# On the Chern-Type Problem in a Complex Projective Space

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**Abstract**

Chern pointed out that it is interesting to study the distribution of the values of the squared norm $\mid \alpha \mid_2 = h_2$ of the second fundamental form $\alpha$ of the Kähler manifold. The purpose of this paper is to investigate the Chern-type problem in a complex projective space.

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## 1 Introduction

Let $M$ be an $n$-dimensional submanifold of an $(n+p)$-dimensional complex space form $M^{n+p}(c)$ of constant holomorphic sectional curvature $c$. Chern pointed out that it is interesting to study the distribution of the values of the squared norm $\mid \alpha \mid_2 = h_2$ of the second fundamental form $\alpha$ of $M$. The first value is of course 0 in the case where $M$ is totally geodesic.

The purpose of this paper is to investigate the Chern-type problem in a complex space form. In this paper, the second fundamental form of complex submanifolds on a complex projective space $CP^{n+p}(c)$ is treated. We prove the following

**Theorem.** *Let $M$ be an $n(\geqq 3)$-dimensional complete complex submanifold of an $(n+p)$-dimensional complex projective space $CP^{n+p}(c)$, $p > 0$, of constant holomorphic sectional curvature $c$. If the squared norm $\mid \alpha \mid_2$ of the second fundamental form $\alpha$ on $M$ satisfies*

$$\mid \alpha \mid_2 \leqq \frac{c(n^2-4)}{12n(n^2-1)},$$

*then $M$ is totally geodesic.*

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## 2 Kähler manifolds

In this section, we shall consider $M$ an $n(\geqq 2)$-dimensional connected Kähler manifold. Then a local unitary frame field $\{E_j\} = \{E_1, \cdots, E_n\}$ on a neighborhood of $M$ can be chosen. This is a complex linear frame which is orthonormal with respect to the Kähler metric $g$ of $M$, that is, $g^c(E_j, \bar{E}_k) = \delta_{jk.}$ Its dual frame field $\{\omega_j\} = \{\omega_1, \cdots, \omega_n\}$ consists of complex valued 1-forms of $(1,0)$ on $M$ such that $\omega_j(E_k) = \delta_{jk}$ and $\omega_1, \cdots, \omega_n, \bar{\omega}_1, \cdots, \bar{\omega}_n$ are linearly independent. Thus the natural extension $g^c$ of the Kähler metric $g$ of $M$ can be expressed as $g^c = 2\sum_j \omega_j \otimes \bar{\omega}_{j.}$ Associated with the frame field $\{E_j\}$, there exist complex valued forms $\omega_{ik,}$ where the indices $i$ and $k$ run over the range $1, \cdots, n$. They are usually called *connection forms* on $M$ such that they satisfy the structure equations of $M$ :

$$d\omega_i + \sum_j \omega_{ij} \wedge \omega_j = 0, \quad \omega_{ij} + \bar{\omega}_{ji} = 0, \tag{2.1}$$

$$d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \tag{2.2}$$

$$\Omega_{ij} = \sum_{k,l} R_{\bar{i}jk\bar{l}} \omega_k \wedge \bar{\omega}_l, \tag{2.3}$$

where $\Omega = (\Omega_{ij})$ (*resp.* $R_{\bar{i}jk\bar{l}}$) denotes the curvature form (*resp.* the components of the Riemannian curvature tensor $R$) of $M$. The second formula of (2.1) means the skew-Hermitian symmetric of $\Omega_{ij,}$ which is equivalent to the symmetric condition

$$R_{\bar{i}jk\bar{l}} = \bar{R}_{\bar{j}il\bar{k}.}$$

Moreover, the first Bianchi identity implies the further symmetric relations

$$R_{\bar{i}jk\bar{l}} = R_{\bar{i}kj\bar{l}} = R_{\bar{l}kj\bar{i}} = R_{\bar{l}jk\bar{i}.} \tag{2.4}$$

Next, relative to the frame field chosen above, the Ricci tensor $S$ of $M$ can be expressed as follows

$$S = \sum_{i,j} (S_{i\bar{j}} \omega_i \otimes \bar{\omega}_j + S_{\bar{i}j} \bar{\omega}_i \otimes \omega_j), \tag{2.5}$$

where $S_{i\bar{j}} = \sum_k R_{\bar{k}ki\bar{j}} = S_{\bar{j}i} = \bar{S}_{\bar{i}j.}$ The scalar curvature $K$ of $M$ is also given by

$$K = 2\sum_j S_{j\bar{j}.} \tag{2.6}$$

Now, the components $R_{\bar{i}jk\bar{l}m}$ and $R_{\bar{i}jk\bar{l}\bar{m}}$ (*resp.* $S_{i\bar{j}k}$ and $S_{i\bar{j}\bar{k}}$) of the covariant derivative of the Riemannian curvature tensor $R$ (*resp.* the Ricci tensor $S$) are obtained by

$$\sum_m (R_{\bar{i}jk\bar{l}m}\omega_m + R_{\bar{i}jk\bar{l}\bar{m}}\bar{\omega}_m) = dR_{\bar{i}jk\bar{l}} -$$

$$-\sum_m (R_{\bar{m}jk\bar{l}}\bar{\omega}_{mi} + R_{\bar{i}mk\bar{l}}\omega_{mj} + R_{\bar{i}jm\bar{l}}\omega_{mk} + R_{\bar{i}jk\bar{m}}\bar{\omega}_{ml}),$$

$$\sum_k (S_{i\bar{j}k}\omega_k + S_{i\bar{j}\bar{k}}\bar{\omega}_k) = dS_{i\bar{j}} - \sum_k (S_{k\bar{j}}\omega_{ki} + S_{i\bar{k}}\bar{\omega}_{kj}).$$

The second Bianchi formula is given by

$$R_{\bar{i}jk\bar{l}m} = R_{\bar{i}jm\bar{l}k},$$

and hence we have

$$S_{i\bar{j}k} = S_{k\bar{j}i} = \sum_l R_{\bar{j}ik\bar{l}l}, \quad K_i = 2\sum_j S_{i\bar{j}j},$$

where $dK = \sum_j (K_j\omega_j + \bar{K}_j\bar{\omega}_j)$. The components $S_{i\bar{j}kl}$ and $S_{i\bar{j}k\bar{l}}$ of the covariant derivative of $S_{i\bar{j}k}$ are expressed by

$$\sum_l (S_{i\bar{j}kl}\omega_l + S_{i\bar{j}k\bar{l}}\bar{\omega}_l) = dS_{i\bar{j}k} - \sum_l (S_{l\bar{j}k}\omega_{li} + S_{i\bar{l}k}\bar{\omega}_{lj} + S_{i\bar{j}l}\omega_{lk}). \tag{2.7}$$

By the exterior differentiation of the definition of $S_{i\bar{j}k}$ and taking account of (2.7), the Ricci formula for the Ricci tensor $S$ is given by

$$S_{i\bar{j}k\bar{l}} - S_{i\bar{j}\bar{l}k} = \sum_m (R_{\bar{l}ki\bar{m}}S_{m\bar{j}} - R_{\bar{l}km\bar{j}}S_{i\bar{m}}).$$

The sectional curvature of the non-degenerate holomorphic plane $P$ spanned by $u$ and $Ju$ is called the *holomorphic sectional curvature,* which is denoted by $H(P) = H(u)$. The Kähler manifold $M$ is said to be *of constant holomorphic sectional curvature* if its holomorphic sectional curvature $H(P)$ is constant for all non-degenerate holomorphic planes $P$ and for all points of $M$. Then $M$ is called a *complex space form*, which is denoted by $M_s^n(c')$ provided that it is of constant holomorphic sectional curvature $c'$, of complex dimension $n$. The standard models of complex space forms are the following three kinds : the complex Euclidean space $C^n$, the complex projective space $CP^n(c')$ or the complex hyperbolic space $CH^n(c')$, according as $c' = 0$, $c' > 0$ or $c' < 0$. It is seen that they are only complete, simply connected and connected complex space forms of dimension $n$.

The Riemannian curvature tensor $R_{\bar{i}jk\bar{l}}$ of $M_s^n(c')$ is given by

$$R_{\bar{i}jk\bar{l}} = \frac{c'}{2}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}). \tag{2.8}$$

## 3 Complex submanifolds

Let $(M', g')$ be an $(n+p)$-dimensional connected Kähler manifold and let $M$ be an $n$-dimensional connected complex submanifold of $M'$. Then $M$ is the Kähler manifold endowed with the induced metric tensor $g$. We choose a local unitary frame field

$\{E_A\} = \{E_1, \cdots, E_{n+p}\}$ on a neighborhood of $M'$ in such a way that restricted to $M$, $E_1, \cdots, E_n$ are tangent to $M$ and the others are normal to $M$. Here and in the sequel, the following convention on the range of indices is used throughout this paper, unless otherwise stated :

$$A, B, \cdots = 1, \cdots, n, n+1, \cdots, n+p,$$
$$i, j, \cdots = 1, \cdots, n,$$
$$x, y, \cdots = n+1, \cdots, n+p.$$

With respect to the unitary frame field $\{E_A\}$, let $\{\omega_A\} = \{\omega_i, \omega_x\}$ be its dual frame field. Then the Kähler metric tensor $g'$ of $M'$ is given by $g' = 2\sum_A \omega_A \otimes \bar{\omega}_A$. The canonical forms $\omega_A$ and the connection forms $\omega_{AB}$ of the ambient space satisfy the structure equations

$$\begin{aligned} &d\omega_A + \sum_B \omega_{AB} \wedge \omega_B = 0, \quad \omega_{AB} + \bar{\omega}_{AB} = 0, \\ &d\omega_{AB} + \sum_C \omega_{AC} \wedge \omega_{CB} = \Omega'_{AB}, \\ &\Omega'_{AB} = \sum_{C,D} R'_{\bar{A}BC\bar{D}} \omega_C \wedge \bar{\omega}_D, \end{aligned} \tag{3.1}$$

where $\Omega' = (\Omega'_{AB})$ (*resp.* $R'_{\bar{A}BC\bar{D}}$) denotes the curvature form with respect to the unitary frame field $\{E_A\}$ (*resp.* the components of the Riemannian curvature tensor $R'$) of $M'$. Restricting these forms to the submanifold $M$, we have

$$\omega_x = 0, \tag{3.2}$$

and the induced Kähler metric tensor $g$ of $M$ is given by $g = 2\sum_j \omega_j \otimes \bar{\omega}_j$. Then $\{E_j\}$ is a local unitary frame field with respect to this metric and $\{\omega_j\}$ is a local dual frame field due to $\{E_j\}$, which consists of complex valued 1-forms of type (1.0) on $M$. Moreover, $\omega_1, \cdots, \omega_n, \bar{\omega}_1, \cdots, \bar{\omega}_n$ are linearly independent. It follows from (3.2) and Cartan's lemma that the exterior derivatives of (3.2) give rise to

$$\omega_{xi} = \sum_j h^x_{ij} \omega_j, \quad h^x_{ij} = h^x_{ji}. \tag{3.3}$$

The quadratic form $\sum_{i,j,x} h^x_{ij} \omega_i \otimes \omega_j \otimes E_x$ with values in the normal bundle is called the *second fundamental form* of the submanifold $M$. From the structure equations of $M'$, it follows that the structure equations for $M$ are similarly given by

$$\begin{aligned} &d\omega_i + \sum_j \omega_{ij} \wedge \omega_j = 0, \quad \omega_{ij} + \bar{\omega}_{ji} = 0, \\ &d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \\ &\Omega_{ij} = \sum_{k,l} R_{\bar{i}jk\bar{l}} \omega_k \wedge \bar{\omega}_l, \end{aligned} \tag{3.4}$$

where $\Omega = (\Omega_{ij})$ (*resp.* $R_{\bar{i}jk\bar{l}}$) denotes the curvature form with respect to the unitary frame field $\{E_A\}$ (*resp.* the component of the Riemannian curvature tensor $R$) of $M$.

Moreover, the following relationships are obtained :

$$(3.5)\qquad \begin{aligned} d\omega_{xy} + \sum_A \omega_{xA} \wedge \omega_{Ay} &= \Omega_{xy}, \\ \Omega_{xy} &= \sum_{k,l} R_{\bar{x}yk\bar{l}}\omega_k \wedge \bar{\omega}_l, \end{aligned}$$

where $\Omega_{xy}$ is called the *normal curvature form* of $M$. For the Riemannian curvature tensors $R$ and $R'$ of $M$ and $M'$, respectively, it follows from (3.1), (3.3) and (3.4) that we have the Gauss equation

$$(3.6)\qquad R_{\bar{i}jk\bar{l}} = R'_{\bar{i}jk\bar{l}} - \sum_x h^x_{jk}\bar{h}^x_{il},$$

and by means of (3.1), (3.3) and (3.5), we have

$$R_{\bar{x}yk\bar{l}} = R'_{\bar{x}yk\bar{l}} + \sum_j h^x_{kj}\bar{h}^y_{jl}.$$

Using (2.5), (2.6) and (3.6), the components of the Ricci tensor $S$ and the scalar curvature $K$ of $M$ are given by

$$(3.7)\qquad \begin{aligned} S_{i\bar{j}} &= \sum_k R'_{\bar{k}ki\bar{j}} - h^2_{i\bar{j}}, \\ K &= 2(\sum_{j,k} R'_{\bar{k}kj\bar{j}} - h_2), \end{aligned}$$

where $h^2_{i\bar{j}} = h^2_{\bar{j}i} = \sum_{k,x} h^x_{ik}\bar{h}^x_{kj}$ and $h_2 = \sum_j h^2_{j\bar{j}}$.

Now, the components $h^x_{ijk}$ and $h^x_{ij\bar{k}}$ of the covariant derivative of the second fundamental form of $M$ are given by

$$\sum_k (h^x_{ijk}\omega_k + h^x_{ij\bar{k}}\bar{\omega}_k) = dh^x_{ij} - \sum_k (h^x_{kj}\omega_{ki} + h^x_{ik}\omega_{kj}) + \sum_y h^y_{ij}\omega_{xy}.$$

Then, substituting $dh^x_{ij}$ into the exterior derivative of (3.3), we have

$$h^x_{ijk} = h^x_{jik} = h^x_{ikj}, \quad h^x_{ij\bar{k}} = -R'_{\bar{x}ij\bar{k}}.$$

Similarly the components $h^x_{ijkl}$ and $h^x_{ijk\bar{l}}$ of the covariant derivative of $h^x_{ijk}$ can be defined by

$$\sum_l (h^x_{ijkl}\omega_l + h^x_{ijk\bar{l}}\bar{\omega}_l) = dh^x_{ijk} - \sum_l (h^x_{ljk}\omega_{li} + h^x_{ilk}\omega_{lj} + h^x_{ijl}\omega_{lk}) + \sum_y h^y_{ijk}\omega_{xy},$$

and by the simple calculation the Ricci formula for the second fundamental form are given by

$$h^x_{ijkl} = h^x_{ijlk},$$
$$h^x_{ijk\bar{l}} - h^x_{ij\bar{l}k} = \sum_r (R_{\bar{l}ki\bar{r}} h^x_{rj} + R_{\bar{l}kj\bar{r}} h^x_{ir}) - \sum_y R_{\bar{x}yk\bar{l}} h^y_{ij}.$$

In particular, let the ambient space be an $(n+p)$-dimensional complex space form $M^{n+p}(c)$ of constant holomorphic sectional curvature $c$. Then, from (2.8), (3.6) and (3.7), we get

$$R_{\bar{i}jk\bar{l}} = \frac{c}{2}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) - \sum_x h^x_{jk}\bar{h}^x_{il}, \tag{3.8}$$

$$S_{i\bar{j}} = \frac{(n+1)c}{2}\delta_{ij} - h^2_{i\bar{j}}, \qquad h^x_{ij\bar{k}} = 0. \tag{3.9}$$

And hence from (3.8) we obtain

$$\begin{aligned} h^x_{ijk\bar{l}} &= \frac{c}{2}(h^x_{ij}\delta_{kl} + h^x_{jk}\delta_{il} + h^x_{ki}\delta_{jl}) \\ &- \sum_{r,y}(h^x_{ri}h^y_{jk} + h^x_{rj}h^y_{ki} + h^x_{rk}h^y_{ij})\bar{h}^y_{rl}. \end{aligned} \tag{3.10}$$

Let us denote by $h_4 = \sum_{i,j} h^2_{i\bar{j}} h^2_{j\bar{i}}$ and $A = (A^x_y)$, where $A^x_y = \sum_{i,j} h^x_{ij}\bar{h}^y_{ij}$. Then, by means of (3.10), the Laplacian $\triangle h_2$ of the function $h_2$ is given by

$$\triangle h_2 = \frac{c}{2}(n+2)h_2 - 2h_4 - TrA^2 + \sum_{i,j,k,x} h^x_{ijk}\bar{h}^x_{ijk}. \tag{3.11}$$

## 4 Totally real bisectional curvature

Let $(M, g)$ be an $n$-dimensional Kähler manifold with almost complex structure $J$. In this section, we consider the totally real bisectional curvature on $M$.

A plane section $P$ in the tangent space $T_xM$ of $M$ at any point $x$ in $M$ is said to be *totally real* if $P$ is orthogonal to $JP$. For the non-degenerate totally real plane $P$ spanned by orthonormal vectors $u$ and $v$, the *totally real bisectional curvature* $B(u, v)$ is defined by

$$B(u,v) = \frac{g(R(u,Ju)Jv,v)}{g(u,u)g(v,v)}. \tag{4.1}$$

For a complex submanifold, using the first Bianchi identity to (4.1) and the fundamental properties of the Riemannian curvature tensor of a complex submanifold, we get

$$B(u,v) = g(R(u,v)v,u) + g(R(u,Jv)Jv,u) = K(u,v) + K(u,Jv), \tag{4.2}$$

where $K(u,v)$ means the sectional curvature of the plane spanned by $u$ and $v$.
**Example 4.1.** Let $Q^n$ be a complex quadric in a complex projective space $CP^{n+1}(c)$ of constant holomorphic sectional curvature $c$. Then it is seen in Kobayashi and Nomizu [4] that its totally real bisectional curvature $B$ satisfies

$$0 \leqq B \leqq \frac{c}{2}.$$

From now on, we suppose that $u$ and $v$ are orthonormal vectors in the non-degenerate totally real plane $P$. If we put $u' = \frac{1}{\sqrt{2}}(u+v)$ and $v' = \frac{1}{\sqrt{2}}(u-v)$, then it is easily seen that

$$g(u',u') = 1, \quad g(v',v') = 1, \quad g(u',v') = 0.$$

Thus we get

$$B(u',v') = g(R(u',Ju')Jv',v') = \frac{1}{4}\{H(u) + H(v) + 2B(u,v) - 4K(u,Jv)\},$$

where $H(u) = K(u,Ju)$ means the holomorphic sectional curvature of the holomorphic plane spanned by $u$ and $Ju$. Hence we have

$$4B(u',v') - 2B(u,v) = H(u) + H(v) - 4K(u,Jv). \tag{4.3}$$

If we put $u'' = \frac{1}{\sqrt{2}}(u+Jv)$ and $v'' = \frac{1}{\sqrt{2}}(Ju+v)$, then we get

$$g(u'',u'') = 1, \quad g(v'',v'') = 1, \quad g(u'',v'') = 0.$$

Using the similar method as (4.3), we have

$$4B(u'',v'') - 2B(u,v) = H(u) + H(v) - 4K(u,v). \tag{4.4}$$

Summing up (4.3) and (4.4) and taking account of (4.2), we obtain

$$2B(u',v') + 2B(u'',v'') = H(u) + H(v). \tag{4.5}$$

Next, let $M$ be an $n(\geqq 3)$-dimensional complex submanifold of $M' = M^{n+p}(c)$ and let $b(M)$ or $a(M)$ be the supremum or the infimum of the set $B$ of the totally real bisectional curvatures on $M$. Suppose that the totally real bisectional curvature is bounded from above (*resp.* below) by a constant $b$ (*resp.* $a$). From the assumption and (4.5), it follows that we have

$$H(u) + H(v) \leqq 4b \ (resp. \ \geqq 4a). \tag{4.6}$$

We can choose an unitary frame field $\{E_1, E_2, \cdots, E_n\}$ on a neighborhood of $M$. Let $\{\omega_1, \omega_2, \cdots, \omega_n\}$ be a dual frame field. If we put $e_j = \frac{1}{\sqrt{2}}(E_j + \bar{E}_j)$, then $\{e_1, e_2, \cdots, e_n\}$ is an orthonormal basis of $T_xM$. Thus the holomorphic sectional curvature $H(e_j)$ of the holomorphic plane defined by $E_j$ is given by

$$H(e_j) = g(R(e_j, Je_j)Je_j, e_j) = R_{\bar{j}j j\bar{j}}.$$

On the other hand, it is easily seen that the plane spanned by $e_j$ and $e_k$ $(j \neq k)$ is totally real and the totally real bisectional curvature $B(e_j, e_k)$ is given by

$$B(e_j, e_k) = g(R(e_j, Je_j)Je_k, e_k) = R_{\bar{j}jk\bar{k}}, \quad j \neq k. \tag{4.7}$$

From the inequality (4.6) for $u = e_j$ and $v = e_k$, we find

$$R_{\bar{j}jj\bar{j}} + R_{\bar{k}kk\bar{k}} \leqq 4b \ (resp. \ \geqq 4a), \quad j \neq k. \tag{4.8}$$

Thus we have

$$\sum_{j<k}(R_{\bar{j}jj\bar{j}} + R_{\bar{k}kk\bar{k}}) \leqq 2bn(n-1) \ (resp. \ \geqq 2an(n-1)), \tag{4.9}$$

which implies that

$$\sum_j R_{\bar{j}jj\bar{j}} \leqq 2bn \ (resp. \ \geqq 2an), \tag{4.10}$$

where the equality holds if and only if $R_{\bar{j}jj\bar{j}} = 2b$ $(resp. \ = 2a)$ for any index $j$.

Since the scalar curvature $K$ is given by

$$K = 2\sum_{j,k} R_{\bar{j}jk\bar{k}} = 2(\sum_j R_{\bar{j}jj\bar{j}} + \sum_{j\neq k} R_{\bar{j}jk\bar{k}}),$$

we have by (4.7)

$$K \leqq 2\sum_j R_{\bar{j}jj\bar{j}} + 2bn(n-1) \ (resp. \ \geqq 2\sum_j R_{\bar{j}jj\bar{j}} + 2an(n-1)),$$

from which it follows that

$$\sum_j R_{\bar{j}jj\bar{j}} \geqq \frac{K}{2} - bn(n-1) \ (resp. \ \leqq \frac{K}{2} - an(n-1)), \tag{4.11}$$

where the equality holds if and only if $R_{\bar{j}jk\bar{k}} = b$ $(resp. \ = a)$ for any distinct indices $j$ and $k$. In this case, $M$ is locally congruent to $M^n(2b)$ $(resp. \ M^n(2a))$ due to Houh [1]. Also (4.8) gives us $\sum_{j\neq k}(R_{\bar{j}jj\bar{j}} + R_{\bar{k}kk\bar{k}}) \leqq 4b(n-1)$ $(resp. \ \geqq 4a(n-1))$, so that

$$(n-2)R_{\bar{j}jj\bar{j}} + \sum_k R_{\bar{k}kk\bar{k}} \leqq 4b(n-1) \ (resp. \ \geqq 4a(n-1)).$$

Combining this with (4.11), we have

$$\begin{aligned} &(n-2)R_{\bar{j}jj\bar{j}} \leqq b(n-1)(n+4) - \tfrac{K}{2} \\ &(resp. \geqq a(n-1)(n+4) - \tfrac{K}{2}), \end{aligned} \tag{4.12}$$

for any index $j$, so that the holomorphic sectional curvature $R_{\bar{j}jj\bar{j}}$ is bounded from above $(resp.$ below) for $n \geqq 3$. Moreover, the equality holds for some index $j$ if and only if $M$ is locally congruent to $M^n(2b)$ $(resp. \ M^n(2a))$.

Since the Ricci curvature $S_{j\bar{j}}$ is given by

$$S_{j\bar{j}} = R_{\bar{j}jj\bar{j}} + \sum_{k \neq j} R_{\bar{j}jk\bar{k}},$$

we have by (4.7)

$$S_{j\bar{j}} \leqq R_{\bar{j}jj\bar{j}} + b(n-1) \ (resp. \ \geqq R_{\bar{j}jj\bar{j}} + a(n-1))$$

and hence, from (4.12), we get

$$(4.13) \qquad \begin{aligned} & S_{j\bar{j}} \leqq \frac{1}{2(n-2)}\{4b(n-1)(n+1) - K\} \\ & (\text{resp.} \geqq \frac{1}{2(n-2)}\{4a(n-1)(n+1) - K\}). \end{aligned}$$

On the other hand, for the scalar curvature $K$, we see by (4.13)

$$\begin{aligned} K \quad & = \quad 2S_{j\bar{j}} + 2\sum_{k \neq j} S_{k\bar{k}} \\ & \leqq \quad 2S_{j\bar{j}} + \frac{1}{n-2}(n-1)\{4b(n-1)(n+1) - K\} \\ (\text{resp.} \quad & \geqq \quad 2S_{j\bar{j}} + \frac{1}{n-2}(n-1)\{4a(n-1)(n+1) - K\}), \end{aligned}$$

and hence we see

$$(4.14) \qquad \begin{aligned} & S_{j\bar{j}} \geqq \frac{1}{2(n-2)}\{(2n-3)K - 4b(n-1)^2(n+1)\} \\ & (resp. \leqq \frac{1}{2(n-2)}\{(2n-3)K - 4a(n-1)^2(n+1)\}). \end{aligned}$$

This together with (4.12) and $R_{\bar{j}ji\bar{i}} \leqq b$ implies

$$(4.15) \qquad \begin{aligned} & R_{\bar{j}jk\bar{k}} \geqq \frac{1}{n-2}\{(n-1)K - (2n^3 - 3n + 2)b\} \\ & (resp. \leqq \frac{1}{n-2}\{(n-1)K - (2n^3 - 3n + 2)a\} \end{aligned}$$

for any distinct indices $j$ and $k$.

## 5 Complex submanifolds of a complex projective space

Let $M' = CP^{n+p}(c)$ be an $(n+p)$-dimensional complex projective space and let $M$ be an $n(\geqq 3)$-dimensional complex submanifold of $CP^{n+p}(c)$. Then by (3.8), we have

$$R_{\bar{j}jk\bar{k}} = \frac{c}{2} - \sum_{x} h^x_{jk}\bar{h}^x_{jk} \leqq \frac{c}{2}, \quad j \neq k.$$

Thus we see that any totally real plane section satisfies $B(u,\ v) \leqq \frac{c}{2}$.

Now, let $a(M)$ be the infimum of the set $B$ of totally real bisectional curvatures of $M$. Though the set $B$ is bounded from above, we have no information on $a(M)$. In their paper [3], Ki and Suh proved the following theorem.

**Theorem 5.1.** *Let $M$ be an $n(\geqq 3)$-dimensional complete complex submanifold of an $(n+p)$-dimensional complex projective space $CP^{n+p}(c)$. If the totally real bisectional curvatures of $M$ are bounded from below, then there exists a constant $a_1 = \dfrac{c}{2n(n^2+2n+3)}(n^3+2n^2+2n-2)$ depending only upon $n$ and $c$ so that if $a(M) > a_1$, then $M$ is congruent to a complex projective space $CP^n(c)$.*

In the following theorem, the above estimate is improved

**Theorem 5.2.** *Let $M$ be an $n(\geqq 3)$-dimensional complete complex submanifold of a complex projective space $CP^{n+p}(c)$. If the totally real bisectional curvatures of $M$ are bounded from below, then there exists a constant $a_2(< a_1)$ such that depending only upon $n$ and $c$ so that if $a(M) > a_2$, then $M$ is congruent to a complex projective space $CP^n(c)$, where $a_1 > a_2$.*

**Proof.** Assume that the set $B$ is bounded from below by a constant $a$. Since the matrix $H = (h^2_{j\bar{k}})$ defined by $h^2_{j\bar{k}} = \sum_{x,r} h^x_{jr}\bar{h}^x_{rk}$ and the matrix $A = (A^x_y)$ defined by $A^x_y = \sum_{j,k} h^x_{jk}\bar{h}^y_{jk}$ are both positive semi-definite Hermitian ones whose all eigenvalues $\mu_j$ and $\mu_x$ are non-negative real valued functions on $M$. Thus we have

$$h_2^2 \geqq h_4 = \sum_j \mu_j^2, \quad h_2^2 \geqq TrA^2 = \sum_x \mu_x^2.$$

By (3.11), we have

$$\Delta h_2 \geqq \frac{c}{2}(n+2)h_2 - 2h_4 - TrA^2,$$

from which together with the above properties about eigenvalues, it follows that

$$\Delta h_2 \geqq \frac{c}{2}(n+2)h_2 - 3h_2^2.$$

A non-negative function $f$ is defined by $h_2$. Then the above inequality is reduced to

$$\Delta f \geqq -3f^2 + \frac{c}{2}(n+2)f. \tag{5.1}$$

On the other hand, since the totally real bisectional curvatures are bounded from below by a constant $a$, we get

$$R_{\bar{j}jk\bar{k}} \geqq a \text{ for any } j,\ k(j \neq k).$$

Hence, by (2.8), (3.7), (4.10) and (4.11), we have

$$2an \leqq \sum_j R_{\bar{j}jj\bar{j}} \leqq \frac{c}{2}n(n+1) - h_2 - an(n-1).$$

Thus we get

$$2h_2 \leqq (c-2a)n(n+1).$$

Therefore we have

$$f = \sum_j \mu_j \leqq \frac{1}{2}(c-2a)n(n+1), \quad \mu_j \geqq 0, \tag{5.2}$$

where the first equality holds if and only if $R_{\bar{j}jj\bar{j}} = 2a$ and $R_{\bar{j}jk\bar{k}} = a$ for any indices $j \neq k$. This means that each eigenvalue $\mu_j$ is bounded. On the other hand, since the Ricci curvature $S_{j\bar{j}}$ of $M$ is given by

$$S_{j\bar{j}} = \frac{c}{2}(n+1) - \mu_j,$$

it is also bounded. Applying the generalized maximum principle due to Omori [6] and Yau [10] to the bounded function $f$, we see that for any sequence $\epsilon_m$ of positive numbers which converges to 0 as $m$ tends to infinity, there exists a point sequence $p_m$ such that

$$|\nabla f(p_m)| < \epsilon_m, \quad \Delta f(p_m) < \epsilon_m, \quad \sup\ f - \epsilon_m < f(p_m).$$

Thus, we have

$$\lim_{m\to\infty} \Delta f(p_m) \leqq \lim_{m\to\infty} \epsilon_m = 0, \quad \lim_{m\to\infty} f(p_m) = \sup\ f. \tag{5.3}$$

By (5.1) and (5.3), we see

$$\sup\ f(\sup\ f - \frac{c}{6}(n+2)) \geqq 0,$$

which means

$$\sup\ f = 0 \quad \text{or} \quad \sup\ f \geqq \frac{c}{6}(n+2).$$

If $\sup\ f = 0$, then $f$ vanishes identically because $f$ is non-negative. Then $M$ is totally geodesic. Suppose that $M$ is not totally geodesic. So, $f$ satisfies $\sup\ f \geqq \frac{c}{6}(n+2)$. On the other hand, by (5.2), we have

$$\sup\ f \leqq \frac{1}{2}(c-2a)n(n+1).$$

Thus, we see that

$$a \leqq \frac{c}{6n(n+1)}(3n^2+2n-2).$$

We denote the right hand side of the above inequality by $a_2$, which is the constant depending on the dimension $n$ and $c$. In this case, we can easily prove that $a_1 > a_2$.

It completes the proof. □

About the value of the squared norm $h_2$ of the second fundamental form of $M$, we assert the following theorem.

**Theorem 5.3.** *Let $M$ be an $n(\geqq 3)$-dimensional complete complex submanifold of a complex projective space $CP^{n+p}(c)$. If the squared norm $h_2$ of the second fundamental form on $M$ satisfies*

$$h_2 \leqq \frac{c}{12n(n^2-1)}(n^2-4),$$

*then $M$ is totally geodesic.*

**Proof.** Suppose that $M$ is not totally geodesic. Then, by Theorem 5.2, there exists a constant $a_2 = \frac{c}{6n(n+1)}(3n^2+2n-2)$ so that the infimum $a(M)$ of the totally real bisectional curvatures of $M$ satisfies $a(M) \leqq a_2$. On the other hand, it is seen that we have

$$R_{\bar{j}jk\bar{k}} = \frac{c}{2} - \sum_x h^x_{jk}\bar{h}^x_{jk} \leqq \frac{c}{2}$$

for any distinct indices $j$ and $k$, and hence it turns out to be

$$b(M) \leqq \frac{c}{2},$$

where the equality holds if and only if $M$ is totally geodesic. Since $M$ is not totally geodesic, we have $b(M) < \frac{c}{2}$. By (2.8), (3.7) and (4.15), we see

$$R_{\bar{j}jk\bar{k}} \geqq \frac{1}{n-2}\{cn(n^2-1) - 2(n-1)h_2 - b(M)(2n^3-3n+2)\}.$$

By the definition of $a(M)$, we get

$$a(M) \geqq \frac{1}{n-2}\{cn(n^2-1) - 2(n-1)h_2 - b(M)(2n^3-3n+2)\},$$

from the fact that $b(M) < \frac{c}{2}$, it follows that we have

$$h_2 > \frac{1}{4(n-1)}(c - 2a(M))(n-2).$$

Since $a(M) \leqq a_2$, we get

$$h_2 > \frac{c}{12n(n^2-1)}(n^2-4).$$

It completes the proof. □

## References


[1] C. S. Houh, *On totally real bisectional curvatures,* Proc. Amer. Math. Soc., 56 (1976), 261-263.

[2] T. Ikawa, H. Nakagawa and A. Romero, *Product complex submanifolds of indefinite complex space forms,* Rocky Mountain J. Math., 18 (1988), 601-615.

[3] U-H, Ki and Y. J. Suh, *On semi-Kaehler manifolds whose totally real bisectional curvature is bounded from below,* J. Korean Math. Soc., 33 (1996), 1009-1038.

[4] S. Kobayashi and K. Nomizu, *Foundations of differential geometry, I and II,* Interscience Publ., 1963 and 1969.

[5] S. Montiel and A. Romero, *Complex Einstein hypersurfaces of indefinite complex space forms,* Math. Proc. Camb. Phil. Soc. 94 (1983), 495-508.

[6] H. Omori, *Isometric immersions of Riemannian manifolds,* J. Math. Soc. Japan, 19(1967), 205-211.

[7] B. O'Neill, *Semi-Riemannian geometry with application to relativity,* Academic Press, 1983.

[8] A. Romero, *Differential geometry of complex hypersurfaces in an indefinite complex space form,* Univ. de Granada, 1986.

[9] A. Romero, *On a certain class of complex Einstein hypersurfaces in indefinite complex space forms,* Math. Z., 192 (1986), 627-635.

[10] S. T. Yau, *Harmonic functions on complete Riemannian manifolds,* Comm. Pure and Appl. Math., 28 (1975), 201-228.

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