Einstein-Like and Conformally Flat Contact Metric Three-Manifolds

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Dedicated to Prof.Dr. Constantin UDRIŞTE
on the occasion of his sixtieth birthday

Abstract

The covariant derivative of the Ricci tensor of a three-dimensional contact metric manifold is computed and it is used to study Enstein-like and conformally flat contact metric three-manifolds . Several classification results are given.

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1 Introduction

As it is well-known, the curvature of a three-dimensional Riemannian manifold is completely determined by its Ricci tensor. For a three-dimensional contact metric manifold $(M, \eta, g, \varphi, \xi)$, D. Perrone [P1] expressed the Ricci tensor ϱ by means of terms related to the contact metric structure of M. Further, D. Perrone, L. Vanhecke and the author [CPV] computed the covariant derivatives of the vector fields forming a φ-basis of a non-Sasakian contact metric three-manifold M. So, this makes possible to compute the covariant derivative of ϱ, by means of its components with respect to a φ-basis, for an arbitrary non-Sasakian contact metric manifold of dimension three. In this paper, such computation is made and it is used in order to classify Einstein-like and conformally flat contact metric three-manifolds.

It is well-known that a three-dimensional Einstein manifold has constant sectional curvature. For contact metric manifolds, D.E. Blair and R. Sharma [BS] proved that locally symmetric (or equivalently, Ricci-parallel) contact metric three-manifolds must have constant sectional curvature 0 or +1.

Einstein-like manifolds have been introduced by A. Gray [G]. Einstein-like metrics are generalizations of both Einstein and Ricci-parallel metrics, so it is worthwhile to try to give a classification of contact metric three-manifolds equipped with an Einstein-like metric.

A Riemannian manifold (M, g) is said to belong to the class $\mathcal{A}$ if its Ricci tensor is *cyclic-parallel*, that is,

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$$(1.1)\qquad (\nabla_X \varrho)(Y,Z) + (\nabla_Y \varrho)(Z,X) + (\nabla_Z \varrho)(X,Y) = 0$$

for all X, Y, Z tangent to M and where ϱ is the Ricci tensor of type $(0,2)$. This condition is equivalent to the fact that ϱ is a Killing tensor, that is,

$$(1.2)\qquad (\nabla_X \varrho)(X,X) = 0.$$

(M,g) is said to be in the class $\mathcal{B}$ if its Ricci tensor is a *Codazzi tensor*, that is,

$$(1.3)\qquad (\nabla_X \varrho)(Y,Z) = (\nabla_Y \varrho)(X,Z).$$

Note that for connected spaces in class $\mathcal{A}$ or in class $\mathcal{B}$ the scalar curvature r is necessarily constant.

In dimension three, Einstein-like manifolds have been studied in the class of homogeneous spaces [AGV] and in some generalizations of such class, like ball-homogeneous spaces [CV] and curvature homogeneous spaces [BuV]. Moreover, Sasakian Einstein-like manifolds have been studied in [AG].

In this paper, we study contact metric three-manifolds belonging to the class $\mathcal{A}$ and we characterize them by proving the following

Theorem 1 *Let (M, η, g) be a three-dimensional, simply connected, complete contact metric manifold. Then the Ricci tensor of M is cyclic-parallel if and only if M is a naturally reductive homogeneous space. If the manifold is not simply connected or complete, then "naturally reductive" has to be replaced by "locally isometric to a naturally reductive space".*

We also give the explicit classification of contact metric three-manifolds in the class $\mathcal{A}$.

The case of a contact metric three-manifold whose Ricci tensor ϱ is a Codazzi tensor seems much more difficult to treat. Many authors ([BaKo], [BS], [Pa], [G-AX1]) studied contact metric three-manifolds having harmonic curvature tensor (a condition which is equivalent to (1.3)), classifying some special classes of such spaces. Using our method, we extend some of these classification results by proving the following

Theorem 2 *Let (M, η, g) be a three-dimensional contact metric manifold whose Ricci tensor is a Codazzi tensor. Suppose that*

$$(1.4)\qquad \nabla_\xi \tau = 2a\tau\varphi,$$

where $\tau = L_\xi g$ is the torsion of (M, η, g) and a is a smooth function which is constant along the geodesic foliation generated by ξ. Then M has constant sectional curvature 0 or $+1$.

As it was proved in [CP], (1.4) with a constant is a necessary but not sufficient condition for the homogeneity of a three-dimensional contact metric manifold.

Theorem 2 is also interesting because it does not hold any more if the scalar curvature of M is not constant, as for example for conformally flat contact metric three-manifolds. The classification of conformally flat contact metric manifolds is another problem which has been investigated by many authors. At one hand, in many cases conformally flat contact metric manifolds must have constant sectional curvature (see [T], [GA-X2], [CPV]). On the other hand, D. E. Blair [B2] constructed examples of conformally flat contact metric three-manifolds which do not have constant sectional curvature. We show that Blair's examples satisfy $\nabla_\xi \tau = 2a\tau\varphi$ with $\xi(a) = 0$. Moreover, we prove the following result, which generalizes Theorem 3.2 of [GA-X2].

Theorem 3 *A three-dimensional conformally flat contact metric manifold such that* $\nabla_\xi \tau = 2a\tau\varphi$, *where* $a \neq 2$ *is a constant, has constant sectional curvature* 0 *or* +1.

2 The covariant derivative of the Ricci tensor for a three-dimensional contact metric manifold

We first collect some basic facts about contact metric manifolds. All manifolds are supposed to be connected and smooth.

A *contact manifold* is a $(2n+1)$-dimensional manifold M equipped with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. It has an underlying almost contact structure (η, φ, ξ) where ξ is a global vector field (called the *characteristic vector field*) and φ a global tensor of type (1.1) such that

$$\eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta\varphi = 0, \quad \varphi^2 = -I + \eta \otimes \xi.$$

A Riemannian metric g can be found such that

$$\eta = g(\xi, \cdot), \quad d\eta = g(\cdot, \varphi\cdot), \quad g(\cdot, \varphi\cdot) = -g(\varphi\cdot, \cdot).$$

We refer to (M, η, g) or to $(M, \eta, g, \xi, \varphi)$ as a contact metric (or Riemannian) manifold.

If L denotes the Lie derivative, we consider the tensors

$$h = \frac{1}{2} L_\xi \varphi, \qquad \tau = L_\xi g.$$

The tensors h and τ are symmetric and satisfy

$$\nabla\xi = -\varphi - \varphi h, \quad \nabla_\xi \varphi = 0, \tag{2.1}$$

$$\tau = 2g(\varphi\cdot, h\cdot), \quad h\varphi = -\varphi h, \quad h\xi = 0 \tag{2.2}$$

and hence,

$$\nabla_\xi \tau = 2g(\varphi, \nabla_\xi h), \quad (\nabla_\xi h)\varphi = -\varphi(\nabla_\xi h), \quad (\nabla_\xi h)\xi = 0. \tag{2.3}$$

A *K-contact* manifold is a contact metric manifold such that ξ is a Killing vector field with respect to g. Clearly, M is K-contact if and only if $\tau = 0$ (or, equivalently, $h = 0$). If the almost complex structure J on $M \times \mathbb{R}$ defined by

$$J(X, f\frac{d}{dt}) = (\varphi X - f\xi, \eta(X)\frac{d}{dt})$$

is integrable, M is said to be *Sasakian*. Any Sasakian manifold is K-contact and the converse is also true for three-dimensional spaces. We refer to [B1] for more information about contact metric manifolds.

From now on, let (M,η,g) be a three-dimensional contact metric manifold and m a point of M. Let U be the open subset of M where $h \neq 0$ and V the open subset of points $m \in M$ such that $h=0$ in a neighbourhood of m. Then $U \cup V$ is an open dense subset of M. For any point $m \in U \cup V$ there exists a local orthonormal basis $\{\xi, e, \varphi e\}$ of smooth eigenvectors of h in a neighbourhood of m. We call $\{\xi, e, \varphi e\}$ a φ-*basis* of (M,η,g). On U we put $he=\lambda e$, where λ is a non-vanishing smooth function. From (2.1), we have $h\varphi e = -\lambda\varphi e$. We recall the following

Lemma 2.1 *[CPV] On U we have*

$$
(2.4) \qquad
\begin{aligned}
&\nabla_\xi e = -a\varphi e\,, && \nabla_\xi \varphi e = ae\,,\\
&\nabla_e \xi = -(\lambda+1)\varphi e\,, && \nabla_{\varphi e}\xi = -(\lambda-1)e\,,\\
&\nabla_e e = \tfrac{1}{2\lambda}\{(\varphi e)(\lambda)+A\}\varphi e\,, && \nabla_{\varphi e}\varphi e = \tfrac{1}{2\lambda}\{e(\lambda)+B\}e\,,\\
&\nabla_e \varphi e = -\tfrac{1}{2\lambda}\{(\varphi e)(\lambda)+A\}e + (\lambda+1)\xi\,, &&\\
&\nabla_{\varphi e} e = -\tfrac{1}{2\lambda}\{e(\lambda)+B\}\varphi e + (\lambda-1)\xi\,, &&
\end{aligned}
$$

$$
(2.5) \qquad \nabla_\xi h = 2ah\varphi + \xi(\lambda)s,
$$

where a is a smooth function, $A=\varrho(\xi,e)$, $B=\varrho(\xi,\varphi e)$ and s is the (1,1)-type tensor defined by $s\xi=0$, $se=e$ and $s\varphi e=-\varphi e$.

From (2.5) we obtain at once the following

Proposition 2.2 *Let (M,η,g) be a three-dimensional contact metric manifold. Then on M we have $\nabla_\xi h = 0$ (equivalently, $\nabla_\xi \tau = 0$) if and only if $a=\xi(\lambda)=0$, while $\nabla_\xi h = 2ah\varphi$ (equivalently, $\nabla_\xi\tau = 2a\tau\varphi$) if and only if $\xi(\lambda)=0$.*

So, the condition given in Theorem 2 for $\nabla_\xi\tau$ is weaker than "$\nabla_\xi\tau = 2a\tau\varphi$, with a constant", and this last condition is weaker than $\nabla_\xi\tau=0$.

In what follows, we shall denote by ∇ the Levi Civita connection of M and by R the corresponding Riemannian curvature tensor given by

$$R_{X,Y} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y].$$

The Ricci tensor of type (0,2), the corresponding operator and the scalar curvature are respectively denoted by ϱ, Q and r.

It was proved in [P1] that the Ricci operator on a three-dimensional contact metric manifold is given by

$$Q = \alpha I + \beta\eta\otimes\xi + \varphi\nabla_\xi h - \sigma(\varphi^2)\otimes\xi + \sigma(e)\eta\otimes e + \sigma(\varphi e)\eta\otimes\varphi e$$

where $\sigma = \varrho(\xi,\cdot)|_{ker\eta}$, $\alpha = \frac{r}{2} - 1 + \lambda^2$ and $\beta = -\frac{r}{2} + 3 - 3\lambda^2$. Using (2.5), we get

$$Q = \alpha I + \beta\eta\otimes\xi + 2ah + \xi(\lambda)\varphi s - \sigma(\varphi^2)\otimes\xi + \sigma(e)\eta\otimes e + \sigma(\varphi e)\eta\otimes\varphi e.$$

Hence, applying Q to the vector fiels of a φ-basis, we get

$$(2.6)\qquad \begin{cases} Q\xi = 2(1-\lambda^2)\xi + Ae + B\varphi e, \\ Qe = A\xi + (\frac{r}{2} - 1 + \lambda^2 + 2a\lambda)e + \xi(\lambda)\varphi e, \\ Q\varphi e = B\xi + \xi(\lambda)e + (\frac{r}{2} - 1 + \lambda^2 - 2a\lambda)\varphi e. \end{cases}$$

If $m \in U$, we can use (2.4) and (2.6) to compute the components of the covariant derivative $\nabla\varrho$ with respect to a φ-basis $\{\xi, e, \varphi e\}$ defined in a neighbourhood of m. After some standard computations, we get the following

Theorem 2.3 *Let* (M, η, g) *be a three-dimensional non-Sasakian contact metric manifold. On* U*, the components of* $\nabla\varrho$ *with respect to a* φ*-basis* $\{\xi, e, \varphi e\}$ *are given by*

$$(2.7)\qquad (\nabla_\xi Q)\xi = -4\lambda\xi(\lambda)\xi + \{\xi(A) + aB\}e + \{\xi(B) - aA\}\varphi e,$$

$$(2.8)\qquad (\nabla_\xi Q)e = \{\xi(A) + aB\}\xi + \{\xi(\alpha + 2a\lambda) + 2a\xi(\lambda)\}e + \\ +\{\xi(\xi(\lambda)) - 4a^2\lambda\}\varphi e,$$

$$(2.9)\qquad (\nabla_\xi Q)\varphi e = \{\xi(B) - aA\}\xi + \{\xi(\xi(\lambda)) - 4a^2\lambda\}e + \\ +\{\xi(\alpha - 2a\lambda) - 2a\xi(\lambda)\}\varphi e,$$

$$(2.10)\qquad (\nabla_e Q)\xi = \{-4\lambda e(\lambda) + 2(\lambda+1)B\}\xi + \\ +\{e(A) + (\lambda+1)\xi(\lambda) - \frac{B}{2\lambda}[(\varphi e)(\lambda) + A]\}e + \{e(B) + \\ +(\lambda+1)(\alpha - 2a\lambda - 2 + 2\lambda^2) + \frac{A}{2\lambda}[(\varphi e)(\lambda) + A]\}\varphi e,$$

$$(2.11)\qquad (\nabla_e Q)e = \{e(A) + (\lambda+1)\xi(\lambda) - \frac{B}{2\lambda}[(\varphi e)(\lambda) + A]\}\xi + \\ +\{e(\alpha + 2a\lambda) - \frac{\xi(\lambda)}{\lambda}[(\varphi e)(\lambda) + A]\}e + \\ +\{e\xi(\lambda) + 2a(\varphi e)(\lambda) + (2a - \lambda - 1)A\}\varphi e,$$

$$(2.12)\qquad (\nabla_e Q)\varphi e = \{e(B) + (\lambda+1)(\alpha - 2a\lambda - 2 + 2\lambda^2) + \\ +\frac{A}{2\lambda}[(\varphi e)(\lambda) + A]\}\xi + \{e\xi(\lambda) + 2a(\varphi e)(\lambda) + (2a - \lambda - 1)A\}e + \\ +\{e(\alpha - 2a\lambda) - 2(\lambda+1)B + \frac{\xi(\lambda)}{\lambda}[(\varphi e)(\lambda) + A]\}\varphi e,$$

$$(2.13)\qquad (\nabla_{\varphi e} Q)\xi = \{-4\lambda(\varphi e)(\lambda) + 2(\lambda-1)A\}\xi + \\ +\{(\varphi e)(A) + (\lambda-1)(\alpha + 2a\lambda - 2 + 2\lambda^2) + \frac{B}{2\lambda}[e(\lambda) + B]\}e + \\ +\{(\varphi e)(B) + (\lambda-1)\xi(\lambda) - \frac{A}{2\lambda}[e(\lambda) + B]\}\varphi e,$$

$$(2.14)\qquad (\nabla_{\varphi e} Q)e = \{(\varphi e)(A) + (\lambda-1)(\alpha + 2a\lambda - 2 + 2\lambda^2) +$$

$$+\frac{B}{2\lambda}[e(\lambda)+B]\}\xi+\{(\varphi e)(\alpha+2a\lambda)-2(\lambda-1)A+$$
$$+\frac{\xi(\lambda)}{\lambda}[e(\lambda)+B]\}e+\{(\varphi e)\xi\lambda-2ae(\lambda)+(1-\lambda-2a)B\}\varphi e,$$

$$(\nabla_{\varphi e}Q)\varphi e=\{(\varphi e)(B)+(\lambda-1)\xi(\lambda)-\frac{A}{2\lambda}[e(\lambda)+B]\}\xi+$$
$$+\{(\varphi e)\xi\lambda-2ae(\lambda)+(1-\lambda-2a)B\}e+$$
$$+\{(\varphi e)(\alpha-2a\lambda)-\frac{\xi(\lambda)}{\lambda}[e(\lambda)+B]\}\varphi e. \tag{2.15}$$

We recall that a contact metric manifold (M,η,g) is said to be *homogeneous* if there exists a connected Lie group of isometries acting transitively on M and leaving η invariant. It is said to be *locally homogeneous* if the pseudogroup of local isometries acts transitively on M and leaves η invariant. Note that a three-dimensional locally homogeneous contact metric manifold is locally isometric to a homogeneous one. The classification of homogeneous contact metric three-manifolds is given by the following

Proposition 2.4 *[**P2**] Let (M,η,g) be a simply connected homogeneous contact metric 3-manifold. Then M is a Lie group and (η,g) is a left invariant contact metric structure.*

The explicit classification of three-dimensional homogeneous contact metric manifolds is also given in [P2] in terms of the Webster scalar curvature and of the torsion invariant $|\tau|$.

We end this section by a lemma which will permit to simplify some of the next proofs. Such lemma can be easily proved by induction.

Lemma 2.5 *Let M be a smooth manifold, $m\in M$, X a vector tangent to M at m and f a real-valued smooth function defined in a neighbourhood of m. Suppose there exist $n\in\mathbb{N}$ and $a_0,...,a_n$ real-valued smooth functions defined in a neighbourhood of m, $a_n\neq 0$, such that $X(a_nf^n+...+a_1f+a_0)=0$ and $X(a_i)=0$ for all i. Then $X(f)=0$.*

3 The class $\mathcal{A}$ for contact metric three-manifolds

We start recalling the characterization of three-dimensional homogeneous manifolds in the class $\mathcal{A}$ given in [AGV].

Proposition 3.1 *[**AGV**] A three-dimensional, connected, simply connected Riemannian manifold is a naturally reductive space if and only if it is a homogeneous manifold with cyclic-parallel Ricci tensor.*

We are now ready to give the
Proof of Theorem 1

Let (M,η,g) be a three-dimensional, simply connected, complete contact metric manifold. If M is naturally reductive, then from Proposition 3.1 it follows at once that M belongs to the class $\mathcal{A}$.

Conversely, assume that the Ricci tensor ϱ of (M,η,g) is cyclic-parallel. If (M,η,g) is Sasakian, since the scalar curvature r is constant, it follows that M is locally φ-symmetric [W] and so, it is homogeneous [Tk]. Therefore, the result follows from Proposition 3.1. From now on, we assume that M is not Sasakian. Then, Theorem 2.3 holds. We now apply equations (1.1) and (1.2) to the components of $\nabla\varrho$ with respect to $\{e_1=\xi, e_2=e, e_3=\varphi e,\}$ on U. First, since $\nabla_1\varrho_{11}=0$, from (2.7) follows

$$\xi(\lambda)=0. \tag{3.1}$$

Next, taking into account (3.1) and the constancy of r, we have

(3.2) $\nabla_2\varrho_{22}=0 \Rightarrow e(\lambda^2+2a\lambda)=0,$

(3.3) $\nabla_3\varrho_{33}=0 \Rightarrow \varphi e(\lambda^2-2a\lambda)=0,$

(3.4) $\nabla_1\varrho_{12}+\nabla_1\varrho_{21}+\nabla_2\varrho_{11}=0 \Rightarrow \xi(A)-2\lambda e(\lambda)+(\lambda+a+1)B=0,$

(3.5) $\nabla_1\varrho_{13}+\nabla_1\varrho_{31}+\nabla_3\varrho_{11}=0 \Rightarrow \xi(B)-2\lambda\varphi e(\lambda)+(\lambda-a-1)A=0,$

(3.6) $\nabla_2\varrho_{21}+\nabla_2\varrho_{12}+\nabla_1\varrho_{22}=0 \Rightarrow e(A)-\dfrac{B}{2\lambda}[(\varphi e)(\lambda)+A]+\lambda\xi(a)=0,$

(3.7) $\nabla_3\varrho_{31}+\nabla_3\varrho_{13}+\nabla_1\varrho_{33}=0 \Rightarrow (\varphi e)(B)-\dfrac{A}{2\lambda}[e(\lambda)+B]-\lambda\xi(a)=0,$

(3.8) $\nabla_2\varrho_{23}+\nabla_2\varrho_{32}+\nabla_3\varrho_{22}=0 \Rightarrow 4a(\varphi e)(\lambda)+(\varphi e)(\lambda^2+2a\lambda)-4(\lambda-a)A=0,$

(3.9) $\nabla_3\varrho_{32}+\nabla_3\varrho_{23}+\nabla_2\varrho_{33}=0 \Rightarrow -4ae(\lambda)+e(\lambda^2-2a\lambda)-4(\lambda+a)B=0.$

Since $e(\lambda^2)=2\lambda e(\lambda)$, (3.2) becomes

$$\lambda e(a)=-(\lambda+a)e(\lambda). \tag{3.10}$$

In the same way, (3.3) gives

$$\lambda(\varphi e)(a)=(\lambda-a)(\varphi e)(\lambda). \tag{3.11}$$

Using (3.10) in (3.9), we have

$$(\lambda-a)e(\lambda)=(\lambda+a)B \tag{3.12}$$

and, using (3.11) in (3.8), we get

$$(\lambda+a)(\varphi e)(\lambda)=(\lambda-a)A. \tag{3.13}$$

Consider $U_1=\{m\in U : \lambda = a \textit{ in a neighbourhood of } m\}$, $U_2=\{m\in U : \lambda = -a \textit{ in a neighbourhood of } m\}$ and $U_3=\{m\in U : \lambda(m)\neq \pm a(m)\}$. It is easy to check that $U_1\cup U_2\cup U_3$ is an open dense subset of U. We now prove that $e(\lambda)=(\varphi e)(\lambda)=0$ on U_1, U_2 and U_3 and hence, on U. On U_1 and on U_2, (3.10) and (3.11) imply at once $e(\lambda)=(\varphi e)(\lambda)=0$. We now prove that $e(\lambda)=(\varphi e)(\lambda)=0$ also holds on U_3. Since $\lambda\neq\pm a$ on U_3, from (3.12) and (3.13) it follows

$$e(\lambda)=\frac{(\lambda+a)}{\lambda-a}B \tag{3.14}$$

and

$$(3.15)\qquad (\varphi e)(\lambda) = \frac{(\lambda - a)}{\lambda + a} A,$$

respectively. So, from (3.10) and (3.11) we also have

$$(3.16)\qquad e(a) = -\frac{(\lambda + a)^2}{\lambda(\lambda - a)} B$$

and

$$(3.17)\qquad (\varphi e)(a) = \frac{(\lambda - a)^2}{\lambda(\lambda + a)} A.$$

Next, (3.6), together with (3.15), gives

$$(3.18)\qquad e(A) = \frac{AB}{\lambda + a} - \lambda \xi(a).$$

In the same way, (3.7) and (3.14) imply

$$(3.19)\qquad (\varphi e)(B) = \frac{AB}{\lambda - a} + \lambda \xi(a).$$

We now differentiate (3.15) with respect to e and we use (3.14), (3.16) and (3.18) to express $e(\lambda)$, $e(a)$ and $e(A)$, respectively. We get

$$e(\varphi e)(\lambda) = \{\frac{2(\lambda + 2a)}{\lambda^2 - a^2} + \frac{\lambda - a}{(\lambda + a)^2}\} AB - \frac{\lambda(\lambda - a)}{\lambda + a} \xi(a).$$

In the same way, differentiating (3.14) with respect to φe and using (3.15), (3.17) and (3.19), we obtain

$$(\varphi e) e(\lambda) = \{\frac{2(\lambda - 2a)}{\lambda^2 - a^2} + \frac{\lambda + a}{(\lambda - a)^2}\} AB + \frac{\lambda(\lambda + a)}{\lambda - a} \xi(a).$$

Therefore, since $[e, \varphi e] = e(\varphi e) - (\varphi e)e$, we have

$$(3.20)\qquad [e, \varphi e](\lambda) = \{\frac{8a}{\lambda^2 - a^2} + \frac{\lambda - a}{(\lambda + a)^2} - \frac{\lambda + a}{(\lambda - a)^2}\} AB - \frac{2\lambda(\lambda^2 + a^2)}{\lambda^2 - a^2} \xi(a).$$

On the other hand, $[e, \varphi e] = \nabla_e(\varphi e) - \nabla_{\varphi e} e$. Thus, using (2.4) together with (3.14) and (3.15), we get

$$(3.21)\qquad [e, \varphi e](\lambda) = -\frac{2a}{\lambda^2 - a^2} AB.$$

Comparing (3.20) and (3.21), we then obtain

$$\frac{2a(\lambda^2 - 3a^2)}{\lambda^2 - a^2} AB - \lambda(\lambda^2 + a^2)\xi(a) = 0$$

and so,

$$\xi(a) = \frac{2a(\lambda^2 - 3a^2)}{\lambda(\lambda^4 - a^4)} AB. \tag{3.22}$$

In the same way, $[e, \varphi e](a) = e(\varphi e)(\lambda) - (\varphi e)e(\lambda)$ can be computed by differentiating (3.16) and (3.17) with respect to φe and e, respectively. On the other hand, $[e, \varphi e](a) = (\nabla_e \varphi e - \nabla_{\varphi e} e)(a)$ can be computed using (2.4). Hence, comparing the two expressions of $[e, \varphi e](a)$, we have

$$\frac{3\lambda^4 - 2\lambda^2 a^2 + 7a^4)}{\lambda(\lambda^2 - a^2)^2} AB + \{\frac{a(3\lambda^2 + a^2)}{\lambda^2 - a^2} - 1\}\xi(a) = 0. \tag{3.23}$$

Using (3.22), (3.23) becomes

$$(3\lambda^6 + 7\lambda^4 a^2 - 11\lambda^2 a^4 - 2\lambda^4 a + 8\lambda^2 a^3 - 6a^5)AB = 0. \tag{3.24}$$

We now prove that (3.24) implies $AB = 0$. In fact, if $AB \neq 0$, (3.24) gives

$$3\lambda^6 + 7\lambda^4 a^2 - 11\lambda^2 a^4 - 2\lambda^4 a + 8\lambda^2 a^3 - 6a^5 = 0. \tag{3.25}$$

Since $\xi(\lambda) = 0$, we can use Lemma 2.5 and (3.25) to conclude that $\xi(a) = 0$. Hence, (3.22) gives $a(\lambda^2 - 3a^2) = 0$. But if $a = 0$, then $e(a) = (\varphi e)(a) = 0$ and if $\lambda^2 = 3a^2$, using (3.10) and (3.11), we can conclude again that $e(a) = (\varphi e)(a) = 0$. So, in both cases (3.16) and (3.17) give $B = 0$ and $A = 0$, respectively, contrary to the assumption $AB \neq 0$. Then, $AB = 0$ and (3.22) gives $\xi(a) = 0$. When $A = 0$, (3.15) and (3.17) imply $(\varphi e)(\lambda) = (\varphi e)(a) = 0$. Since $\xi(\lambda) = 0$, we also get $[\xi, \varphi e](\lambda) = 0$. On the other hand, from (2.4) we obtain

$$[\xi, \varphi e](\lambda) = (\lambda + a - 1)e(\lambda)$$

and so, $(\lambda + a - 1)e(\lambda) = 0$, from which it follows easily $e(\lambda) = 0$. In fact, if $e(\lambda) \neq 0$, then $\lambda + a - 1 = 0$ and so, differentiating with respect to e, $e(\lambda) + e(a) = 0$. This, together with (3.10), gives $a = 0$, since $e(\lambda) \neq 0$. In particular, $e(a) = 0$ and so, (3.10) gives $e(\lambda) = 0$, contrary to the assumption. So, $e(\lambda) = 0$ and, from (3.14) and (3.16), we also get $B = 0$ and $e(a) = 0$. In the same way, it is possible to show that when $B = 0$ we have $e(\lambda) = e(a) = 0$ and also $(\varphi e)(\lambda) = A = (\varphi e)(a) = 0$. Thus, we proved that λ and a are constant on U_3 and hence, on U. Since λ and a are continuous and M is connected, we can conclude that λ and a are globally constant on M.

So, using (2.4) we obtain

$$[e, \varphi e] = c_1 \xi, \quad [\varphi e, \xi] = c_2 e, \quad [\xi, e] = c_3 \varphi e$$

where $c_1 = 2$, $c_2 = 1 - \lambda - a$ and $c_3 = \lambda + 1 - a$ are constant. From this we may conclude that (M, η, g) is isometric to a unimodular Lie group with a left-invariant contact metric structure. In particular, M is homogeneous and so, Proposition 3.1 implies that M is a naturally reductive space □

Theorem 3.4 of [CPV] gives the classification of contact metric three-manifolds with cyclic-parallel Ricci tensor, satisfying $\sigma = \varrho(\xi, \cdot)_{|Ker\eta} = 0$. As it follows from the proof of Theorem 1, on a contact metric three-manifold (M, η, g) with cyclic-parallel Ricci tensor we always have $A = B = 0$, that is, $\sigma = 0$. Therefore, the following classification holds.

Theorem 3.2 *Let (M,η,g) be a three-dimensional contact metric manifold. Then ϱ is cyclic-parallel if and only if the manifold is locally isometric to a unimodular Lie group G equipped with a left-invariant metric structure and satisfying one of the following conditions, given in terms of τ and the Webster scalar curvature W:*

(i) $\tau = 0$ *(that is, the structure is Sasakian). Then G is the Heisenberg group H_3 if $W=0$, $SU(2)$ if $W>0$ or $\widetilde{SL}(2,\mathbb{R})$ if $W<0$;*

(ii) $\tau \neq 0$ *and* $W = 1 + \frac{\|\tau\|}{4\sqrt{2}}$. *In this case $G = SU(2)$;*

(iii) $\tau \neq 0$ *and* $W = 1 - \frac{\|\tau\|}{4\sqrt{2}}$. *Then G is $SU(2)$ if $\|\tau\| < 2\sqrt{2}$, $\widetilde{SL}(2,\mathbb{R})$ if $\|\tau\| > 2\sqrt{2}$ or $\tilde{E}(2)$ if $\|\tau\| = 2\sqrt{2}$. (In this last case, g is a flat metric.)*

4 The class $\mathcal{B}$ for contact metric three-manifolds

The curvature of a Riemannian manifold (M,g) is said to be *harmonic* if the divergence of its curvature tensor is zero. A Riemannian manifold has harmonic curvature if and only if its Ricci tensor satisfies

$$(\nabla_X Q)Y = (\nabla_Y Q)X, \tag{4.1}$$

for all tangent vector fields X and Y. Clearly, (4.1) is equivalent to (1.3), that is, a Riemannian manifold has harmonic curvature if and only if it belongs to the class $\mathcal{B}$.

Several authors studied contact metric manifolds with harmonic curvature tensor. In [Pa], generalizing a previous result of [BaKo], the classification of $(2n+1)$-dimensional contact metric manifolds with harmonic curvature tensors and such that ξ belongs to the (k,μ)-nullity distribution has been given. The three-dimensional case is particularly interesting because D. E. Blair and R. Sharma [BS] proved that a locally symmetric contact metric three-manifold is either flat or a Sasakian manifold of constant sectional curvature $+1$. This result was extended in [G-AX1] to the broader class of contact metric three-manifolds with harmonic curvature, assuming $\nabla_\xi \tau = 0$.

A three-dimensional Riemannian manifold (M,g) is conformally flat if and only if its Ricci tensor satisfies

$$\nabla_i \varrho_{jk} - \nabla_k \varrho_{ij} = \frac{1}{4}(\delta_{jk}\nabla_i r - \delta_{ij}\nabla_k r), \tag{4.2}$$

where $\{e_i\}$ is a local orthonormal frame on M. So, (M,g) has harmonic curvature if and only if it is conformally flat and has constant scalar curvature. With respect to a local φ-basis, taking into account Proposition 2.2 and applying (4.2) to (2.7)-(2.15), we can easily prove the following Lemma, which will be used in the proof of Theorems 2 and 3.

Lemma 4.1 *Let (M,η,g) be a conformally flat non-Sasakian contact metric three-manifold and $\{e_1=\xi, e_2=e, e_3=\varphi e\}$ a local φ-basis. If (1.4) holds with a constant along the geodesic foliation generated by ξ, then*

$$\xi(A) = -4\lambda e(\lambda) + (2\lambda - a + 2)B - \frac{1}{4}e(r), \tag{4.3}$$

$$\xi(B) = -4\lambda(\varphi e)(\lambda) + (2\lambda + a - 2)A - \frac{1}{4}(\varphi e)(r), \tag{4.4}$$

$$e(A) = \frac{B}{2\lambda}[(\varphi e)(\lambda) + A] + \frac{1}{4}\xi(r), \tag{4.5}$$

$$(\varphi e)(B) = \frac{A}{2\lambda}[e(\lambda) + B] + \frac{1}{4}\xi(r), \tag{4.6}$$

$$(\varphi e)(A) = -4a^2\lambda - \frac{B}{2\lambda}[e(\lambda) + B] - (\lambda - 1)(\frac{r}{2} + 2a\lambda - 3 + 3\lambda^2), \tag{4.7}$$

$$e(B) = -4a^2\lambda - \frac{A}{2\lambda}[(\varphi e)(\lambda) + A] - (\lambda + 1)(\frac{r}{2} - 2a\lambda - 3 + 3\lambda^2), \tag{4.8}$$

$$2\lambda e(\lambda - a) = (\lambda - 2a + 3)B - \frac{1}{4}e(r), \tag{4.9}$$

$$2\lambda(\varphi e)(\lambda + a) = (\lambda + 2a - 3)A - \frac{1}{4}(\varphi e)(r). \tag{4.10}$$

Before proving Theorem 2, we recall the following characterization of three-dimensional homogeneous manifolds belonging to class $\mathcal{B}$, given in [AGV].

Proposition 4.2 *[AGV] A three-dimensional, connected, simply connected Riemannian manifold is a symmetric space if and only if it is a homogeneous space whose Ricci tensor is a Codazzi tensor.*

We are now ready to give the

Proof of Theorem 2

Let (M, η, g) be a three-dimensional contact metric manifold whose Ricci tensor satisfies (1.3). Suppose first that (M, η, g) is Sasakian. The scalar curvature r being constant, M is locally φ-symmetric [W] and so, it is homogeneous [Tk]. Therefore, Proposition 4.2 implies that M is locally symmetric, from which it follows that M has constant sectional curvature 0 or +1, as proved in [BS].

From now on, we assume that M is non-Sasakian. Then, (4.3)-(4.10) hold with $\xi(r) = e(r) = (\varphi e)(r) = 0$. We compute $\xi e(\lambda - a)$ differentiating (4.9) with respect to ξ. On the other hand, Proposition 2.2 gives $\xi(\lambda) = \xi(a) = 0$ ans hence, $\xi(\lambda - a) = 0$. So, $\xi e(\lambda - a) = [\xi, e](\lambda - a)$, which can be computed using (2.4). Comparing the two expressions of $\xi e(\lambda - a)$, we get

$$(\lambda - a + 1)(\varphi e)(\lambda - a) = \frac{\lambda - 2a + 3}{2\lambda}\{-4\lambda(\varphi e)(\lambda) + (2\lambda + a - 2)A\}. \tag{4.11}$$

From (4.10) it follows

$$(\varphi e)(a) = \frac{\lambda + 2a - 3}{2\lambda}A - (\varphi e)(\lambda). \tag{4.12}$$

Using (4.12), (4.11) becomes

$$4\lambda(2\lambda - 3a + 4)(\varphi e)(\lambda) = \{3\lambda^2 - 2(a - 1)\lambda - (2a - 3)^2\}A. \tag{4.13}$$

We differentiate (4.13) with respect to ξ and we get

(4.14) $$4\lambda(2\lambda - 3a + 4)\xi(\varphi e)(\lambda) = \{3\lambda^2 - 2(a-1)\lambda - (2a-3)^2\}\xi(A).$$

But $\xi(\varphi e)(\lambda) = [\xi, \varphi e](\lambda)$ can be computed using (2.4), while $\xi(A)$ is given by (4.3). From (4.14) it then follows

$$\begin{aligned}&4\lambda\{5\lambda^2 - (3a-4)\lambda - (a-1)(3a-4) - (2a-3)^2\}e(\lambda) = \\ &(2\lambda - a + 2)\{3\lambda^2 - 2(a-1)\lambda - (2a-3)^2\}B.\end{aligned}$$

On the other hand, differentiating (4.10) with respect to ξ and taking into account $\xi(\varphi e)(\lambda + a) = [\xi, \varphi e](\lambda + a)$ and (4.3), we get

$$2\lambda(\lambda + a - 1)e(\lambda + a) = (\lambda + 2a - 3)\{-4\lambda e(\lambda) + (2\lambda - a + 2)B\}$$

from which, using (4.9) to compute $e(a)$, it follows

(4.15) $$4\lambda(2\lambda + 3a - 4)e(\lambda) = \{3\lambda^2 + 2(a-1)\lambda - (2a-3)^2\}B.$$

As we did for (4.13), we differentiate (4.15) with respect to ξ and use $\xi(e)(\lambda) = [\xi, e](\lambda)$ and (4.4), to obtain

$$\begin{aligned}&4\lambda\{5\lambda^2 + (3a-4)\lambda - (a-1)(3a-4) - (2a-3)^2\}(\varphi e)(\lambda) = \\ &(2\lambda + a - 2)\{3\lambda^2 + 2(a-1)\lambda - (2a-3)^2\}A.\end{aligned}$$

Thus, $e(\lambda)$ and B satisfy

(4.16) $$\begin{cases} 4\lambda(2\lambda + 3a - 4)e(\lambda) = \{3\lambda^2 + 2(a-1)\lambda - (2a-3)^2\}B, \\ \\ 4\lambda\{5\lambda^2 - (3a-4)\lambda - (a-1)(3a-4) - (2a-3)^2\}e(\lambda) = \\ (2\lambda - a + 2)\{3\lambda^2 - 2(a-1)\lambda - (2a-3)^2\}B, \end{cases}$$

while $(\varphi e)(\lambda)$ and A satisfy

(4.17) $$\begin{cases} 4\lambda(2\lambda - 3a + 4)(\varphi e)(\lambda) = \{3\lambda^2 - 2(a-1)\lambda - (2a-3)^2\}A, \\ \\ 4\lambda\{5\lambda^2 + (3a-4)\lambda - (a-1)(3a-4) - (2a-3)^2\}e(\lambda) = \\ (2\lambda + a - 2)\{3\lambda^2 + 2(a-1)\lambda - (2a-3)^2\}B. \end{cases}$$

To end the proof, we shall show that if $e(\lambda) = B = 0$, then $(\varphi e)(\lambda) = A = 0$, and conversely. So, if some of these functions are different from zero, then necessarily the determinants of the matrices of both systems (4.16) and (4.17) must vanish. But we prove that also in this case $e(\lambda) = B = 0$ and so, $(\varphi e)(\lambda) = A = 0$. Thus, in any case we can conclude that $e(\lambda) = (\varphi e)(\lambda) = A = B = 0$ on M. Moreover, (4.9) and (4.10) also give $e(a) = (\varphi e)(a) = 0$, that is, λ and a are globally constant on M. Then, as in the proof of Theorem 1, we can conclude that M is locally homogeneous and Proposition 4.2 implies that M is locally symmetric. So, M has constant sectional curvature 0 or +1, as proved in [BS]. The following steps complete the proof.

Step 1: We prove that if $e(\lambda) = B = 0$, then $(\varphi e)(\lambda) = A = 0$, and conversely.

Let $e(\lambda) = B = 0$ and suppose that $A \neq 0$. Since also $\xi(\lambda) = 0$, using (2.4) we have

$$0 = [\xi, e](\lambda) = (\lambda - a + 1)(\varphi e)(\lambda).$$

So, either $(\varphi e)(\lambda) = 0$ or $a = \lambda - 1$.

If $(\varphi e)(\lambda) = 0$, since $A \neq 0$, from (4.4) and (4.13) we get

$$\begin{cases} 2\lambda + a - 2 = 0, \\ 3\lambda^2 - 2(a-1)\lambda - (2a-3)^2 = 0, \end{cases}$$

from which $\lambda = 1/3$ and $a = 4/3$ follow. But this case can not occur. In fact, if $\lambda = 1/3$ and $a = 4/3$, taking also into account $B = 0$, (4.8) gives

$$\frac{3}{2}A^2 + (\frac{2}{3}r - \frac{64}{27}) = 0. \tag{4.18}$$

Differentiating (4.18) with respect to φe, since $A \neq 0$, we get $(\varphi e)(A) = 0$. Then, using $\lambda = 1/3$, $a = 4/3$ and $B = 0$, (4.7) gives $r = 32/7$. So, from (4.18), $\frac{3}{2}A^2 + \frac{128}{27} = 0$ follows, which is impossible. Therefore, in this case $A = 0$.

If $a = \lambda + 1$, (4.10) becomes

$$(\varphi e)(\lambda) = \frac{(3\lambda - 1)}{4\lambda}A \tag{4.19}$$

and (4.8) gives

$$(7\lambda - 1)A^2 + 40\lambda^5 + 56\lambda^4 + 4(r-2)\lambda^3 + 4(r-6)\lambda^2 = 0. \tag{4.20}$$

We differentiate (4.20) with respect to φe. Taking into account (4.19) and $A \neq 0$, we obtain

$$\begin{aligned} &8\lambda(7\lambda - 1)(\varphi e)(A) + 7(3\lambda - 1)A^2 + \\ &+(3\lambda - 1)\{200\lambda^4 + 224\lambda^3 + 12(r-2)\lambda^2 + 8(r-6)\lambda\} = 0. \end{aligned} \tag{4.21}$$

We can express $(\varphi e)(A)$ using (4.7). So, (4.21) becomes

$$7(3\lambda - 1)A^2 + \lambda\{96\lambda^4 + 264\lambda^3 + 8(r-25)\lambda^2 + 4(11r - 74)\lambda\} = 0. \tag{4.22}$$

Note that if $\lambda = 1/3$, then $a = \lambda + 1 = 4/3$ and we proved already that this leads to a contradiction. So, $3\lambda - 1 \neq 0$ and we can use (4.22) to compute A^2. Comparing (4.20) and (4.22) we then get

$$168\lambda^4 - 856\lambda^3 + 4(7r + 276)\lambda^2 - 4(61r + 88)\lambda + 16(r - 8) = 0.$$

Lemma 2.5 then implies $(\varphi e)(\lambda) = 0$ from which it follows $A = 0$, as we already proved. So, in any case $A = 0$, from which, using (4.17), it follows easily $(\varphi e)(\lambda) = 0$.

In the same way, assuming $(\varphi e)(\lambda) = A = 0$, we obtain $e(\lambda) = B = 0$.

Step 2: We prove that $e(\lambda) = B = 0$ also holds when the determinants of the matrices of both systems (4.16) and (4.17) vanish.

Computing these determinants, after some easy but long computations we get

$$(4.23) \qquad \begin{cases} 3\lambda^4 - (14a^2 - 37a + 24)\lambda^2 + (4a^2 - 12a + 9)(4a^2 - 9a + 5) = 0, \\ \\ 3(2 - a)\lambda^2 + 8a^3 - 38a^2 + 59a - 3 = 0. \end{cases}$$

Note that if $a = 2$, the last equation of (4.23) gives $32 = 0$, which is impossible. Therefore, $a \neq 2$ and we can use the second equation of (4.23) to express λ^2 by means of a. Substituting in the first equation of (4.23), we get

$$(4.24) \qquad -16a^5 + 676a^4 + 4338a^3 + 4698a^2 - 4464a = 0.$$

Using Lemma 2.5, from (4.24) we obtain $e(a) = (\varphi e)(a) = 0$. Differentiating the second equation of (4.23) by e and by φe, we then get at once $e(\lambda) = 0$ and $(\varphi e)(\lambda) = 0$, respectively. Next, from (4.9) and (4.10) it follows $(\lambda - 2a + 3)B = 0$ and $(\lambda + 2a - 3)A = 0$, respectively. Since $\lambda \neq 0$, if $\lambda - 2a + 3 = 0$ then $\lambda + 2a - 3 \neq 0$ and conversely. So, either $A = 0$ or $B = 0$. In any case, proceeding as in Step 1, we can conclude that $A = B = 0$ and this ends the proof □

5 Conformally flat contact metric three-manifolds

Conformally flat contact metric manifolds have been studied by several authors. S. Tanno [T] proved that a conformally flat K-contact space has constant sectional curvature $+1$. In [O] it was proved that Sasakian conformally flat manifolds of dimension ≥ 5 have constant sectional curvature $+1$. Conformally flat contact metric manifolds such that $Q\varphi = \varphi Q$ have constant sectional curvature 0 or $+1$, as proved in [BKo]. For three-dimensional manifolds, this result was generalized in [GA-X2] where it was proved that a conformally flat contact metric manifold with $\nabla_\xi \tau = 0$ has constant sectional curvature 0 or $+1$. We now prove Theorem 3, which extends this last result.

Proof of Theorem 3

We assume M is non-Sasakian, since the Sasakian case has been already studied in [T]. Then, Lemma 4.1 holds, with $e(a) = (\varphi e)(a) = 0$, since a is constant. Using (4.9) in (4.3) and (4.10) in (4.4), we get

$$(5.1) \qquad \xi(A) = -2\lambda e(\lambda) + (\lambda + a - 1)B$$

and

$$(5.2) \qquad \xi(B) = -2\lambda(\varphi e)(\lambda) + (\lambda - a + 1)A,$$

respectively. We now compute $[e, \varphi e](\lambda) = e(\varphi e)(\lambda) - (\varphi e)e(\lambda)$ by differentiating (4.9) with respect to φe and (4.10) with respect to e, respectively. Moreover, $[e, \varphi e](\lambda) = (\nabla_e \varphi e - \nabla_{\varphi e} e)(\lambda)$ can also be computed using (2.4). Comparing the two expressions of $[e, \varphi e](\lambda)$, we obtain

$$(5.3) \qquad \frac{2a-3}{4\lambda}\xi(r) + \frac{2a-3}{2\lambda^2}AB + \frac{1}{8\lambda}[\varphi e, e](r) + \frac{\lambda-2a+3}{4\lambda^2}Ae(\lambda) + \\ -\frac{\lambda+2a-3}{4\lambda^2}B(\varphi e)(\lambda) + \frac{e(\lambda)(\varphi e)(r) - e(r)(\varphi e)(\lambda)}{8\lambda^2} = 0.$$

Using (2.4), we get

$$[\varphi e, e](r) = \frac{1}{2\lambda}[(\varphi e)(\lambda) + A]e(r) - \frac{1}{2\lambda}[e(\lambda) + B](\varphi e)(r) - 2\xi(r)$$

and so, (5.3) becomes

$$(5.4) \qquad \frac{a-2}{2\lambda}\xi(r) + \frac{2a-3}{2\lambda^2}AB + \frac{\lambda-2a+3}{4\lambda^2}Ae(\lambda) - \frac{\lambda+2a-3}{4\lambda^2}B(\varphi e)(\lambda) + \\ +\frac{1}{16\lambda^2}[A - (\varphi e)(\lambda)]e(r) - \frac{1}{16\lambda^2}[B - e(\lambda)](\varphi e)(r) = 0.$$

Using (4.9) and (4.10) to express $e(r)$ and $(\varphi e)(r)$, from (5.4) it follows

$$\frac{a-2}{2\lambda}\xi(r) = 0$$

and so, $\xi(r) = 0$, since $a \neq 2$.

Next, we differentiate (4.9) with respect to ξ. Using (4.4), we get

$$\xi e(\lambda) = -(\lambda - 2a + 3)(\varphi e)(\lambda) - \frac{\xi e(r)}{8\lambda} + + \frac{(\lambda - 2a + 3)(\lambda - a + 1)}{2\lambda}A.$$

Since $\xi(\lambda) = 0$ (see Proposition 2.2), using (2.4), we also get

$$\xi e(\lambda) = [\xi, e](\lambda) = (\lambda - a + 1)(\varphi e)(\lambda).$$

Hence,

$$\xi e(r) = 4(\lambda - 2a + 3)(\lambda - a + 1)A - 8\lambda(2\lambda - 3a + 4)(\varphi e)(\lambda),$$

that is, using (4.10) to express $(\varphi e)(\lambda)$,

$$(5.5) \qquad \xi e(r) = 4\{-\lambda^2 - 2(2a-3)\lambda + (2a-3)(4a-5)\}A + \\ +(2\lambda - 3a + 4)(\varphi e)(r).$$

In the same way, we compute $\xi(\varphi e)(\lambda)$ by differentiating (4.10) with respect to ξ and we compare with $\xi(\varphi e)(\lambda) = [\xi, \varphi e](\lambda) = (\lambda + a - 1)$ obtained using (2.4). So, we have

$$\xi(\varphi e)(r) = 4(\lambda + 2a - 3)(\lambda + a - 1)B - 8\lambda(2\lambda + 3a - 4)e(\lambda),$$

that is, using (4.9),

(5.6) $\xi(\varphi e)(r) = 4\{-\lambda^2 + 2(2a-3)\lambda + (2a-3)(4a-5)\}B + (2\lambda + 3a - 4)e(r)$.

Since $\xi(r) = 0$, we also have

$$\xi e(r) = [\xi, e](r) = (\lambda - a + 1)(\varphi e)(r)$$

and

$$\xi(\varphi e)(r) = [\xi, \varphi e](r) = (\lambda + a - 1)e(r).$$

Comparing with (5.5) and (5.6), we then get

$$(5.7) \qquad 4\{-\lambda^2 - 2(2a-3)\lambda + (2a-3)(4a-5)\}A + (\lambda - 2a + 3)(\varphi e)(r) = 0$$

and

$$(5.8) \qquad 4\{-\lambda^2 + 2(2a-3)\lambda + (2a-3)(4a-5)\}B + (\lambda + 2a - 3)e(r) = 0.$$

We now differentiate (5.7) with respect to ξ. Taking into account $\xi(\varphi e)(r) = (\lambda + a - 1)e(r)$, we obtain

$$\begin{aligned} &4(3a-4)\{-\lambda^2 - 2(2a-3)\lambda + (2a-3)(4a-5)\}B + \\ &+\{(8-5a)\lambda + (-2a+3)(3a-4)\}e(r) = 0. \end{aligned}$$

Therefore, B and $e(r)$ satisfy

$$(5.9) \qquad \begin{cases} 4\{-\lambda^2 + 2(2a-3)\lambda + (2a-3)(4a-5)\}B + (\lambda + 2a - 3)e(r) = 0, \\ \\ 4(3a-4)\{-\lambda^2 - 2(2a-3)\lambda + (2a-3)(4a-5)\}B + \\ +\{(8-5a)\lambda + (2a-3)(3a-4)\}e(r) = 0. \end{cases}$$

If $B = e(r) = 0$, then (4.9) implies $e(\lambda) = 0$. We now prove that $e(\lambda) = 0$ also holds when (5.9) admits other solutions. In fact, in this case the determinant of the matrix of the system (5.9) vanishes, that is,

$$\begin{aligned} &4\{-\lambda^2 + 2(2a-3)\lambda + (2a-3)(4a-5)\}\{(8-5a)\lambda - (2a-3)(3a-4)\} + \\ &-4(3a-4)(\lambda + 2a - 3)\{-\lambda^2 - 2(2a-3)\lambda + (2a-3)(4a-5)\} = 0. \end{aligned}$$

from which it follows

$$(5.10) \qquad 4(2a-3)\lambda\{\lambda^2 + (2-a)\lambda + (2a-3)(1-a)\} = 0$$

Note that if $a = 3/2$, then (5.8) gives $e(r) = 4\lambda B$ which, together with (4.9), gives at once $e(\lambda) = 0$. If $a \neq 3/2$, using Lemma 2.5, from (5.10) it follows again $e(\lambda) = 0$.

A very similar argument also shows that $(\varphi e)(\lambda) = 0$ on M. Therefore, λ is locally constant and hence, M being connected, it is globally constant on M.

Using the constancy of λ, we now prove that A and B are constant on M. On the one hand, we compute $[\xi, e](A) = \xi e(A) - e\xi(A)$ by differentiating (4.3) and (4.5) with respect to e and ξ, respectively. On the other hand, using (2.4) and (4.7), we can compute $[\xi, e](A) = (\nabla_\xi e - \nabla_e \xi)(A)$. Comparing the two expressions, we then obtain

$$A^2 + B^2 + 6\lambda^4 + (r + 4a^2 + 6a - 12)\lambda^2 + (a-1)(r-6) = 0. \tag{5.11}$$

We now differentiate (5.11) with respect to ξ. We take into account the constancy of λ and a and $\xi(r) = 0$ and we use (4.3) and (4.4) to express $\xi(A)$ and $\xi(B)$, respectively. So, we get $4\lambda AB = 0$, that is, $AB = 0$.

If $A = 0$, (5.1) gives $(\lambda + a - 1)B = 0$. Suppose $B \neq 0$. From (5.2) and (4.6) we get $\xi(B) = 0$ and $(\varphi e)(B) = 0$. Differentiating (4.7) with respect to e, we obtain

$$\frac{B}{\lambda} e(B) + \frac{\lambda - 1}{2} e(r) = 0, \tag{5.12}$$

from which, using $B \neq 0$ and (4.3), it follows $e(B) = 2\lambda(\lambda-1)(3\lambda+1)$. Comparing with (4.8) and differentiating with respect to e, we get $\frac{\lambda+1}{2}e(r) = 0$. Note that $a = 1 - \lambda \neq 2$ and so, $\lambda \neq -1$. Thus, $e(r) = 0$ and (5.12) gives $e(B) = 0$, that is, B is constant. In the same way, assuming $B = 0$, we get that A is constant on M. In any case, we have

$$[\xi, e] = (\lambda - a + 1)\varphi e, \quad [e, \varphi e] = 2\xi - \frac{A}{2\lambda} e + \frac{B}{2\lambda}\varphi e, \quad [\varphi e, \xi] = -(\lambda + a - 1)e,$$

whith λ, a, A and B constant on M. Therefore, M is locally homogeneous, since it is locally isometric to a Lie group. Hence, M is locally symmetric [Tg] and, according to [BS], we conclude that M has constant sectional curvature 0 or +1 □

Remark 5.1. In Theorem 3 we assumed $a \neq 2$, but this assumption is not very restrictive, because there do not exist conformally flat locally homogeneous non-Sasakian contact metric three-manifolds with $a = 2$. In fact, if (M, η, g) is such a manifold, then (4.9) and (4.10) give $(\lambda - 1)B = 0$ and $(\lambda + 1)A = 0$, respectively. If $B = 0$, (4.4) also gives $A = 0$ and conversely, if $A = 0$, then $B = 0$ by (4.3). Thus, $A = B = 0$ and, comparing (4.7) and (4.8), we eventually get $\lambda^2 + 8 = 0$, which cannot occur.

Remark 5.2. Even if the conformal flatness implies the constancy of the sectional curvature for many classes of contact metric manifolds, there also exist examples of conformally flat contact metric three-manifolds which do not have constant sectional curvature. These examples were constructed by D.E. Blair in [B2]. We now show that such examples satisfy $\nabla_\xi \tau = 2a\tau\varphi$ with $\xi(a) = 0$. On the one hand, this implies that Theorem 2 cannot be extended considering conformal flatness instead of harmonicity of the curvature. On the other hand, this also means that Theorem 3 is somehow the "maximal" result which can be proved in this direction for conformally flat contact metric manifolds.

We consider $\mathbb{R}^3$ with cylindrical coordinates (r, θ, z). Let η be the contact form on $\mathbb{R}^3$ given by

$$\eta = \frac{1}{2}(\beta r d\theta + \gamma dz),$$

where β and γ are smooth functions depending only on r. β and γ satisfy

$$\begin{cases} \frac{1}{r}\beta + \beta' = \sqrt{\beta^2 + \gamma^2}\,\gamma, \\ -\gamma' = \sqrt{\beta^2 + \gamma^2}\,\beta. \end{cases}$$

Moreover, $\beta \neq 0$ and $\gamma \neq 0$ [B2].

The Riemannian metric

$$g = \frac{e^{2s}}{4}(dr^2 + r^2 d\theta^2 + dz^2),$$

where $e^{2s} = \beta^2 + \gamma^2$, is a conformally flat metric associated to η. The tensor φ is given by

$$\varphi = e^{-2s} \begin{pmatrix} 0 & rf\gamma & -f\beta \\ -\frac{f}{r}\gamma & 0 & 0 \\ f\beta & 0 & 0 \end{pmatrix}, \tag{5.13}$$

where $f = e^s$. The characteristic vector field is

$$\xi = 2e^{-2s}(\frac{\beta}{r}\frac{\partial}{\partial\theta} + \gamma\frac{\partial}{\partial z}). \tag{5.14}$$

For more details, see [B2]. A vector field X belongs to Kerη if and only if it is orthogonal to ξ. Therefore, if $X \in Ker\eta$, then $X = X_1\frac{\partial}{\partial r} + \frac{X_2}{r}\frac{\partial}{\partial\theta} - \frac{\beta}{\gamma}X_2\frac{\partial}{\partial z}$ for some real functions X_1 and X_2 on $\mathbb{R}^3$.

We now compute the Christoffel coefficients of the Riemannian connection of $(\mathbb{R}^3, g)$ with respect to the basis $\{e_1 = \frac{\partial}{\partial r}, e_2 = \frac{\partial}{\partial\theta}, e_3 = \frac{\partial}{\partial z}\}$. Using the well-known formula

$$\Gamma_{ij}^k = \frac{1}{2}\sum_r \{\partial_i g_{jr} + \partial_j g_{ir} - \partial_r g_{ij}\}g^{rk},$$

where g_{ij} are the coefficients of g with respect to the basis $\{e_i\}$ and $g^{ij} = (g_{ij})^{-1}$, after some easy computations we obtain

$$\begin{cases} \nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r} = s'\frac{\partial}{\partial r}, & \nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial\theta} = (\frac{1}{r} + s')\frac{\partial}{\partial\theta}, & \nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial z} = s'\frac{\partial}{\partial z}, \\ \nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial r} = (\frac{1}{r} + s')\frac{\partial}{\partial\theta}, & \nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\theta} = -(r + r^2 s')\frac{\partial}{\partial r}, & \nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial z} = 0 \\ \nabla_{\frac{\partial}{\partial z}}\frac{\partial}{\partial r} = s'\frac{\partial}{\partial z}, & \nabla_{\frac{\partial}{\partial z}}\frac{\partial}{\partial\theta} = 0, & \nabla_{\frac{\partial}{\partial z}}\frac{\partial}{\partial z} = -s'\frac{\partial}{\partial r}. \end{cases} \tag{5.15}$$

We now compute the tensor h. Since $h\xi = 0$, we get

$$\frac{\beta}{r}h(\frac{\partial}{\partial\theta}) + \gamma h(\frac{\partial}{\partial z}) = 0.$$

Next, we compute $\nabla_{\frac{\partial}{\partial z}}\xi$ using (2.1) and (5.17). We then get

$$\nabla_{\frac{\partial}{\partial z}}\xi = e^{-2s}f\beta\frac{\partial}{\partial r} - e^{-2s}f\beta h(\frac{\partial}{\partial r}).$$

On the other hand, taking into account (5.18) and (5.19), we also have

$$\nabla_{\frac{\partial}{\partial z}}\xi = -2e^{-2s}\gamma s'\frac{\partial}{\partial r}.$$

Comparing the previous expressions of $\nabla_{\frac{\partial}{\partial z}}\xi$, we obtain

$$h(\frac{\partial}{\partial r}) = (1 + \frac{2\gamma s'}{f\beta})\frac{\partial}{\partial r}.$$

Therefore, $\frac{\partial}{\partial r}$ is an eigenvector of h. Using (5.17), we finally obtain

$$\varphi\frac{\partial}{\partial r} = e^{-s}(-\frac{\gamma}{r}\frac{\partial}{\partial \theta} + \beta\frac{\partial}{\partial z}).$$

So, $\{\xi, e = 2e^{-s}\frac{\partial}{\partial r}, \varphi e = 2e^{-2s}(-\frac{\gamma}{r}\frac{\partial}{\partial\theta} + \beta\frac{\partial}{\partial z})\}$ is a φ-basis for $(I\!\!R^3, \eta, g)$. Note that $he = \lambda e$, with $\lambda = 1 + \dfrac{2\gamma s'}{\beta e^{3\sigma}}$. Since β, γ and s only depend on r, we have $\frac{\partial}{\partial\theta}(\lambda) = \frac{\partial}{\partial z}(\lambda) = 0$ and so, by (5.18), $\xi(\lambda) = 0$.

From (2.4) it follows that a satisfies $\nabla_\xi e = -a\varphi e$. Thus, we can use (5.19) to compute a. We get

$$a = \frac{2\beta}{f\gamma}(\frac{1}{r} + s') = -\frac{2\gamma s'}{f\beta} = \frac{2\beta\gamma}{re^{3s}}.$$

As we noted for λ, $\xi(a) = 0$, because β, γ and σ only depend on r. Therefore, we can conclude that $(I\!\!R^3, \eta, g)$ is a conformally flat contact metric three-manifold satisfying $\nabla_\xi\tau = 2a\tau\varphi$ (since $\xi(\lambda) = 0$, see Proposition 2.2), with $\xi(a) = 0$. However, $(I\!\!R^3, \eta, g)$ is not locally homogeneuo s [B2] and so, it does not have constant sectional curvature.
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