On Conformally Flat Pseudosymmetric Spaces

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Dedicated to Prof.Dr. Constantin UDRIŞTE
on the occasion of his sixtieth birthday

Abstract

In a recent paper [1] M. C. Chaki introduced and studied a type of non-flat Riemannian space $(M^n, g)(n \geq 2)$ whose curvature tensor R^h_{ijk} satisfies the condition

$$R^h_{ijk,l} = 2\lambda_l R^h_{ijk} + \lambda^h R_{lijk} + \lambda_i R^h_{ilk} + \lambda_k R^h_{ijl} \tag{1}$$

where λ_i is a non-zero vector and comma denotes covarient differentiation with respect to the metric g_{ij}. Such a space was called by him a pseuso symmetric space, the vector λ_i was called its associated vector and an n-dimensional space of this kind has been denoted $(PS)_n$. Tarafder[2] proved that a conformally flat $(PS)_n$ $(n \geq 3)$ with non-zero constant scalar curvature is a subprojective space in the sense of Kagan[3], if the associated vector is gradient. In the present paper we obtain the above result without assuming any restriction on the scalar curvature. Among others it is shown that a conformally flat $(PS)_n$ can be expressed as a warped product $I \times e^q M^*$ where M^* is an Einstein space and such space is a space of quasi-constant curvature [4].

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1 Conformally flat $(PS)_n$ $(n \geq 3)$

It is known [1] that a conformally flat $(PS)_n$ $(n \geq 3)$ can not be of zero scalar curvature and also it is known [2] that in a conformally flat $(PS)_n$

$$R_{ij} = \frac{R-t}{n-1} g_{ij} + \frac{nt-R}{(n-1)\lambda_p \lambda^p} \lambda_i \lambda_j$$

where R denotes the scalar curvature and t is a scalar.

The above expression can be written as

$$R_{ij} = \alpha g_{ij} + \beta v_i v_j \tag{1.1}$$

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where $\alpha = \dfrac{R-t}{n-1}$, $\beta = \dfrac{nt-R}{n-1}$ are two scalars and $v_i = \dfrac{\lambda_i}{\sqrt{\lambda_i \lambda^i}}$ is a unit vector. On the otherhand, a conformally flat space is conformally symmetric, that is, $C^h_{ijk,l} = 0$. The above equation is equivalent to

$$R_{jl,k} - R_{jk,l} = \frac{1}{2(n-1)}(g_{jl}R_{,k} - g_{jk}R_{,l}). \tag{1.2}$$

The relation (1.1) implies

$$R_{ij,k} = \alpha_k g_{ij} + \beta_k v_i v_j + \beta(v_j v_{i,k} + v_i v_{j,k}), \text{ where } \alpha_{,k} = \alpha_k \text{ and } \beta, k = \beta_k. \tag{1.3}$$

Substituting (1.3) into (1.2) we obtain

$$\begin{aligned} \alpha_k g_{jl} &+ \beta_k v_j v_l + \beta(v_l v_{j,k} + v_j v_{l,k}) - \alpha_l g_{jk} - \beta_l v_j v_k - \beta(v_k v_{j,l} + v_j v_{k,l}) \\ &= \frac{1}{2(n-1)}(g_{jl}R_k - g_{jk}R_l) \end{aligned} \tag{1.4}$$

where $R_k = R_{,k}$.

Since $v^i v_i = 1$ and $(v_{i,k})v^i = 0$, so by transvecting with g^{jl}, (1.4) reduces to

$$(n-1)\alpha_k + \beta_k - (\beta_a v^a)v_k - \beta(v_k v^a_{,a} + v^a v_{k,a}) = \frac{1}{2} R_k. \tag{1.5}$$

Transvecting (1.4) with v^j we obtain

$$(\alpha_k v_l - \alpha_l v_k) + (\beta_k v_l - \beta_l v_k) + \beta(v_{l,k} - v_{k,l}) = \frac{1}{2(n-1)}(v_l R_k - v_k R_l). \tag{1.6}$$

Transvecting again with v^l we have

$$(\alpha_k + \beta_k) - (\alpha_a v^a)v_k - (\beta_a v^a)v_k - \beta v^a v_{k,a} = \frac{1}{2(n-1)}\{R_k - (v^a R_a)v_k\}$$

Substituting this into (1.5) we find

$$(n-2)\alpha_k - \beta v_k v^a_{,a} + (\alpha_a v^a)v_k - \frac{1}{2(n-1)}v_k(v^a R_a) = \frac{1}{2}\,\frac{n-2}{(n-1)}R_k \tag{1.7}$$

Transvecting (1.7) with v^k, we get

$$(n-1)(\alpha_a v^a) - \beta v^a_{,a} = \frac{1}{2}(R_a v^a).$$

Thus (1.7) reduces to

$$R_k = \lambda v_k + 2(n-1)(\alpha_k - \mu v_k) \tag{1.8}$$

where $\lambda = R_a v^a$ and $\mu = \alpha_a v^a$.

Substituting this into (1.6), we obtain

$$(\beta_k v_l - \beta_l v_k) + \beta(v_{l,k} - v_{k,l}) = 0. \tag{1.9}$$

Now if v_i is gradient, that is, $v_{l,k} - v_{k,l} = 0$, then

$$\beta_k v_l - \beta_l v_k = 0. \; That \; is, \tag{1.10}$$

$$\beta_k = a v_k \;\; where \; a \; is \; a \; scalar. \tag{1.10a}$$

Now by (1.8), (1.9) and (1.10) the equation (1.4) reduces to

$$\beta(v_l v_{j,k} - v_k v_{j,l}) = \frac{1}{2(n-1)}\phi(v_k g_{jl} - v_l g_{jk})$$

where $\phi = \lambda - 2(n-1)\mu$.

Transvecting the above equation with v^l and using $v_{j,l} = v_{l,j}$, we get

$$v_{j,k} = \frac{1}{2(n-1)} \frac{\phi}{\beta} (v_k v_j - g_{jk}). \tag{1.11}$$

Let us consider the scalar function

$$f = \frac{1}{2(n-1)} \frac{\phi}{\beta} \neq 0.$$

We have

$$f_k = -\frac{1}{2(n-1)} \frac{\phi}{\beta^2} \beta_k + \frac{1}{2(n-1)} \frac{\phi_k}{\beta} \text{ where } f_k = f_{,k} \text{ and } \phi_k = \phi_{,k}.$$

Again (1.8) implies

$$R_{k,j} = \phi_j v_k + \phi v_{k,j} + 2(n-1)(\alpha_{k,j} - \alpha_{j,k})$$

from which we get $\phi_j v_k = \phi_k v_j$, that is, $\phi_k = A v_k$, where A is a scalar function.

Thus form (1.10a) and (1.12) $f_k = B v_k$ where

$$B = \frac{1}{2(n-1)\beta} \left(-\frac{\phi_a}{\beta} + A\right)$$

Using (1.12), it is easy to show that $\omega_i = \frac{1}{2(n-1)\beta}\phi v_i$ is a gradiant vector field. In fact, $\omega_{i,j} = v_i f_j + f v_{ij} = \beta v_j v_i + f v_{ij} = \omega_{j,i}$. Thus (1.11) can be written as follows: $v_{j,k} = -f g_{jk} + \omega_k v_j$ where ω_k is gradient.

Hence v_i is a concircular vector field. Since $f \neq 0$, v_i is a proper concircular vector.

Hence λ_i is a proper concircular vector field.

It is known [3] that if a conformally flat space admits a proper concircular vector field, then the space is a subprojective space in the sence of Kagan. Thus we can state

Theorem 1. *If the associated vector of a conformally flat $(PS)_n$ is gradient, then the space is a subprojective space.*

In [6] K. Yano proved that in order that a Riemannian space admits a concircular vector field, it is necessary and sufficient that there exists a coordinate system with respect to which the fundamental quadratic differential form may be written in the form

$$ds^2 = (dx^1)^2 + c^q g^*_{\alpha\beta} dx^\alpha dx^\beta$$

where $g^*_{\alpha\beta} = g^*_{\alpha\beta}(x^\nu)$ are the functions of x^γ only (α, β, $\gamma = 2, 3..........n$) and $q = q(x^1) \neq$ constant is a function of x^1 only. Since conformally flat $(PS)_n$ admits proper concircular vector field v_i the space under consideration is the warped product $1 \times e^q M^*$ where (M^*, g^*) is an $(n-1)$-dimensional Riemannian space. Gebarowski [6] proved that the warped product $1 \times e^q M^*$ satisfies (1.2) iff M^* is an Einstein space. Thus we state the following theorem :

Theorem 2. *A conformally flat $(PS)_n$ is the warped product $1 \times e^q M^*$ where M^* is an Einstein space.*

A conformally flat Riemannian space is said to be of quasi-constant curvature [4] if the curvature tensor R_{hijk} is given by

$$R_{hijk} = a(g_{hj}g_{ik} - g_{hk}g_{ij}) + b(g_{hj}\theta_i\theta_k - g_{hk}\theta_i\theta_j - g_{ij}\theta_h\theta_k + g_{ik}\theta_h\theta_j) \tag{1.11}$$

where a and b are differentiable functions and θ_i is a unit vector. Since our space is conformally flat, the curvature tensor is given by

$$R_{hijk} = \frac{1}{n-2}(R_{hk}g_{ij} - R_{hj}g_{ik} + R_{ij}g_{hk} - R_{ik}g_{hj}) - \frac{R}{(n-1)(n-2)}(g_{ij}g_{hk} - g_{ik}g_{hj})$$

Now on account of (1.1) the above equation reduces to (1.11), where

$$\theta_i = v_i, \; a = \frac{R}{(n+1)(n-2)} - \frac{2\alpha}{(n-2)} \text{ and } b = -\frac{\beta}{n-2}$$

Hence we obtain

Theorem 3. *A conformally flat $(PS)_n$ is a space of quasi-constant curvature.*

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