Cross Ratios of Points and Lines in Moufang Planes

B. Celik and S. Ciftci

Dedicated to Prof.Dr. Constantin UDRIŞTE on the occasion of his sixtieth birthday

Abstract

In this paper, first we extend the known definition of cross-ratio of collinear points to whole Moufang plane. Later we introduce the cross-ratios for lines and the known results about the cross-ratios of points which are adapted to cross-ratios of lines without using the principle of duality. Finally, we give a theorem which describes the relation between the cross-ratios of points and lines.

Mathematics Subject Classification: 51A35
Key words: Cross ratio, Moufang planes

1 Introduction

Let M be a projective plane coordinated by an alternative field $\mathcal{A}$, $char\mathcal{A} \neq 2$. If $\mathcal{A}$ is associative, then M is Desarguesian and if $\mathcal{A}$ is non-associative then $\mathcal{A}$ is Cayley division algebra over its center $\mathcal{Z}$ and M is a Moufang plane (see [4]). Then, $\mathcal{A}$ is equipped with the involution $\gamma : x \to \bar{x}$, the norm form $n : x \to x\bar{x}$ and the trace form $t : x \to \frac{1}{2}(x + \bar{x})$ (see [1]). In this case, the ranges of the norm and trace forms are $\mathcal{Z}$, but the range of the γ is A. Also norm form is multiplicative and trace form is both symmetric and associative (i.e. $n(xy) = n(x)n(y)$, $t(xy) = t(yx)$, $t(x(yz)) = t((xy)z)$).

There is an equivalence relation $\equiv$ on $\mathcal{A}$ which is defined by "$a \equiv b \Leftrightarrow \exists c \in \mathcal{A}\backslash\{0\}, a = c^{-1}bc$" and this equivalence relation is called conjugate. For any element x of $\mathcal{A}$, the equivalence class of x is called the conjugacy class of x and it is denoted by $[x]$. It was shown in [5] and [3] that

$$(1) \qquad \text{“}n(x) = n(y), t(x) = t(y)\text{”} \Leftrightarrow \text{“}[x] = [y]\text{”}$$

and this property will be used frequently in this paper.

$\mathcal{A} \cup \{\infty\}$ is denoted by $\hat{\mathcal{A}}$, $\infty \notin \mathcal{A}$ and the transformations $t_u(x) = x + u$, $r_u(x) = xu$, $l_u(x) = ux$, $i(x) = x^{-1}, \infty \longleftrightarrow 0$, which are defined on $\hat{\mathcal{A}}$, are called translation with u (translation), right multiplication with u (right multiplication), left multiplication with u (left multiplication), and inverse transformation respectively.

Balkan Journal of Geometry and Its Applications, Vol.5, No.2, 2000, pp. 37-46

2 Definition and properties of the cross-ratio of points

Let M be a Moufang plane which is coordinated with an alternative field $\mathcal{A}$ such that *char* $\mathcal{A} \neq 2$. Any point of $l = [0,0]$ which is different from (0) is denoted by $X = (x,0) = x$ and $(0) := \infty$, $\quad x, 0 \in \mathcal{A}$, $\infty \notin \mathcal{A}$. Let $A = (a,0)$, $B = (b,0)$, $C = (c,0)$, $D = (d,0)$ be four arbitrary affine points of l. The cross-ratio $(A,B:C,D)$ of A,B,C,D is defined by

$$(A,B:C,D) = \left[\left((a-d)^{-1}(b-d) \right) \left((b-c)^{-1}(a-c) \right) \right] = (a,b:c,d)$$

and $[x]$ denotes the conjugacy class $\{ \, y^{-1}xy \mid y \in \mathcal{A} \}$ of x. If one of the A,B,C,D is ∞, then omit the factors containing it.

The proofs of Theorem 2.1 and Theorem 2.2 can be found in [3] with some calculating errors.

Theorem 2.1. *If a,b,c,d are distinct elements of $\mathcal{A}$, then*

$$(a,b:c,d) = \left[\left((a-b)^{-1} - (a-d)^{-1} \right) \left((a-b)^{-1} - (a-c)^{-1} \right)^{-1} \right].$$

Proof. Since $n(x)$ is multiplicative and $t(x)$ is associative

$$u' = \left((a-d)^{-1}(b-d) \right) \left((b-c)^{-1}(a-c) \right)$$

is conjugate to

$$u = \left(\left((a-d)^{-1}(b-d) \right) (b-c)^{-1} \right) (a-c).$$

Thus $(a-d)^{-1}(b-d) = \left(u\,(a-c)^{-1} \right) (b-c)$ so

$$(2) \qquad \left((a-d)^{-1}(b-d) \right) (a-b)^{-1} = \left(\left(u\,(a-c)^{-1} \right) (b-c) \right) (a-b)^{-1}$$

The left hand side of this equation can be viewed as

$$\begin{aligned} \left((a-d)^{-1}(b-d) \right) (a-b)^{-1} &= \left((a-d)^{-1}((a-d)-(a-b)) \right) (a-b)^{-1} \\ &= \left(1-(a-d)^{-1}(a-b) \right) (a-b)^{-1} \\ &= (a-b)^{-1} - \left((a-d)^{-1}(a-b) \right) (a-b)^{-1} \\ &= (a-b)^{-1} - (a-d)^{-1} \end{aligned}$$

substituting this into (2), we have

$$\begin{aligned} &(a-b)^{-1} - (a-d)^{-1} = \left(\left(u\,(a-c)^{-1} \right) (b-c) \right) (a-b)^{-1} \\ &\Rightarrow \left(\left(\left((a-b)^{-1} - (a-d)^{-1} \right) (a-b) \right) (b-c)^{-1} \right) (a-c) = u \\ &\Rightarrow [u] = \left[\left(\left(\left((a-b)^{-1} - (a-d)^{-1} \right) (a-b) \right) (b-c)^{-1} \right) (a-c) \right]. \end{aligned}$$

And from (1),

$$\begin{aligned}
[u] &= \left[\left((a-b)^{-1}-(a-d)^{-1}\right)\left((a-b)\left((b-c)^{-1}(a-c)\right)\right)\right] \\
&= \left[\left((a-b)^{-1}-(a-d)^{-1}\right)\left(\left((a-c)^{-1}(b-c)\right)(a-b)^{-1}\right)^{-1}\right] \\
&= \left[\left((a-b)^{-1}-(a-d)^{-1}\right)\left(\left((a-c)^{-1}((a-c)-(a-b))\right)(a-b)^{-1}\right)^{-1}\right] \\
&= \left[\left((a-b)^{-1}-(a-d)^{-1}\right)\left(\left(1-(a-c)^{-1}(a-b)\right)(a-b)^{-1}\right)^{-1}\right] \\
&= \left[\left((a-b)^{-1}-(a-d)^{-1}\right)\left((a-b)^{-1}-\left((a-c)^{-1}(a-b)\right)(a-b)^{-1}\right)^{-1}\right],
\end{aligned}$$

so

$$[u]=[u']=\left[\left((a-b)^{-1}-(a-d)^{-1}\right)\left((a-b)^{-1}-(a-c)^{-1}\right)^{-1}\right]$$

is obtained. □

If $[x]^{-1}$ and $1-[x]$ are defined to be $\left[x^{-1}\right]$ and $[1-x]$, respectively, the following statements are valid and cross-ratio is invariant under the identical permutation and (12)(34), (13)(24), (14)(23). Let a,b,c,d distinct elements of $\hat{\mathcal{A}}$ and $w\in(a,b:c,d)$. Then

$$(a,b:c,d)=(b,a:c,d)^{-1}, 1-(a,b:c,d)=(a,c:b,d)$$

$$(a,b:c,d)=[w], \qquad (b,a:c,d)=[w]^{-1}, \qquad (a,c:b,d)=1-[w]$$

$$(b,c:a,d)=1-[w]^{-1}, \qquad (c,a:b,d)=[1-w]^{-1}, (c,b:a,d)=\left[1-w^{-1}\right]^{-1}.$$

Theorem 2.2. *Let $r\in\mathcal{A}, r\neq 0, r\neq 1$. If $a,b,c\in\hat{\mathcal{A}}$ are distinct. Then there exists $d\in\hat{\mathcal{A}}$ such that $(a,b:c,d)=[r]$. If r is in the center $\mathcal{Z}$ of $\mathcal{A}$, then d is unique.*

Proof. Suppose first that $a,b,c\in\mathcal{A}$. Then we must determine $d\in\mathcal{A}$ such that

$$(a,b:c,d)=\left[\left((a-d)^{-1}(b-d)\right)\left((b-c)^{-1}(a-c)\right)\right]=[r].$$

Let $u=(b-c)^{-1}(a-c)$. For any $s\in[r]$, $s\neq u$, $d=\left(a\left(su^{-1}\right)-b\right)\left(u\,(s-u)^{-1}\right)$, if $s=u=(b-c)^{-1}(a-c)$, then $d=\infty$ is the desired element of $\mathcal{A}$.

If $s\in[r]$ and $c=\infty$, since $s\neq 1$, $d=(as-b)(s-1)^{-1}$ satisfies $(a,b:c,d)=[r]$. The remaining cases $b=\infty$, $a=\infty$ are reduced to the case $c=\infty$.

If $r\in\mathcal{Z}$, so $[r]=\{r\}$, $s=r$ and the solution $d\in\hat{\mathcal{A}}$ is unique. □

Now we give a theorem related to the transformations preserving cross-ratio

Theorem 2.3. *If $\sigma=t_u, r_u, i$ or γ, then $(a,b:c,d)=(\sigma(a),\sigma(b):\sigma(c),\sigma(d))$ for all $a,b,c,d\in\hat{\mathcal{A}}$ (see [3]).*

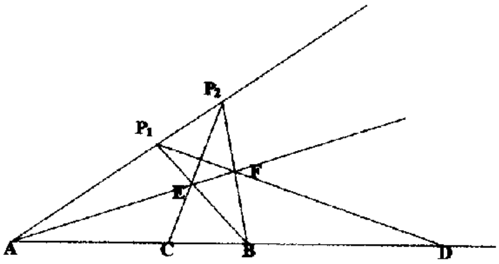

Fig. 1

A quadruple a, b, c, d of elements of $\hat{\mathcal{A}}$ is said to be *harmonic* if $(a, b : c, d) = [-1]$ and we let $h(a, b, c, d)$ represents the statement: a, b, c, d *are harmonic*.

It is trivial from Theorem 2.2 that if a, b, c are different elements of $\hat{\mathcal{A}}$, there is a unique element $d \in \hat{\mathcal{A}}$ such that $h(a, b, c, d)$ and the relation $h(a, b, c, d)$ is invariant under the elements of group which is generated by the permutations (12), (13), (24). Also when $\sigma = t_u, r_u, i$ or γ and $h(a, b, c, d)$ then by Theorem 2.3 it is easy to see that $h(\sigma(a), \sigma(b), \sigma(c), \sigma(d))$. And since $l_u = ir_{u^{-1}}i$, the transformation l_u also preserves harmonicity.

If A, B and C are distinct points of the line $l = [0, 0]$ and D is constructed from A, B, C, P_1 and P_2 via Fig. 1, then the point D is uniquely determined by A, B, C (i.e. independent of the choice of P_1 and P_2). The points A, B, C, D of l are called in *harmonic position* if they can be embedded as in Fig. 1. The distinct points a, b, c, d (possibly ∞) are in harmonic position if and only if $h(a, b, c, d)$ (see [3]).

In this paper we denote by $G_i(l)$ the group of all projectivities of l and by $T(l)$ the group which is generated by t_u, r_u and i. Since the transformation $\varphi : l \to l$ given by

$$\varphi = \begin{cases} r_{(b-a)^{-1}} t_{-a} & \text{if } c = \infty \\ r_{(b^{-1}-a^{-1})^{-1}} t_{-a^{-1}} i & \text{if } c = 0 \\ r_{\left((b-c)^{-1}-(a-c)^{-1}\right)^{-1}} t_{-(a-c)^{-1}} i t_{-c} & \text{otherwise} \end{cases}$$

transforms the points a, b, c to $0, 1, \infty$, respectively, $T(l\ )$ is transitive on ordered triples of distinct points of l.

In [3] Theorem 7, Ferrar shows that $G_i(l\ ) = T(l\ )$.

The cross-ratio definition of different points of l is extended to whole Moufang plane in [2].

3 The Cross-ratio of concurrent lines

Let $\mathcal{L}_{(0,0)}$ denote the set of lines which are passing through the point $(0, 0)$ in Moufang plane $\mathcal{M}$ which is coordinated by an alternative field $\mathcal{A}$ with $char\mathcal{A} \neq 2$. In this case,

$$\mathcal{L}_{(0,0)} = \{m := [m, 0] \mid m \in \mathcal{A}\} \cup \{\tilde{\infty} := [0]\}.$$

If p, q, r, s are distinct elements of $\mathcal{L}_{(0,0)}$, different from $\tilde{\infty}$, we denote the cross-ratio $\langle p, q : r, s\rangle$ as a conjugacy class as follows:

$$\langle p, q : r, s\rangle = \left[\left((p-s)^{-1}(q-s)\right)\left((q-r)^{-1}(p-r)\right)\right].$$

If one of the lines p, q, r, s is $\tilde{\infty}$, then omit the factors containing it.

After this definition every result about the cross-ratio of points on $l = [0, 0]$ can be adapted to the cross-ratio of lines of $\mathcal{L}_{(0,0)}$ easily. For instance, if $p, q, r, s \in \mathcal{L}_{(0,0)}$ are different lines and $\langle p, q : r, s\rangle = [u]$ then the following equalities are valid:

$$\langle p, q : r, s\rangle = \left[\left((p-q)^{-1} - (p-s)^{-1}\right)\left((p-q)^{-1} - (p-r)^{-1}\right)^{-1}\right]$$

$$\langle p, q : r, s\rangle = \langle q, p : r, s\rangle^{-1}, \qquad 1 - \langle p, q : r, s\rangle = \langle p, r : q, s\rangle,$$

$$\langle p, q : s, r\rangle = [u]^{-1}, \ \langle p, r : q, s\rangle = [1-u], \ \langle p, r : s, q\rangle = \left[(1-u)^{-1}\right],$$

$$\langle p, s : r, q\rangle = \left[-u(1-u)^{-1}\right], \qquad \langle p, s : q, r\rangle = \left[1 - u^{-1}\right]$$

and elements of the group which is generated by the identic permutation and (12)(34), (13)(24), (14)(23) preserve the cross-ratio of lines.

Theorem 3.1. *Let $u \in \mathcal{A}$, $u \neq 0, u \neq 1$. If $p, q, r, s \in \mathcal{L}_{(0,0)}$ are different elements, then there exist an $s \in \mathcal{L}_{(0,0)}$ such that $\langle p, q : r, s\rangle = [u]$ and if u is an element of center $\mathcal{Z}$ of $\mathcal{A}$, then s is unique.*

The proof of this theorem can be done by means of the process in the proof of Theorem 2.2.

Definition 3.1. A quadruple p, q, r, s of elements of $\mathcal{L}_{(0,0)}$ is said to be *harmonic* if $\langle p, q : r, s\rangle = [-1]$ and we let $H(p, q, r, s)$ for "the lines p, q, r, s are called *harmonic*". The distinct lines p, q, r, s are called to be in *harmonic position* if any quadrilateral l_1, l_2, l_3, l_4 exists as in Fig. 2.

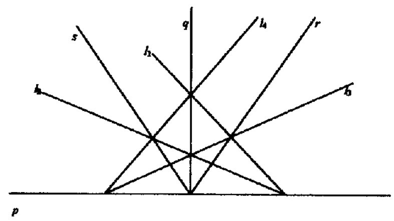

Fig. 2

The transformations

$$t_u : [x, 0] \to [x+u, 0], \quad \tilde{\infty} \to \tilde{\infty}$$

$$l_u : [x, 0] \to [ux, 0], \tilde{\infty} \to \tilde{\infty}$$

$$r_u : [x, 0] \to [xu, 0], \tilde{\infty} \to \tilde{\infty}$$

and

$$i : [x, 0] \to \left[x^{-1}, 0\right], [0, 0] \to \tilde{\infty}$$

which are defined on $\mathcal{L}_{(0,0)}$ are called *translation* (by u), *left multiplication* (by u), *right multiplication* (by u) and *inverse transformations* respectively.

Now we can state a theorem which can be proved by using the methods of the proof of the Theorem 2.6 in [3].

Theorem 3.2. *Distinct lines p, q, r, s are in harmonic position iff $H(p, q, r, s)$.*
Proof. Let the lines p, q, r, s be in harmonic position with respect to the quadrilateral l_1, l_2, l_3, l_4 (Fig. 2). In this case without lose of generality, we may assume $l_1 = [\infty]$ and $l_2 = [p, 1]$ (since, if P is a point and l_1 and l_2 are lines not incident with P, then there is an elation fixing all lines passing through P and mapping l_1 to l_2). So we obtain

$$q \wedge l_2 = [q, 0] \wedge [p, 1] = \left((q-p)^{-1},\ q(q-p)^{-1}\right),$$

$$r \wedge l_1 = [r, 0] \wedge [\infty] = (r)$$

and

$$l_3 = (q \wedge l_2) \vee\ (r \wedge l_1) = \left[r,\ (q-r)(q-p)^{-1}\right].$$

And since $s \wedge l_2 = \left((s-p)^{-1},\ s(s-p)^{-1}\right)$ any line $[x, y]$ passing through $s \wedge l_2$ has a form

$$y = (s-x)(s-p)^{-1} \tag{3}$$

Similarly any line $[x, y]$ passing through $p \wedge l_3$ has the form

$$y = (p-x)\left((p-r)^{-1}\left((q-r)(q-p)^{-1}\right)\right) \tag{4}$$

and any line $[x, y]$ passing through $q \wedge l_1 = (q)$ has the form

$$x = q \tag{5}$$

Since $s \wedge l_2,\ p \wedge l_3$ and $q \wedge l_1$ are collinear, from the equations (3), (4) and (5)

$$(s-q)(s-p)^{-1} = (p-q)\left((p-r)^{-1}\left((q-r)(q-p)^{-1}\right)\right)$$

is obtained. Then

$$(p-q)^{-1}\left((s-q)(s-p)^{-1}\right) = (p-r)^{-1}\left((q-r)(q-p)^{-1}\right),$$

and substituting $s - q = (s-p) + (p-q)$ and $q - r = (q-p) + (p-r)$ by simple calculations we arrive at the equality

$$\left((p-q)^{-1} - (p-s)^{-1}\right)\left((p-q)^{-1} - (p-r)^{-1}\right)^{-1} = -1,$$

which is equivalent to $H(p, q, r, s)$.

If $s = \tilde{\infty}$ we utilize the same computations with the exception $s \wedge l_2 = (0, 1)$. In this case any line passing through $s \wedge l_2$ has the form $y = 1$ and using (4), (5) we obtain

$$1 = (p-q)\left((p-r)^{-1}\left((q-r)(q-p)^{-1}\right)\right).$$

So $(p-r)(p-q)^{-1} = (q-r)(q-p)^{-1}$ and then $(q-r)^{-1}(p-r) = -1$. Therefore $H(p, q, r, s)$. Other cases (i.e. $r = \tilde{\infty}$ or $q = \tilde{\infty}$ or $p = \tilde{\infty}$) can be shown by similar calculations, and the proof is complete, the converse following from Theorem 3.1.

□

Lemma 3.3. *The transformations t_u l_u $(u \neq 0)$ and i are projectivities of $\mathcal{L}_{(0,0)}$.*
Proof. By the calculations,

$$t_u = \sigma(0, [\infty], (1,0))\,\sigma((1,0), \tilde{\infty}, (1,-u))\,\sigma((1,-u), [\infty], 0)$$

$$l_u = \sigma(0, [\infty], (1,1))\,\sigma\left((1,1), [0,0], \left(1, u^{-1}\right)\right)\sigma\left(\left(1, u^{-1}\right), [\infty], 0\right)$$

$$i = \sigma(0, [\infty], (1,1))\,\sigma((1,1), [0,0], (1))\,\sigma((1), [1], 0)$$

are obtained and these complete the proof. □

We denote by $T\left(\mathcal{L}_{(0,0)}\right)$ the group of transformations of $\mathcal{L}_{(0,0)}$ generated by $\{t_u\} \cup \{l_u\} \cup \{i\}$. Note that since $r_u = i l_{u^{-1}} i$ the transformation

$$r_u : [x,0] \to [xu,0],\ \tilde{\infty} \to \tilde{\infty}$$

is a projectivity of $\mathcal{L}_{(0,0)}$ and element of $T\left(\mathcal{L}_{(0,0)}\right)$. And we denote by $G\left(\mathcal{L}_{(0,0)}\right)$ the group of all projectivities of $\mathcal{L}_{(0,0)}$.
Lemma 3.4. $T\left(\mathcal{L}_{(0,0)}\right)$ *is a triply transitive subgroup of* $G\left(\mathcal{L}_{(0,0)}\right)$.
Proof. By Lemma 3.3, $T\left(\mathcal{L}_{(0,0)}\right)$ is subgroup of $G\left(\mathcal{L}_{(0,0)}\right)$. Therefore it must be shown that there exists a transformation Ψ in $T\left(\mathcal{L}_{(0,0)}\right)$ which transforms the distinct lines $a, b, c \in \mathcal{L}_{(0,0)}$ to $0, 1, \tilde{\infty}$, respectively. We give the proof in three cases:

Case 1: If $c = \tilde{\infty}$, then $\Psi = l_{(b-a)^{-1}} t_{-a}$ since

$$l_{(b-a)^{-1}} t_{-a}(a) = l_{(b-a)^{-1}}(a-a) = l_{(b-a)^{-1}}(0) = 0$$

$$l_{(b-a)^{-1}} t_{-a}(b) = l_{(b-a)^{-1}}(b-a) = (b-a)^{-1}(b-a) = 1$$

$$l_{(b-a)^{-1}} t_{-a}(c) = l_{(b-a)^{-1}} t_{-a}(\tilde{\infty}) = l_{(b-a)^{-1}}(\tilde{\infty}) = \tilde{\infty}.$$

Case 2: If $c = 0$, then applying i we can return to the case 1.

Case 3: If $c \neq \tilde{\infty}$ and $c \neq 0$, then applying t_{-c} we can return to the case2. □.

Theorem 3.5. $G\left(\mathcal{L}_{(0,0)}\right) = T\left(\mathcal{L}_{(0,0)}\right)$
Proof. From Lemma 3.4 we must only show that $G\left(\mathcal{L}_{(0,0)}\right) \subset T\left(\mathcal{L}_{(0,0)}\right)$. Let

$$\mu = \prod_{i=0}^{n-1} \sigma(P_{i+1}, l_i, P_i),\ P_0 = P_n = (0,0).$$

There is a line l such that l$\neq l_i$ and l$\notin P_i$ for all i. Then

$$\mu = \prod_{i=0}^{n-1} \sigma((0,0), l, P_{i+1})\,\sigma(P_{i+1}, l_i, P_i)\,\sigma(P_i, l, (0,0))$$

and for this reason it suffices to show that

$$\sigma((0,0), l, P_{i+1})\,\sigma(P_{i+1}, l_i, P_i)\,\sigma(P_i, l, (0,0)) \in T\left(\mathcal{L}_{(0,0)}\right).$$

Thus we thus consider a general element

$$\sigma((0,0), l, P'')\,\sigma(P'', d, P')\,\sigma(P', l, (0,0)). \tag{6}$$

There are two cases, $d \notin (0,0)$ and d$\in (0,0)$.

Case 1: Let d$\notin (0,0)$. Then

$$\sigma\left(P'', d, P'\right) = \sigma\left(P'', d, (0,0)\right)\sigma\left((0,0), d, P'\right)$$

and (6) becomes to

$$\sigma\left((0,0), l, P''\right)\sigma\left(P'', d, (0,0)\right)\sigma\left((0,0), d, P'\right)\sigma\left(P', l, (0,0)\right)$$

of which first two and last two factor forms to

$$\eta = \sigma\left((0,0), l, P''\right)\sigma\left(P'', d, (0,0)\right).$$

Thus it suffices to show that $\eta \in T\left(\mathcal{L}_{(0,0)}\right)$. Since $T\left(\mathcal{L}_{(0,0)}\right)$ is triply transitive on $\mathcal{L}_{(0,0)}$ there exists a $\sigma \in T\left(\mathcal{L}_{(0,0)}\right)$ such that $\sigma(r) = 0,\ \sigma(s) = 1$ and $\sigma(t) = \tilde{\infty}$. And now it suffices to show that $\eta\sigma \in T\left(\mathcal{L}_{(0,0)}\right)$. Thus we obtain a new mapping defined by the Fig. 3 where $r = 0,\ s = 1$ and $t = \tilde{\infty}$.

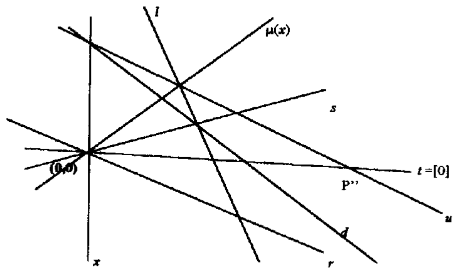

Fig. 3

This mapping will not be altered if the entire configuration in Figure 3 is acted upon by an elation with center $(0,0)$ mapping l to $[\infty]$. So without lose of generality we can take l= $[\infty]$. Thus, since l= $[\infty]$, s and d are concurrent, $d = [1,q]$, and since $P'' \in t = [0] = \tilde{\infty}$, $P'' = (0,s)$. Let $x = [a,0]$. Then

$$\begin{aligned} u &= (x \wedge d) \vee P'' = ([a,o] \wedge [1,q]) \vee (0,s) \\ &= \left((a-1)^{-1} q,\ a(a-1)^{-1} q\right) \vee (0,s) \\ &= \left[a - s\left(q^{-1}a\right) + sq^{-1},\ s\right] \end{aligned}$$

and

$$\begin{aligned} u \wedge l &= \left[a - s\left(q^{-1}a\right) + sq^{-1},\ s\right] \wedge [\infty] \\ &= \left(a - s\left(q^{-1}a\right) + sq^{-1}\right). \end{aligned}$$

Finally

$$\begin{aligned} \eta(x) &= (0,0) \vee (u \wedge l) = (0,0) \vee \left(a - s\left(q^{-1}a\right) + sq^{-1}\right) \\ &= \left[a - s\left(q^{-1}a\right) + sq^{-1},\ 0\right]. \end{aligned}$$

Consequently, we have

$$\eta(x) = t_{sq^{-1}} l_s l_{s^{-1}-q^{-1}}(x).$$

which is the desired result.

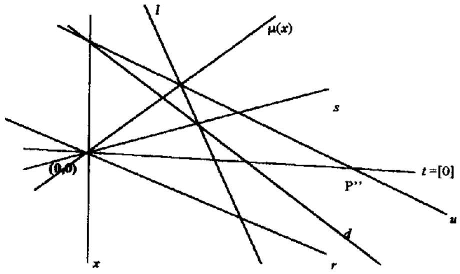

Fig. 4

Case 2: Let $d \in (0,0)$. Then we must show that $\mu \in T\left(\mathcal{L}_{(0,0)}\right)$, where

$$\mu = \sigma\left((0,0), l, P''\right)\sigma\left(P'', d, P'\right)\sigma\left(P', l, (0,0)\right)$$

is the mapping defined by the Fig. 4. Because of the same reasons with Case 1, to take $d = 0$, $s = 1$, $t = \tilde{\infty} = [0]$ and $l = [\infty]$ is not a lose generality. Since $P' \in t = [0]$, $P' = (0, s)$ and since $P'' \in s = 1$, $P'' = (q, q)$. If $x = [a, 0]$, $e = P' \vee (x \wedge l) = (0, s) \vee (a) = [a, s]$ and

$$f = P'' \vee (d \wedge e) = (q, q) \vee \left(-a^{-1}s, 0\right) = \left[q\left(q + a^{-1}s\right)^{-1}, \left(q\left(q + a^{-1}s\right)^{-1}\right)\left(a^{-1}s\right)\right].$$

So

$$\mu(x) = (f \wedge l) \vee (0,0) = \left(q\left(q + a^{-1}s\right)^{-1}\right) \vee (0,0) = \left[q\left(q + a^{-1}s\right)^{-1}, 0\right]$$

and therefore we have the desired result

$$\mu(x) = l_q i t_q r_s i(x).$$

□

Now we can extend the definition of the cross-ratio of lines which are passing through $(0,0)$ to whole Moufang plane as follows:

Let p, q, r, s be distinct lines passing through the point P. There are three cases:

i) If $P \notin [\infty]$, considering the perspectivity $\sigma(0, [\infty], P)$, we have

$$\langle p, q : r, s\rangle = \langle p', r' : s', q'\rangle$$

where

$$p' = \sigma(p) = (p \wedge [\infty]) \vee 0,\ q' = \sigma(q) = (q \wedge [\infty]) \vee 0,$$
$$r' = \sigma(r) = (r \wedge [\infty]) \vee 0,\ s' = \sigma(s) = (s \wedge [\infty]) \vee 0.$$

ii) If $P \in [\infty]$, $P \neq (\infty)$, then applying the perspectivity $\sigma(0, [1], P)$, we have

$$\langle p, q : r, s\rangle = \langle p', r' : s', q'\rangle,$$

where

$$p' = \sigma(p) = (p \wedge [1]) \vee 0,\ q' = \sigma(q) = (q \wedge [1]) \vee 0,$$
$$r' = \sigma(r) = (r \wedge [1]) \vee 0,\ s' = \sigma(s) = (s \wedge [1]) \vee 0.$$

iii) If $P \in [\infty]$, $P = (\infty)$, then considering the perspectivity $\sigma(0, [0,1], P)$, we have

$$\langle p, q : r, s\rangle = \langle p', r' : s', q'\rangle,$$

where

$$p' = \sigma(p) = (p \wedge [0,1]) \vee 0,\ \ q' = \sigma(q) = (q \wedge [0,1]) \vee 0,$$
$$r' = \sigma(r) = (r \wedge [0,1]) \vee 0,\ \ s' = \sigma(s) = (s \wedge [0,1]) \vee 0.$$

Now we can give the final theorem:

Theorem 3.6. *If P, Q, R, S are distinct collinear points and p,q,r,s are distinct concurrent lines such that $P \in p,\ Q \in q, R \in r, S \in s$, then $\langle p, q : r, s\rangle = (P, Q : R, S)$.*

Proof. We can denote, by a, the line incident with all of P, Q, R, S and, by A, the point on which all of the lines p, q, r and s pass through. Then there are four cases:

Case 1: If $A = (\infty)$ and $a = [0, 0]$, then p, q, r, s are in form $p = [p]$, $q = [q]$, $r = [r]$, $s = [s]$ and $P = (p, 0)$, $Q = (q, 0)$, $P = (r, 0)$, $P = (s, 0)$. If one of the lines p, q, r, s is $\tilde{\infty}$ then one of the points P, Q, R, S is (0). Thus

$$\langle p, q : r, s\rangle = \left(p^{-1}, q^{-1} : r^{-1}, s^{-1}\right) = (p, q : r, s) = (P, Q : R, S)$$

Case 2: If $A = (\infty)$ and $a \neq [0, 0]$ the proof follows by case 1, considering the perspectivity $\sigma([0, 0], (\infty), a)$.

Case 3: If $A \neq (\infty)$ and $(\infty) \notin a$ the proof follows by previous two cases, considering the perspectivity $\sigma((\infty), a, A)$.

Case 4: If $A \neq (\infty)$ and $(\infty) \in A$, we can take a line b such that $(\infty), A \notin b$. Considering the perspectivity $\sigma(b, A, a)$, the proof follows by case 3. □

As a consequence of the last theorem we can give the following statement:

If P, Q, R and S are distinct collinear points and p, q, r and s are distinct concurrent lines such that $P \in p,\ Q \in q, R \in r, S \in s$, then $H(p, q, r, s) = h(P, Q, R, S)$.

References

[1] Blunck,A, *Cross-ratios in Moufang Planes*, J. Geom., 40, (1991), pp.20-25.

[2] Çiftçi,S., *On 6-figures in Moufang projective planes*, Commun. Fac. Sci. Univ. Ank. Series A_1, (1989), pp.21-28.

[3] Ferrar, J.C., *Cross-ratios in projective and affine planes*, in: Plaumann, P., Strambach: Geometry-von Staudt's Point of viev (Proceedings Bad Windsheim 1980), Reidel, Dordrecht, (1981), pp. 101-125.

[4] Pickert,G., *Projective Ebenen*, Springer, Berlin, (1995).

[5] Schleiermacher,A., *Duppelverhaltnisse auf einer Geraden in einer Moufangebene (char $\neq 2$)*, Indag. Math., 27, (1965), pp.482-496.

Uludag University,
Faculty of Sciences and Art,
Department of Mathematics,
Görükle/BURSA
TURKEY