Ricci-Semisymmetric Hypersurfaces

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Dedicated to Prof.Dr. Constantin UDRIŞTE
on the occasion of his sixtieth birthday

Abstract

The set of all manifolds which are Ricci-semisymmetric and satisfy $R \cdot S = 0$ contains the set of manifolds which are semisymmetric and satisfy $R \cdot R = 0$ as a proper subset. However, considering only those manifolds (M^n, g) which can be immersed as a hypersurface of some ambient space, one might ask whether this can lead to nonsemisymmetric Ricci-semisymmetric hypersurfaces. In particular for Euclidean ambient spaces $\mathbf{E}^{n+1}$, this is commonly known as the Problem of P.J. Ryan, and has been an open question since 1972. We discuss a number of contributions to the study of the equivalence of semisymmetry and Ricci-semisymmetry for hypersurfaces.

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1 Introduction

A semi-Riemannian manifold (M^n, g), $n = \dim\ M \geq 3$, is called semisymmetric if

$$R \cdot R = 0\,, \tag{1}$$

holds on M. It is well known that the class of semisymmetric manifolds includes the set of locally symmetric manifolds ($\nabla R = 0$) as a proper subset. For precise definitions of the symbols used, we refer to Section 2.

A semi-Riemannian manifold (M^n, g), $n \geq 3$, is said to be Ricci-semisymmetric, if the following condition is satisfied

$$R \cdot S = 0\,. \tag{2}$$

Again, the class of Ricci-semisymmetric manifolds includes the set of Ricci-symmetric manifolds ($\nabla S = 0$) as a proper subset. It is clear that every semisymmetric manifold is Ricci-semisymmetric. The converse statement is however not true.

Although the conditions (1) and (2) do not coincide for manifolds in general, there has been a long standing question:

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Question 1.1 *Are the conditions $R\cdot R=0$ and $R\cdot S=0$ equivalent for hypersurfaces of Euclidean spaces?*

This question has been first raised by P.J. Ryan in 1972 (cfr. Problem P 808 of [15] and references therein), and has been an open problem ever since. Question 1.1 is commonly refered to as the Problem of P.J. Ryan. We discuss a number of results which contributed to the solution of the above mentioned question, and situate a selection of results concerning generalised problems which are closely related to the original question.

The present paper is organised as follows. In Section 3 we recall how a negative answer to Question 1.1 was obtained. Indeed, in [2] it has been established that the conditions of semisymmetry and Ricci-semisymmetry are not equivalent for hypersurfaces in Euclidean spaces by giving an example of a hypersurface M^5 of $\mathbf{E}^6$ which satisfies $R\cdot S=0$, but does not fulfill $R\cdot R=0$; this result will be the subject of Theorem 3.1. In [3] it has been shown that this example of [2] is not an isolated case, but belongs to an infinite family of which it is the simplest representative. By Theorem 3.2 we thus show how to construct for all dimensions $n\geq 5$ nontrivial hypersurfaces M^n of $\mathbf{E}^{n+1}$ for which $R\cdot S=0$ but $R\cdot R\neq 0$. In Section 4 we broaden the scope and enlarge the original question to the study of the equivalence of more general curvature conditions in more general ambient spaces; we list a number of results in this context. Finally, we also give a few explicit examples of solved equivalence problems in the ambient spaces $\mathbf{E}^{n+1}$ and S^{n+1}.

2 Preliminaries

Let (M^n,g) be an n-dimensional, $n\geq 3$, semi-Riemannian connected manifold of class C^∞. We denote by ∇, S and κ the Levi-Civita connection, the Ricci tensor and the scalar curvature of (M^n,g), respectively. We define on M^n the endomorphisms $\tilde{R}(X,Y)$, $X\wedge Y$ and $\tilde{\mathcal{C}}(X,Y)$ by

$$\begin{aligned}\tilde{R}(X,Y)Z &= [\nabla_X,\nabla_Y]Z-\nabla_{[X,Y]}Z\,,\\ (X\wedge Y)Z &= g(Y,Z)X-g(X,Z)Y\ ,\end{aligned}$$

$$\tilde{\mathcal{C}}(X,Y)=\tilde{\mathcal{R}}(X,Y)+\frac{1}{n-2}\left(\frac{\kappa}{n-1}X\wedge Y-(X\wedge\tilde{\mathcal{S}}Y+\tilde{\mathcal{S}}X\wedge Y)\right)\ ,$$

respectively, where $X,Y,Z\in\Xi(M^n)$, $\Xi(M^n)$ being the Lie algebra of vector fields on M^n, and the Ricci operator $\tilde{\mathcal{S}}$ of (M^n,g) is defined by $S(X,Y)=g(X,\tilde{\mathcal{S}}Y)$. The $(0,4)$-tensor G is defined by $G(X_1,\ldots,X_4)=g((X_1\wedge X_2)X_3,X_4)$. The Riemann curvature tensor R and the Weyl curvature tensor C of (M^n,g) are defined by

$$\begin{aligned}R(X_1,X_2,X_3,X_4) &= g(\tilde{R}(X_1,X_2)X_3,X_4)\,,\\ C(X_1,X_2,X_3,X_4) &= g(\tilde{\mathcal{C}}(X_1,X_2)X_3,X_4)\,,\end{aligned}$$

respectively. Further, for a symmetric $(0,2)$-tensor field D on M^n, we define the endomorphism $X\wedge_D Y$ of $\Xi(M^n)$ by

$$(X\wedge_D Y)Z=D(Z,Y)X-D(Z,X)Y\,,$$

where $X, Y, Z \in \Xi(M^n)$. Evidently, we have $X \wedge_g Y = X \wedge Y$. For a $(0,k)$-tensor field T on M^n, $k \geq 1$, and a symmetric $(0,2)$-tensor field D on M, we define the $(0,k+2)$-tensor fields $R \cdot T$ and $Q(D,T)$ by

$$\begin{aligned}(R \cdot T)(X_1, \ldots, X_k; X, Y) &= -T(\tilde{\mathcal{R}}(X,Y)X_1, X_2, \ldots, X_k) \\ &\quad - \cdots - T(X_1, \ldots, X_{k-1}, \tilde{\mathcal{R}}(X,Y)X_k)\ , \\ Q(D,T)(X_1, \ldots, X_k; X, Y) &= -T((X \wedge_D Y)X_1, X_2, \ldots, X_k) \\ &\quad - \cdots - T(X_1, \ldots, X_{k-1}, (X \wedge_D Y)X_k)\ .\end{aligned}$$

Curvature conditions involving tensors of the form $R \cdot T$ only, are called curvature conditions of semisymmetric type; examples are $R \cdot R = 0$, $R \cdot S = 0$, but also e.g. $C \cdot R = 0$, which is easily constructed following the same pattern. Curvature conditions involving tensors of both the forms $R \cdot T$ and $Q(D,T)$, are called curvature conditions of pseudosymmetric type. In the sequel we will touch upon some results for certain generalizations of the semisymmetric and Ricci-semisymmetric manifolds, namely the pseudosymmetric and Ricci-pseudosymmetric manifolds respectively.

A semi-Riemannian manifold M is said to be pseudosymmetric if at every point of M the following condition is satisfied

($*$) the tensors $R \cdot R$ and $Q(g,R)$ are linearly dependent.

This condition is equivalent with the existence of a real-valued function L_R, defined on the set $U_R = \{x \in M | R - \frac{\kappa}{n(n-1)} G \neq 0 \text{ at } x\}$, such that

$$R \cdot R = L_R Q(g,R) \tag{3}$$

holds on U_R. The class of pseudosymmetric manifolds contains the semisymmetric manifolds as a proper subset.

Manifolds satisfying the condition

$$R \cdot S = L_S Q(g,S)\,, \tag{4}$$

on the set $U_S = \{x \in M | S - \frac{\kappa}{n} g \neq 0 \text{ at } x\}$, with S the Ricci tensor, are called Ricci-pseudosymmetric. Manifolds satisfying (4) are equivalently characterized by the fact that at every point of M the following condition is satisfied

($**$) the tensors $R \cdot S$ and $Q(g,S)$ are linearly dependent.

Again, the class of Ricci-pseudosymmetric manifolds includes the set of Ricci-semisymmetric manifolds as a proper subset. It is clear that every pseudosymmetric manifold is Ricci-pseudosymmetric; the converse statement is however not true.

For a concise introduction to the geometrical motivation for the concept of pseudosymmetry, and a survey of properties with references to more detailed literature, see e.g. [8].

3 The problem of P.J. Ryan in Euclidean spaces

Whereas the conditions $R \cdot R = 0$ and $R \cdot S = 0$ are equivalent on any 3-dimensional manifold, for $n > 3$ we have the following results. It had been proved in [16] that (1) and (2) are equivalent for hypersurfaces which have positive scalar curvature in a Euclidean space $\mathbf{E}^{n+1}$, $n > 3$. In [14] this result was generalized to hypersurfaces of a Euclidean space $\mathbf{E}^{n+1}$, $n > 3$, which have nonnegative scalar curvature and also to hypersurfaces of constant scalar curvature. [14] also proves that (1) and (2) coincide for hypersurfaces of Riemannian space forms with nonzero constant sectional curvature. Further, in [13] it was proved that (1) and (2) are equivalent for hypersurfaces of a Euclidean space $\mathbf{E}^{n+1}$, $n > 3$, under the additional global condition of completeness. In [6], it has been shown that the conditions (1) and (2) are equivalent for hypersurfaces of the Euclidean space $\mathbf{E}^5$.

In [2] a negative answer to Question 1.1 was given for hypersurfaces of a Euclidean space $\mathbf{E}^{n+1}$, $n \geq 5$. Indeed, [2] gives an example of a hypersurface M^5 of $\mathbf{E}^6$ which satisfies $R \cdot S = 0$, but which is not semisymmetric. The existence of such a hypersurface M^5 of $\mathbf{E}^6$ which is Ricci-semisymmetric, but does not fulfill $R \cdot R = 0$, is recalled in Theorem 3.1 here below. This proves that the conditions $R \cdot R = 0$ and $R \cdot S = 0$ are not equivalent for hypersurfaces of Euclidean space in general, thus solving the Problem of P.J. Ryan.

W.r.t. a local orthonormal frame $\{e_i\}_{i=1}^n$ which diagonalises the shape operator $\mathcal{A}$, with principal curvatures $\lambda_i (i = 1, ...n)$, the only nonzero components of the Riemann-Christoffel curvature tensor R and the Ricci tensor S (which is diagonal) are

$$\begin{aligned} R_{ijji} &= \lambda_i \lambda_j \,, \qquad\qquad i \neq j \,, \quad 1 \leq i, j \leq n \,, \\ S_{ii} &= \lambda_i \left(\sum_{i \neq j} \lambda_j \right) . \end{aligned}$$

The set of equations for $R \cdot R = 0$ (1) amounts to:

$$\lambda_i \lambda_j \lambda_k (\lambda_i - \lambda_j) = 0 \,, \qquad i \neq j, j \neq k, k \neq i, \qquad 1 \leq i, j \leq n \,. \tag{5}$$

Analogously, the set of equations for $R \cdot S = 0$ (2) amounts to:

$$\lambda_i \lambda_j (\lambda_i - \lambda_j) \left(\sum_{k \neq i, k \neq j} \lambda_k \right) = 0 \,, \qquad i \neq j \,, \quad 1 \leq i, j, k \leq n \,. \tag{6}$$

We remark that a solution of (5) is indeed automatically a solution of (6). Theorem 3.1 shows that there exists a 5-dimensional hypersurface of $\mathbf{E}^6$, for which the principal curvatures are a solution of (6), but do not satisfy (5).

Theorem 3.1 *There exists an isometric immersion of a 5-dimensional manifold M^5 into $\boldsymbol{E}^6$ with a metric*

$$\begin{aligned} ds^2 &= e^{2x^1} \big((dx^1)^2 + \cos^2 \phi(x^2, x^3)(dx^2)^2 + \sin^2 \phi(x^2, x^3)(dx^3)^2 \\ &\qquad + \cos^2 \psi(x^4, x^5)(dx^4)^2 + \sin^2 \psi(x^4, x^5)(dx^5)^2 \big) \,, \end{aligned} \tag{7}$$

and principal curvatures $(0, b, b, -b, -b)$; where $b(x^1) = e^{-x^1}$, and ϕ and ψ are solutions of the equation

$$\frac{\partial^2 \zeta}{(\partial x^i)^2} - \frac{\partial^2 \zeta}{(\partial x^j)^2} = -\sin(2\zeta) \,, \tag{8}$$

for $(i, j) = (2, 3)$, and $(4, 5)$, respectively. M^5 satisfies $R \cdot S = 0$, but is not semisymmetric.

Before proceeding, we observe that in Euclidean spaces there do not exist Ricci-pseudosymmetric hypersurfaces which are not already pseudosymmetric or Ricci-pseudosymmetric. We organise this observation in the following

Proposition 3.1 *A nonpseudosymmetric Ricci-pseudosymmetric hypersurface M^n of a Euclidean space $\boldsymbol{E}^{n+1}$ $(n \geq 5)$ must necessarilly be Ricci-semisymmetric.*

Proof. We recall the fact that the Ricci-pseudosymmetric manifolds (4) include the pseudosymmetric manifolds (3) as a proper subset. A Ricci-pseudosymmetric manifold which is not pseudosymmetric is called properly Ricci-pseudosymmetric. Remark 3.1 of [4] indicates the form of the diagonalized shape operator for properly Ricci-pseudosymmetric hypersurfaces of Euclidean spaces:

$$(9)\ \mathcal{A} = O_p \oplus (1-r)\beta\, I_q \oplus -(1-q)\beta\, I_r \quad \text{with } \ p+q+r=n,\ p>0,\ q>1,\ r>1\,.$$

However, by inspection, one can verify that this set of principal curvatures actually satisfies the equations (6). Indeed, 6 possibilities have to be checked, corresponding to

$$(\lambda_i, \lambda_j) \quad = \quad \begin{array}{cc} (0,0)\,, & (0,-(1-q)\beta)\,, \\ (0,(1-r)\beta)\,, & ((1-r)\beta,-(1-q)\beta)\,, \\ ((1-r)\beta,(1-r)\beta)\,, & (-(1-q)\beta,-(1-q)\beta)\,. \end{array}$$

As soon as one of the principal curvatures is zero, or both principal curvatures are equal, (6) is fulfilled since either a factor λ_i or the factor $(\lambda_i - \lambda_j)$ vanishes. The only remaining situation to be verified is when $(\lambda_i, \lambda_j) = ((1-r)\beta, -(1-q)\beta)$; but in this case, a straightforward calculation shows that then the factor $\left(\sum_{k\neq i, k\neq j} \lambda_k\right)$ vanishes. Consequently, a hypersurface with diagonalized shape operator (9) would in fact be Ricci-semisymmetric. □

Corollary 3.1 *A Ricci-semisymmetric manifold which is not semisymmetric is called properly Ricci-semisymmetric. Hypersurfaces of Euclidean spaces with diagonalized shape operator of the form (9) are properly Ricci-semisymmetric.*

Proof. Indeed, in view of Proposition 3.1, hypersurfaces of Euclidean spaces with diagonalised shape operator of the form (9) are Ricci-semisymmetric. On the other hand, they cannot be semisymmetric, since otherwise the hypersurface would automatically be pseudosymmetric; this however contradicts the known fact (Remark 3.1 of [4]) that hypersurfaces with diagonalised shape operator of the form (9) are properly Ricci-pseudosymmetric and thus nonpseudosymmetric. Alternatively, one can also verify directly that the set of principal curvatures in (9) does not satisfy the equations (5); this indeed confirms that the hypersurface is not semisymmetric. □

Corollary 3.2 *A properly Ricci-semisymmetric hypersurface of a Euclidean space must necessarily have a diagonalized shape operator of the form (9).*

Proof. Therefore, we use the fact that every Ricci-semisymmetric manifold is also Ricci-pseudosymmetric, and consequently either properly Ricci-pseudosymmetric or pseudosymmetric. According to Proposition 3.1 a properly Ricci-pseudosymmetric hypersurface of $\mathbf{E}^{n+1}$ has a diagonalised shape operator of the form (9) and is Ricci-semisymmetric, and in view of Corollary 3.1 properly Ricci-semisymmetric. In all other cases, the Ricci-semisymmetric hypersurface M^n of $\mathbf{E}^{n+1}$ has to be pseudosymmetric and thus satisfies $R \cdot R = LQ(g, R)$. By contraction, it follows that $R \cdot S = LQ(g, S)$. Since M^n is Ricci-semisymmetric and thus satisfies $R \cdot S = 0$, we deduce that $LQ(g, S) = 0$. If at a point $L = 0$, then $R \cdot R = 0$ and the hypersurface is semisymmetric. If $L \neq 0$, then $Q(g, S)$ has to vanish. From $Q(g, S) = 0$, it follows that the Ricci tensor S has to be proportional to the metric tensor g. Following a result by A. Fialkow [12], the shape operator $\mathcal{A}$ of an Einstein hypersurface of $\mathbf{E}^{n+1}$ takes one of the following forms:

$$\mathcal{A} = aI_p \oplus O_q\,, \qquad p + q = n\,,$$

and is locally a hyperplane, a cylinder, or a hypersphere. In all these cases, the condition for semisymmetry is trivially satisfied. Summarizing, since all possible cases have been exhaustively considered, we see that a Ricci-semisymmetric hypersurface which is not semisymmetric must indeed have a diagonalised shape operator of the form (9); this finishes the proof of our statement. □

The result recalled in Theorem 3.1 was generalized in [3], where it was proven that Ricci-semisymmetric hypersurfaces M^n which are not semisymmetric exist in Euclidean spaces $\mathbf{E}^{n+1}$ for all dimensions $n \geq 5$. Indeed, according to Corollary 3.1, hypersurfaces with diagonalized shape operator given by (9) would provide examples of nonsemisymmetric Ricci-semisymmetric hypersurfaces of the Euclidean spaces, provided they exist. In [3] it was proven that nonsemisymmetric Ricci-semisymmetric hypersurfaces $M^n_{(p,q,r)}$ of $\mathbf{E}^{n+1}$ with diagonalized shape operator given by (9) do really exist in all dimensions $n \geq 5$ and for all possible choices of (p, q, r). The existence of the immersions of M^n in $\mathbf{E}^{n+1}$ for which $R \cdot S = 0$ but $R \cdot R \neq 0$, relies on the (complete) integrability of a system of partial differential equations of Bourlet type. In particular [3] thus show that the example of Theorem 3.1 is not an isolated case, but belongs to an infinite family of which it is the simplest representative. The construction for all dimensions $n \geq 5$ of nontrivial hypersurfaces M^n of $\mathbf{E}^{n+1}$ for which $R \cdot S = 0$ but $R \cdot R \neq 0$ relies on Theorem 3.2 here below which stems from [3]. The approach identifies links with the theory of completely integrable systems, and thus gives insight into the nonlinearity underlying the geometry.

Theorem 3.2 *There exists an isometric immersion of the n-dimensional manifold* $M^n_{(1,q,r)}$*, with* $q \geq 3$*,* $r \geq 3$*, and* $q + r + 1 = n$*, into* $\boldsymbol{E}^{n+1}$ *with the metric*

$$\begin{aligned} ds^2 &= e^{2hx^1}\Bigg((dx^1)^2 + B^2 \sum_{i=2}^{q+1} {l_i}^2(x^2, \cdots, x^{q+1})(dx^i)^2 \\ &\qquad + C^2 \sum_{i=n-r+1}^{n} {l_i}^2(x^{n-r+1}, \cdots, x^n)(dx^i)^2\Bigg)\,, \end{aligned} \tag{10}$$

and principal curvatures,

$$\lambda_1 = 0\,, \quad \lambda_i = (1-r)\beta\, e^{-hx^1}(2 \leq i \leq q+1)\,, \quad \lambda_i = -(1-q)\beta e^{-hx^1}(n-r+1 \leq i \leq n)\,.$$

The parameters h, β*, B, and C are related by the following conditions*

$$(1-q)(1-r)\beta^2 = h^2\,, \tag{11}$$

$$\left(h^2+(1-r)^2\beta^2\right)B^2 = 1\,, \tag{12}$$

$$\left(h^2+(1-q)^2\beta^2\right)C^2 = 1\,, \tag{13}$$

and the functions $\{l_i(x^{\alpha+1},\cdots,x^{\alpha+m})\}_{i=\alpha+1}^{\alpha+m}$ *are a solution of the completely integrable system*

$$\frac{\partial\gamma_{ij}}{\partial x^i}+\frac{\partial\gamma_{ji}}{\partial x^j}+\sum_{k\neq i,k\neq j}\gamma_{ki}\gamma_{kj}+l_il_j=0 \qquad (\alpha+1\leq i,j\leq\alpha+m)(i\neq j)\,, \tag{14}$$

$$\frac{\partial\gamma_{jk}}{\partial x^i}=\gamma_{ji}\gamma_{ik} \qquad (\alpha+1\leq i,j,k\leq\alpha+m)(i\neq j,j\neq k,k\neq i)\,, \tag{15}$$

with $\gamma_{ij}=\dfrac{1}{l_i}\dfrac{\partial l_j}{\partial x^i}$ $(\alpha+1\leq i,j\leq\alpha+m)(i\neq j)$ *and for* $(\alpha,m)=(1,q)$*, and* $(\alpha,m)=(q+1,r)$*, respectively.* $M^n_{(1,q,r)}$ *satisfies* $R\cdot S=0$*, but is not semisymmetric.*

It is now clear how to construct genuine Ricci-semisymmetric nonsemisymmetric hypersurfaces of all Euclidean spaces $\mathbf{E}^{n+1}$ $(n\geq 5)$ corresponding to all possible (p,q,r), thus with $p>0,q>1,r>1$ and $p+q+r=n$. First, when $p>1$, take a product immersion in $\mathbf{E}^{n+1}$ of $\mathbf{E}^{p-1}$ with a hypersurface $M^{n-p+1}_{(1,q,r)}$ of $\mathbf{E}^{n-p+2}$; if both $q\geq 3$ and $r\geq 3$, Theorem 3.2 proves the existence of this hypersurface $M^{n-p+1}_{(1,q,r)}$ of $\mathbf{E}^{n-p+2}$. When e.g. $q=2$, then i,j,k range over only 2 possible values, and consequently equations of the type (15) cannot occur. For the same reason (14) gives only 1 single equation. If we make the Ansatz

$$l_2(x^2,x^3)=\cos\phi(x^2,x^3)\,, \qquad \text{and} \qquad l_3(x^2,x^3)=\sin\phi(x^2,x^3)\,,$$

the remaining Gauss equation (14) turns into

$$\frac{\partial^2\phi}{(\partial x^2)^2}-\frac{\partial^2\phi}{(\partial x^3)^2}=-\sin(\phi)\,. \tag{16}$$

This is the sine Gordon equation and essentially (upon adjustment of the normalisation, which is conventional) the equation (8) which was encountered in Theorem 3.1.

4 Generalisations and further developments

The examples constructed in [2] (see Theorem 3.1) and in [3] (see Theorem 3.2) answer Question 1.1 and thus solve the problem of P.J. Ryan: nonsemisymmetric hypersurfaces of Euclidean spaces which satisfy $R\cdot S=0$ do exist. Although the fundamental question has now been solved, a number of new questions can be raised. Indeed, one may e.g. ask for still more examples of nonsemisymmetric Ricci-semisymmetric hypersurfaces of Euclidean spaces $\mathbf{E}^{n+1}$, or even for a classification of all Ricci-semisymmetric hypersurfaces of the Euclidean spaces which are not semisymmetric. More general than Question 1.1, one can therefore state the following:

Problem 4.1 *Study the equivalence of semisymmetry and Ricci-semisymmetry for hypersurfaces of Euclidean spaces* $\boldsymbol{E}^{n+1}$, $n\geq 5$.

In our analysis, we used the concept of pseudosymmetry and subsequent structural results for hypersurfaces, as for example formula (9), as a technical tool to isolate appropriate candidates for counterexamples. Perhaps (9) might also provide a starting point for the associated classification problem.

4.1 More general ambient spaces

Going beyond Problem 4.1, one can also consider more general ambient spaces, and for example state the following:

Problem 4.2 *Study the equivalence of semisymmetry and Ricci-semisymmetry for hypersurfaces of semi-Euclidean spaces* $\boldsymbol{E}_s^{n+1}$, $n \geq 4$.

One possibility to tackle such problems, and gain more insight is searching for sufficient conditions on hypersurfaces for both concepts (1) and (2) to be equivalent. We quote some results of this kind which thus contribute to the solution of Problem 4.2:

Theorem 4.1 *(1) and (2) are equivalent for Lorentzian hypersurfaces of a Minkowski space* $\boldsymbol{E}_1^{n+1}$*,* $n \geq 4$ *[9].*

Theorem 4.2 *(1) and (2) are equivalent for para-Kähler hypersurfaces of a semi-Euclidean space* $\boldsymbol{E}_s^{2m+1}$*,* $m \geq 2$ *[9].*

Another result along this line of thought is the following theorem from [10]. For hypersurfaces with pseudosymmetric Weyl tensor $C \cdot C = LQ(g, C)$ of a semi-Euclidean space $\mathbf{E}_s^{n+1}$, the conditions of $R \cdot R = 0$ and $R \cdot S = 0$ are equivalent. Finally, we also quote the following result which stems from [1].

Theorem 4.3 *For hypersurfaces of a semi-Euclidean space* $\boldsymbol{E}_s^{n+1}$*,* $n \geq 4$*, which satisfy the curvature condition* $C \cdot R = 0$*, the conditions of* $R \cdot R = 0$ *and* $R \cdot S = 0$ *are equivalent.*

One can still further generalize Problem 4.2 to even more general ambient spaces, and e.g. state the following:

Problem 4.3 *Study the equivalence of semisymmetry and Ricci-semisymmetry for hypersurfaces of semi-Riemannian spaces of constant sectional curvature* $\tilde{N}^{n+1}(c)$, $n \geq 4$.

In this respect, e.g. [14] proves that (1) and (2) coincide for hypersurfaces of Riemannian space forms with nonzero constant sectional curvature. Problem 4.3 on the equivalence of (1) and (2) was solved for the 4-dimensional case in [7]; more precisely:

Theorem 4.4 *For hypersurfaces of a semi-Riemannian space of constant sectional curvature* $\tilde{N}^5(c)$*, the conditions* $R \cdot R = 0$ *and* $R \cdot S = 0$ *are equivalent.*

This generalizes a result from [6] where the above was proven for hypersurfaces of a Euclidean space $\mathbf{E}^5$. We also quote the following result from [5] which generalizes Theorem 4.3.

Theorem 4.5 *For hypersurfaces of a semi-Riemannian space form* $\tilde{N}^{n+1}(c)$*,* $n \geq 4$*, which satisfy the curvature condition* $C \cdot R = 0$*, the conditions of semisymmetry and Ricci-semisymmetry are equivalent.*

4.2 More general curvature conditions

Analogously to P.J. Ryan's problem for the conditions (1) and (2), one could ask a similar question for pseudosymmetric and Ricci-pseudosymmetric hypersurfaces, respectively. Although the conditions (3) and (4) do not coincide for manifolds in general, one could state the problem whether or not the conditions $R \cdot R = L_R Q(g, R)$ and $R \cdot S = L_S Q(g, S)$ are equivalent for hypersurfaces of semi-Riemannian spaces of constant sectional curvature. But, it is known that this question has a negative answer in general by the existence of nonpseudosymmetric, Ricci-pseudosymmetric hypersurfaces of $S^{n+1}(c)$. Namely, in [11] it was shown that Cartan hypersurfaces of $S^{n+1}(c)$, $n = 6, 12, 24$, are such hypersurfaces. This however does not exclude that the conditions (3) and (4) may be equivalent for hypersurfaces in some special cases. As an example, we can e.g. recall Proposition 4.1 from [7].

Theorem 4.6 *Every Ricci-pseudosymmetric hypersurface M^4 of a 5-dimensional semi-Riemannian space of constant sectional curvature $N^5(c)$ is pseudosymmetric.*

Otherwise stated, this proves that the conditions (3) and (4) are equivalent for 4-dimensional hypersurfaces of a semi-Riemannian space of constant sectional curvature.

4.3 Some explicit examples

The inclusions among the 4 curvature conditions, semisymmetry (1), Ricci-semisymmetry (2), pseudosymmetry (3), and Ricci-pseudosymmetry (4), can be summarized in the following table.

$$\begin{array}{ccc} R \cdot S = L_S Q g, S & \supset & R \cdot R = L_R Q(g, R) \\ \cup & & \cup \\ R \cdot S = 0 & \supset & R \cdot R = 0 \end{array}$$

In general, all inclusions in the table are strict for manifolds M^n with $n \geq 4$. However, for hypersurfaces this picture can be refined.

Indeed, for hypersurfaces of Euclidean spaces, Proposition 3.1 shows that nonpseudosymmetric Ricci-pseudosymmetric hypersurfaces M^n of Euclidean spaces $\mathbf{E}^{n+1}$ ($n \geq 4$) must necessarily be Ricci-semisymmetric. Whence there follows that in this particular case Ricci-pseudosymmetric hypersurfaces which are neither Ricci-semisymmetric nor pseudosymmetric do not exist. However, [2] and [3] prove the existence of nonsemisymmetric Ricci-semisymmetric hypersurfaces M^n of the Euclidean spaces $\mathbf{E}^{n+1}$ ($n \geq 4$).

For hypersurfaces in other ambient spaces, the situation can look quite different. For example, for hypersurfaces of spheres S^{n+1} ($n \geq 4$), [14] shows that the conditions (1) and (2) coincide, and that consequentely there do not exist nonsemisymmetric Ricci-semisymmetric hypersurfaces. On the other hand, [11] shows that Cartan hypersufaces M^n of S^{n+1} ($n = 6, 12, 24$) satisfy $R \cdot S = L_S Q(g, S)$ with $L_S \neq 0$, but do not fulfil $R \cdot R = L_R Q(g, R)$. This proves the existence of Ricci-pseudosymmetric hypersurfaces M^n of spheres S^{n+1} ($n \geq 4$) which are neither pseudosymmetric nor Ricci-semisymmetric.

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