM\backslash\{b|A(b) = 0\} \to R$,

$$g_{\alpha\beta}(a,b) = \frac{[A(b)]^2}{\|A\|^2_\varphi}\varphi_{\alpha\beta}(a) = \varphi_{\alpha\beta}(a)\exp\left[2\ln\frac{|A(b)|}{\|A\|_\varphi}\right].$$

It follows that the functional $\mathcal{L}_T$ becomes a $\begin{pmatrix} & P & \\ g & \varphi & h \end{pmatrix}$-energy.

The equations of harmonic maps can be derived, putting

$$\tau = \ln\|\xi\|_\psi,\ \sigma(a,b) = \ln(\|A\|_\varphi/|A(b)|),\ P^\gamma_{\beta i} = \delta^\gamma_\beta \xi^b_i(x),$$

in $(**)$, where $\xi^b_i = \psi_{ij}\xi^j$.

In the particular case when we take $M = (R^n, \varphi = \delta)$ and $N = (R, \psi = 1)$, supposing that $(grad\ f)(a) \neq 0$, $\forall a \in M$, the solutions of the above PDE system are the well known *pseudolinear functions*. These functions have the property that all hypersurfaces of constant level $M_{f(a)}$ are totally geodesic [7]. Consequently, the pseudolinear functions are examples of harmonic maps between the generalized Lagrange spaces $(M, g_{\alpha\beta}(a,b) = \delta_{\alpha\beta}\{[A(b)]^2/\|A\|^2\})$ and $(N, h(x) = \xi^2(x))$. For example, the function $f(a) = e^{<v,a>+w}$, where $v \in M$, $w \in R$, is solution for the above PDE system with $\xi(a) = 1$ and $A(f(a)) = f(a)v$.

4. Continuous groups of transformations

The fundamental PDE system of the group having the infinitesimal generators ξ_r is

$$\frac{\partial f^i}{\partial a^\alpha} = \sum_{r=1}^t \xi^i_r(f)A^r_\alpha(a), \quad i = \overline{1,n},\ \alpha = \overline{1,m},$$

where $\{\xi_r\}_{r=\overline{1,t}} \subset \mathcal{X}(N)$ are vector fields on N and $\{A^r\}_{r=\overline{1,t}} \subset \Lambda^1(M)$ is a family of covector fields on M. The geometrical interpretation of solutions via harmonic maps theory is still an open problem. It can be attacked in two ways:

- like in the preceding examples,
- using the Lagrangian

$$L(a^\alpha, f^i, f^i_\alpha) = \frac{1}{2}g^{\gamma\mu}(a)h_{kl}(x)(f^k_\gamma - \xi^i_r A^r_\gamma)(f^l_\mu - \xi^i_s A^s_\mu)$$

and the prolongation by differentiation [12].

3 Maxwell and Einstein equations

Finally, we remark that, in all above cases, the solutions of the PDE system $\delta f = T$ are harmonic maps between generalized Lagrange spaces of type $(M^n, e^{2\sigma(x,y)}\gamma_{ij}(x))$, where $\sigma : TM\backslash\{\text{Hyperplane}\} \to R$ is a smooth function. These spaces, endowed with the non-linear connection $N^i_j(x,y) = \Gamma^i_{jk}(x)y^k$, where $\Gamma^i_{jk}(x)$ are the Christoffel symbols for the Riemannian metric $\gamma_{ij}(x)$, verify a constructive axiomatic formulation of General Relativity due to Ehlers, Pirani and Schild [4]. Moreover, such spaces represent convenient relativistic models because they have the same conformal and projective properties as the Riemannian space (M, γ_{ij}).

Denoting by r^i_{jkl} the curvature tensor field of the metric γ_{ij}, by γ^{ij} the inverse tensor field of γ_{ij}, and $r_{ij} = r^k_{ijk}$, $r = \gamma^{ij} r_{ij}$, $\delta/\delta x^i = \partial/\partial x^i - N^j_i(\partial/\partial y^j)$, $\sigma_i = \delta\sigma/\delta x^i$, $\dot{\sigma}_i = \partial\sigma/\partial y^i$, we shall use the following notations

$$\sigma^H = \gamma^{kl}\sigma_k\sigma_l,\ \sigma_{ij} = \sigma_{i|j} + \sigma_i\sigma_j - \gamma_{ij}\sigma^H/2,\ \bar{\sigma} = \gamma^{ij}\sigma_{ij}$$
$$\sigma^V = \gamma^{ab}\dot{\sigma}_a\dot{\sigma}_b,\ \dot{\sigma}_{ab} = \dot{\sigma}_a|_b + \dot{\sigma}_a\dot{\sigma}_b - \gamma_{ab}\sigma^V/2,\ \dot{\sigma} = \gamma^{ab}\dot{\sigma}_{ab},$$

where $_{|i}$ (resp. $|_a$) represents the h- (resp. v-) covariant derivative induced by the non-linear connection N^i_j.

Developing the formalism presented in [4, 5], the folllowing Maxwell equations hold,

$$\begin{cases} F_{ij|k} + F_{jk|i} + F_{ki|} = \sum_{(ijk)} g_{ip} r^h_{qjk}\dot{\sigma}_h y^p y^q \\ F_{ij}|_k + F_{jk}|_i + F_{ki}|_j = -(f_{ij|k} + f_{jk|i} + f_{ki|}) \\ f_{ij}|_k + f_{jk}|_i + f_{ki}|_j = 0, \end{cases}$$

where the electromagnetic tensors F_{ij} and f_{ij} are

$$F_{ij} = (g_{ip}\sigma_j - g_{jp}\sigma_i)y^p,\ \ f_{ij} = (g_{ip}\dot{\sigma}_j - g_{jp}\dot{\sigma}_i)y^p.$$

Also, the Einstein equations will take the form

$$\begin{cases} r_{ij} - \frac{1}{2} r\gamma_{ij} + t_{ij} = \mathcal{K}T^H_{ij} \\ (2-n)(\dot{\sigma}_{ab} - \dot{\sigma}\gamma_{ab}) = \mathcal{K}T^V_{ab}, \end{cases}$$

where T^H_{ij} and T^V_{ab} are the h- and v- components of the energy momentum tensor field, $\mathcal{K}$ is the gravific constant and

$$t_{ij} = (n-2)(\gamma_{ij}\bar{\sigma} - \sigma_{ij}) + \gamma_{ij} r_{st} y^s \gamma^{tp}\dot{\sigma}_p + \dot{\sigma}_i r^a_{tja} y^t - \gamma_{is}\gamma^{ap}\dot{\sigma}_p r^s_{tja} y^t.$$

Remark. For the form of generalized Einstein-Yang-Mills equations in the Lagrange space $(M, e^{2\sigma(x,y)}\gamma_{ij}(x))$, see [1, 5].

Open problem. Is it possible to build a unique generalized Lagrange geometry naturally asociated to a given PDE system, in the large ?

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