# Symmetric Curvature-Like Tensors on a Semi-Definite Kähler Manifold

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***Dedicated to Prof.Dr. Constantin UDRIŞTE on the occasion of his sixtieth birthday***

**Abstract**

The purpose of this paper is to investigate properties and relations of symmetric or locally symmetric semi-definite Kähler manifolds.

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**Key words**: Semi-definite Kähler manifold, concircular curvature tensor, projective curvature tensor, second curvature-like tensor, locally symmetric, symmetric

## 1 Introduction

As is well known, there exist some special tensors on the Kähler manifold extending the Riemannian curvature tensor on the Riemannian manifold, for example, the concircular, projective or conformal curvature tensor. In his paper [3], Bochner introduced three kinds of curvature-like tensors on the Kähler manifold which are closely related to the projective curvature tensor, the concircular curvature tensor and the conformal curvature tensor, and calculated the Betti numbers on a compact Kähler manifold under suitable conditions for the above curvature-like tensors. See Yano and Bochner [7]. The first tensor is called the Weyl curvature tensor and the third tensor is known as the Bochner curvature tensor. On the other hand, in their paper [4] Matsumoto and Tanno treated Kähler manifolds with parallel Bochner curvature tensor.

In this note, we investigate properties of curvature-like tensors $H$, $Z$ and $W$ on a semi-definite Kähler manifold.

Let $(M,\ g)$ be an $n(\geqq 2)$-dimensional semi-definite Kähler manifold with Kähler connection $\nabla$. Let $R$ or $S$ be the Riemannian curvature tensor or the Ricci tensor with components $K_{\bar{i}jk\bar{l}}$ or $S_{i\bar{j}}$ and let $r$ be the scalar curvature. We denote by $H$ the second curvature-like tensor on $M$ with components $H_{\bar{i}jk\bar{l}}$ defined by

$$H_{\bar{i}jk\bar{l}} = K_{\bar{i}jk\bar{l}} - \frac{1}{2(n+1)}\{\epsilon_j(\delta_{ji}S_{k\bar{l}} + \delta_{jl}S_{k\bar{i}}) + \epsilon_k(\delta_{ki}S_{j\bar{l}} + \delta_{kl}S_{j\bar{i}})\},$$

$Z$ the concircular curvature tensor on $M$ with components $Z_{\bar{i}jk\bar{l}}$ defined by

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$$Z_{\bar{i}jk\bar{l}} = K_{\bar{i}jk\bar{l}} - \frac{r}{n(n+1)}\epsilon_{jk}(\delta_{ji}\delta_{kl} + \delta_{jl}\delta_{ki})$$

and $W$ the projective curvature tensor on $M$ with components $W_{\bar{i}jk\bar{l}}$ defined by

$$W_{\bar{i}jk\bar{l}} = K_{\bar{i}jk\bar{l}} - \frac{1}{2(n+1)}\epsilon_j(\delta_{ji}S_{k\bar{l}} + \delta_{jl}S_{k\bar{i}}).$$

The Riemannian curvature tensor $R$ (resp. the second curvature-like tensor $H$, the concircular curvature tensor $Z$ or the projective curvature tensor $W$) is said to be *parallel* if it satisfies $\nabla R = 0$ (resp. $\nabla H = 0$, $\nabla Z = 0$ or $\nabla W = 0$). If the Riemannian curvature tensor $R$ (resp. the second curvature-like tensor $H$, the concircular curvature tensor $Z$ or the projective curvature tensor $W$) is parallel, then $M$ is said to be *locally symmetric* (resp. *$H$-symmetric*, *$Z$-symmetric* or *$W$-symmetric*). Simply $M$ is called to be *symmetric* when $M$ is $H$-symmetric, $Z$-symmetric or $W$-symmetric. The class of semi-definite $H$-symmetric (resp. $Z$-symmetric or $W$-symmetric) Kähler manifolds contains semi-definite Kähler manifolds with vanishing second curvature-like tensor (resp. the concircular curvature tensor or the projective curvature tensor). From these facts it seems to be natural to consider properties of semi-definite symmetric Kähler manifolds, and to study relations of locally symmetric semi-definite Kähler manifolds and symmetric semi-definite Kähler manifolds.

The purpose of this paper is to investigate properties and relations of symmetric or locally symmetric semi-definite Kähler manifolds.

## 2 Semi-definite Kähler manifolds

This section is concerned with recalling basic formulas on a semi-definite Kähler manifold.

Let $M$ be an $n(\geqq 2)$-dimensional semi-definite Kähler manifold with the semi-definite Kähler metric tensor $g$ and the almost complex structure $J$. From the semi-definite Kähler structure $\{g, J\}$, it follows that $J$ is integrable and the index of $g$ is even, say $2s(0 \leqq s \leqq n)$. In the case where the index $2s$ is contained in the range $0 < s < n$, the structure $\{g, J\}$ is said to be *indefinite Kähler structure* and, in particular, in the case where $s = 0$ or $n$, it is said to be *definite Kähler structure*.

In this section, we shall consider $M$ an $n(\geqq 2)$-dimensional connected semi-definite Kähler manifold of index $2s$, $0 \leqq s \leqq n$. Then a local unitary frame field $\{U_j\} = \{U_1, \cdots, U_n\}$ on a neighborhood of $M$ can be chosen. This is a complex linear frame which is orthonormal with respect to the semi-definite Kähler metric $g$ of $M$, that is, $g(U_j, U_k) = \epsilon_j\delta_{jk}$, where

$$\epsilon_j = -1 \text{ or } 1 \text{ according as } 0 \leqq j \leqq s \text{ or } s+1 \leqq j \leqq n.$$

Its dual frame field $\{\omega_j\} = \{\omega_1, \cdots, \omega_n\}$ consists of complex valued 1-forms of $(1,0)$ on $M$ such that $\omega_j(U_k) = \epsilon_j\delta_{jk}$ and $\omega_1, \cdots, \omega_n, \bar{\omega}_1, \cdots, \bar{\omega}_n$ are linearly independent. Thus the natural extension $g^c$ of the semi-definite Kähler metric $g$ of $M$ can be expressed as $g^c = 2\sum_j \epsilon_j\omega_j \otimes \bar{\omega}_j$. Associated with the frame field $\{U_j\}$, there exist complex valued forms $\omega_{ik}$, where the indices $i$ and $k$ run over the range

1, $\cdots$ , $n$. They are usually called *connection forms* on $M$ such that they satisfy the structure equations of $M$ :

$$d\omega_i + \sum_j \epsilon_j \omega_{ij} \wedge \omega_j = 0, \quad \omega_{ij} + \bar{\omega}_{ji} = 0,$$

(2.1)
$$d\omega_{ij} + \sum_k \epsilon_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij},$$

$$\Omega_{ij} = \sum_{k,l} \epsilon_{kl} K_{\bar{i}jk\bar{l}} \omega_k \wedge \bar{\omega}_l,$$

where $\epsilon_{k\ldots l} = \epsilon_k \ldots \epsilon_l$ and $\Omega = (\Omega_{ij})$ (*resp.* $K_{\bar{i}jk\bar{l}}$) denotes the curvature form (*resp.* components of the semi-definite Riemannian curvature tensor $R$) of $M$ in Kobayashi and Nomizu [5]. The second formula of (2.1) means the skew-Hermitian symmetricity of $\Omega_{ij,}$ which is equivalent to the symmetric condition

$$K_{\bar{i}jk\bar{l}} = \bar{K}_{\bar{j}il\bar{k}}. \tag{2.2}$$

Moreover, substitution the third equation of (2.1) into the exterior differential of the first equation of (2.1), the first Bianchi identity

$$\sum_j \epsilon_j \Omega_{ij} \wedge \omega_j = 0 \tag{2.3}$$

is given. It implies further symmetric relations

$$K_{\bar{i}jk\bar{l}} = K_{\bar{i}kj\bar{l}} = K_{\bar{l}kj\bar{i}} = K_{\bar{l}jk\bar{i}}. \tag{2.4}$$

Now, relative to the frame field chosen above, the Ricci tensor $S$ of $M$ can be expressed as follows :

$$S = \sum_{i,j} \epsilon_{ij} (S_{i\bar{j}} \omega_i \otimes \bar{\omega}_j + S_{\bar{i}j} \bar{\omega}_i \otimes \omega_j), \tag{2.5}$$

where $S_{i\bar{j}} = \sum_k \epsilon_k K_{\bar{k}ki\bar{j}} = S_{\bar{j}i} = \bar{S}_{\bar{i}j.}$ The scalar curvature $r$ of $M$ is also given by

$$r = 2 \sum_j \epsilon_j S_{j\bar{j}}. \tag{2.6}$$

An $n$-dimensional semi-definite Kähler manifold $M$ is said to be *Einstein*, if the Ricci tensor $S$ is given by

$$S_{i\bar{j}} = \frac{r}{2n} \epsilon_i \delta_{ij}, \quad S = \frac{r}{2n} g. \tag{2.7}$$

The components $K_{\bar{i}jk\bar{l}m}$ and $K_{\bar{i}jk\bar{l}\bar{m}}$ (*resp.* $S_{i\bar{j}k}$ and $S_{i\bar{j}\bar{k}}$) of the covariant derivative of the Riemannian curvature tensor $R$ (*resp.* the Ricci tensor $S$) are obtained by

$$\begin{aligned}(2.8)\qquad &\sum_m \epsilon_m (K_{\bar{i}jk\bar{l}m}\omega_m + K_{\bar{i}jk\bar{l}\bar{m}}\bar{\omega}_m) = dK_{\bar{i}jk\bar{l}} \\ &-\sum_m \epsilon_m (K_{\bar{m}jk\bar{l}}\bar{\omega}_{mi} + K_{\bar{i}mk\bar{l}}\omega_{mj} + K_{\bar{i}jm\bar{l}}\omega_{mk} + K_{\bar{i}jk\bar{m}}\bar{\omega}_{ml}),\end{aligned}$$

and

$$(2.9)\qquad \sum_k \epsilon_k (S_{i\bar{j}k}\omega_k + S_{i\bar{j}\bar{k}}\bar{\omega}_k) = dS_{i\bar{j}} - \sum_k \epsilon_k (S_{k\bar{j}}\omega_{ki} + S_{i\bar{k}}\bar{\omega}_{kj}).$$

The second Bianchi identity is given by the exterior derivative of the third equation of the structure equations (2.1), this is, we have

$$(2.10)\qquad d\Omega_{ij} = \sum_k \epsilon_k (\Omega_{ik}\wedge\omega_{kj} - \omega_{ik}\wedge\Omega_{kj}).$$

In fact, we find

$$\begin{aligned} d\Omega_{ij} &= \sum_k \epsilon_k d(\omega_{ik}\wedge\omega_{kj}) \\ &= \sum_k \epsilon_k (d\omega_{ik}\wedge\omega_{kj} - \omega_{ik}\wedge d\omega_{kj}) \\ &= \sum_j \epsilon_k \left\{ (\Omega_{ik} - \sum_l \epsilon_l \omega_{il}\wedge\omega_{lk})\wedge\omega_{kj} - \omega_{ik}\wedge(\Omega_{kj} - \sum_l \epsilon_l \omega_{kl}\wedge\omega_{lj}) \right\} \\ &= \sum_j \epsilon_k (\Omega_{ik}\wedge\omega_{kj} - \omega_{ik}\wedge\Omega_{kj}), \end{aligned}$$

where the first equality is derived from the property that $d^2 = 0$, the second one follows from the fact that the complex connection form is an 1-form and the property of the exterior derivative and the third one is derived from the structure equations (2.1). We can regard $\Omega = (\Omega_{ij})$ and $\omega = (\omega_{ij})$ as complex matrices of order $n$. Then (2.10) can be reformed as

$$(2.11)\qquad d\Omega = \Omega\wedge\omega - \omega\wedge\Omega.$$

By the straightforward calculation we have

$$(2.12)\qquad K_{\bar{i}jk\bar{l}m} = K_{\bar{i}jm\bar{l}k},$$

and hence

$$(2.13)\qquad S_{i\bar{j}k} = S_{k\bar{j}i} = \sum_l \epsilon_l K_{\bar{j}ik\bar{l}l.}$$

On the other hand, the exterior differential $dr$ of the scalar curvature $r$ on $M$ is given by

$$(2.14)\qquad dr = \sum_m \epsilon_m (r_m\omega_m + r_{\bar{m}}\bar{\omega}_m).$$

Putting $i = j$ in (2.13) and summing up with respect to $i$, we have

$$\sum_r \epsilon_r S_{r\bar{r}k} = \sum_{r,s} \epsilon_{rs} K_{\bar{r}rk\bar{s}s} = \sum_{r,s} \epsilon_{rs} K_{\bar{r}rs\bar{s}k} = \frac{r_k}{2}. \tag{2.15}$$

Now, a semi-definite Kähler manifold $M$ of constant holomorphic sectional curvature is called a *semi-definite complex space form*. An $n$-dimensional semi-definite complex space form of constant holomorphic sectional curvature $c$ and of index $2s$, $0 \leqq s \leqq n$, is denoted by $M^n_s(c)$. The standard models of semi-definite complex space forms are the following three kinds : the semi-definite complex Euclidean space $\mathbf{C}^n_s$, the semi-definite complex projective space $\mathbf{CP}^n_s(c)$ or the semi-definite complex hyperbolic space $\mathbf{CH}^n_s(c)$, according as $c = 0$, $c > 0$ or $c < 0$. For any integer $s$ ($0 \leqq s \leqq n$), it is seen that they are only complete and simply connected semi-definite complex space forms of dimension $n$ and of index $2s$. The Riemannian curvature tensor $K_{\bar{i}jk\bar{l}}$ of $M^n_s(c)$ is given by

$$K_{\bar{i}jk\bar{l}} = \frac{c}{2}\epsilon_{jk}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}). \tag{2.16}$$

# 3 Curvature-like tensors

This section is concerned with curvature-like tensors $H$, $Z$ and $W$ on a semi-definite Kähler manifold.

Let $M$ be an $n(\geqq 2)$-dimensional semi-definite Kähler manifold of index $2s$ with a semi-definite Kähler metric $g$. The *second curvature-like tensor* $H$ with components $H_{\bar{i}jk\bar{l}}$ is defined by

$$H_{\bar{i}jk\bar{l}} = K_{\bar{i}jk\bar{l}} - \frac{1}{2(n+1)}\{\epsilon_j(\delta_{ji}S_{k\bar{l}} + \delta_{jl}S_{k\bar{i}}) + \epsilon_k(\delta_{ki}S_{j\bar{l}} + \delta_{kl}S_{j\bar{i}})\}. \tag{3.1}$$

As is easily seen (cf. Yano and Bocher [7] in the definite case), the semi-definite Kähler manifold $M$ with vanishing second curvature-like tensor $H$ is of constant holomorphic sectional curvature.

On the other hand, let $Z$ be a tensor with components $Z_{\bar{i}jk\bar{l}}$ such that

$$Z_{\bar{i}jk\bar{l}} = K_{\bar{i}jk\bar{l}} - \frac{r}{n(n+1)}\epsilon_{jk}(\delta_{ji}\delta_{kl} + \delta_{jl}\delta_{ki}). \tag{3.2}$$

In the case where the semi-definite Kähler manifold $M$ is Einstein, the tensor $Z$ is eqivalent to the second curvature-like tensor $H$. It is trivial that the semi-definite Kähler manifold $M$ with $Z = 0$ is of constant holomorphic sectional curvature. The tensor $Z$ is the formal analogue to the concircular curvature tensor on the Riemannian manifold [7]. It is said to be the *concircular curvature tensor* on $M$.

Moreover, a tensor $W$ with components $W_{\bar{i}jk\bar{l}}$ is defined by

$$W_{\bar{i}jk\bar{l}} = K_{\bar{i}jk\bar{l}} - \frac{1}{2(n+1)}\epsilon_j(\delta_{ji}S_{k\bar{l}} + \delta_{jl}S_{k\bar{i}}). \tag{3.3}$$

It is the formal analogue to the Weyl projective curvature tensor on the Riemannian manifold. This is introduced by Bochner. It is also called the *projective curvature* *tensor* on $M$. As is easily seen in definite case (cf. Yano and Bochner [7]), the semi-definite Kähler manifold $M$ with vanishing projective curvature tensor is of constant holomorphic sectional curvature.

**Lemma 3.1.** *Let $M$ be an $n(\geqq 2)$-dimensional semi-definite Kähler manifold of index $2s$. Then the second curvature-like tensor $H$ with components $H_{\bar{i}jk\bar{l}}$ satisfies*

$$H_{\bar{i}jk\bar{l}} = H_{\bar{i}kj\bar{l}} = H_{\bar{l}jk\bar{i}} = H_{\bar{l}kj\bar{i}}, \tag{3.4}$$

$$H_{i\bar{j}} = H_{\bar{j}i} = \bar{H}_{j\bar{i}} = \frac{1}{4(n+1)}\{2nS_{i\bar{j}} - r\epsilon_i\delta_{ij}\}, \tag{3.5}$$

*where $H_{i\bar{j}}$ is the components of the Ricci-like tensor of $H$ defined by $H_{i\bar{j}}=\sum_r \epsilon_r\, H_{\bar{r}ri\bar{j}}$.*

**Proof.** Owing to the definition (3.1) of $H$, (3.4) is clear.

Since $S_{i\bar{j}} = \sum_k \epsilon_k K_{\bar{k}ki\bar{j}}$, we have $S_{i\bar{j}} = S_{\bar{j}i} = \bar{S}_{j\bar{i}}$ and hence we obtain $H_{i\bar{j}} = H_{\bar{j}i} = \bar{H}_{j\bar{i}}$.

Moreover, putting $i = j$ in (3.1) and summing up with respect to $j$, we get

$$H_{i\bar{j}} = \frac{1}{4(n+1)}\{2nS_{i\bar{j}} - r\epsilon_i\delta_{ij}\},$$

where we have used (2.6).

It completes the proof. □

**Proposition 3.2.** *Let $M$ be an $n(\geqq 2)$-dimensional semi-definite Kähler manifold of index $2s$. Then the second curvature-like tensor $H$ has vanishing $t$, where $t$ is the scalar-like curvature of $H$ defined by $t=2\sum_{r,s} \epsilon_{rs}\, H_{\bar{r}rs\bar{s}}$.*

**Proof.** Putting $i = j$ in (3.5) and summing up $j$, we obtain

$$t = \frac{1}{2(n+1)}(nr - nr) = 0.$$

It completes the proof. □

**Lemma 3.3.** *Let $M$ be an $n(\geqq 2)$-dimensional semi-definite Kähler manifold of index $2s$. Then the concircular curvature tensor $Z$ with components $Z_{\bar{i}jk\bar{l}}$ satisfies*

$$Z_{\bar{i}jk\bar{l}} = Z_{\bar{i}kj\bar{l}} = Z_{\bar{l}jk\bar{i}} = Z_{\bar{l}kj\bar{i}}, \tag{3.6}$$

$$Z_{i\bar{j}} = Z_{\bar{j}i} = \bar{Z}_{j\bar{i}} = S_{i\bar{j}} - \frac{r}{n}\epsilon_i\delta_{ij}, \tag{3.7}$$

*where $Z_{i\bar{j}}$ is the components of the Ricci-like tensor of $Z$ defined by $Z_{i\bar{j}}=\sum_r \epsilon_r\, Z_{\bar{r}ri\bar{j}}$.*

**Proof.** Owing to the definition (3.2) of $Z$, (3.6) is clear.

Since $S_{i\bar{j}} = \sum_k \epsilon_k K_{\bar{k}ki\bar{j}}$, we have $S_{i\bar{j}} = S_{\bar{j}i} = \bar{S}_{j\bar{i}}$ and hence we obtain $Z_{i\bar{j}} = Z_{\bar{j}i} = \bar{Z}_{j\bar{i}}$. Putting $k = l$ in (3.2) and summing up with respect to $k$, we find

$$Z_{i\bar{j}} = S_{i\bar{j}} - \frac{r}{n}\epsilon_i\delta_{ij}.$$

It completes the proof. □

**Proposition 3.4.** *Let $M$ be an $n(\geqq 2)$-dimensional semi-definite Kähler manifold of index $2s$. Then the concircular curvature tensor $Z$ satisfies $u = -r$, where $u$ is the scalar-like curvature of $Z$ defined by $u{=}2\sum_{r,s}\epsilon_{rs}\ Z_{\bar{r}rs\bar{s}}$.*

**Proof.** Putting $i = j$ in (3.7) and summing up $j$, we find

$$u = r - 2r = -r.$$

It completes the proof. □

**Lemma 3.5.** *Let $M$ be an $n(\geqq 2)$-dimensional semi-definite Kähler manifold of index $2s$. Then the projective curvature tensor $W$ with components $W_{\bar{i}jk\bar{l}}$ satisfies*

$$W_{\bar{i}jk\bar{l}} = W_{\bar{l}jk\bar{i}}, \tag{3.8}$$

$$W_{i\bar{j}} = \frac{1}{2}S_{i\bar{j}}, \tag{3.9}$$

*where $W_{i\bar{j}}$ are the components of the Ricci-like tensor of $W$ defined by $W_{i\bar{j}}{=}\sum_r \epsilon_r\ W_{\bar{r}ri\bar{j}}$.*

**Proof.** Owing to the definition (3.3) of $W$, (3.8) is clear.

Since $S_{i\bar{j}} = \sum_k \epsilon_k K_{\bar{k}ki\bar{j}}$, if we put $i = j$ in (3.3) and summing up with respect to $j$, we obtain

$$W_{i\bar{j}} = \frac{1}{2}S_{i\bar{j}}.$$

It completes the proof. □

**Remark 3.1.** Let $M$ be an $n(\geqq 2)$-dimensional semi-definite Kähler manifold of index $2s$. Then the projective curvature tensor $W$ with components $W_{\bar{i}jk\bar{l}}$ has the following properties:

$$W_{\bar{i}jk\bar{l}} \neq W_{\bar{i}kj\bar{l}},$$

$$W_{\bar{i}jk\bar{k}} = \frac{1}{4(n+1)}\{2(2n+1)S_{\bar{i}j} - r\epsilon_j\delta_{ji}\}.$$

**Proposition 3.6.** *Let $M$ be an $n(\geqq 2)$-dimensional semi-definite Kähler manifold of index $2s$. Then the projective curvature tensor $W$ satisfies $2v = r$, where $v$ is the scalar-like curvature of $W$ defined by $v{=}2\sum_{r,s}\epsilon_{rs}\ W_{\bar{r}rs\bar{s}}$.*

**Proof.** Putting $i = j$ in (3.9) and summing up $j$, we deduce

$$v = \frac{1}{2}r.$$

It completes the proof. □

# 4 Symmetric semi-definite Kähler manifolds

Let $M$ be an $n(\geqq 2)$-dimensional semi-definite Kähler manifold of index $2s$ with the Kähler connection $\nabla$. We choose a local frame field $\{U_j\}$ of unitary frames on a neighborhood of $M$. Let $H$ be the second curvature-like tensor with components $H_{\bar{i}jk\bar{l}}$ defined by

$$H_{\bar{i}jk\bar{l}} = K_{\bar{i}jk\bar{l}} - \frac{1}{2(n+1)}\{\epsilon_j(\delta_{ji}S_{k\bar{l}} + \delta_{jl}S_{k\bar{i}}) + \epsilon_k(\delta_{kl}S_{j\bar{i}} + \delta_{ki}S_{j\bar{l}})\}.$$

And let $Z$ be the concircular curvature tensor with components $Z_{\bar{i}jk\bar{l}}$ such that

$$Z_{\bar{i}jk\bar{l}} = K_{\bar{i}jk\bar{l}} - \frac{r}{n(n+1)}\epsilon_{jk}(\delta_{ji}\delta_{kl} + \delta_{jl}\delta_{ki}).$$

Moreover, the projective curvature tensor $W$ with components $W_{\bar{i}jk\bar{l}}$ is defined by

$$W_{\bar{i}jk\bar{l}} = K_{\bar{i}jk\bar{l}} - \frac{1}{2(n+1)}\epsilon_j(\delta_{ji}S_{k\bar{l}} + \delta_{jl}S_{k\bar{i}}).$$

Now, we suppose that the curvature-like tensor $Z$ vanishes identically. It is then easily seen that $M$ is of constant holomorphic sectional curvature. The curvature-like tensor $Z$ (resp. $H$ or $W$) is said to be *parallel* if it satisfies $\nabla Z = 0$ (resp. $\nabla H = 0$ or $\nabla W = 0$), namely, $Z_{\bar{i}jk\bar{m}n} = 0$ (resp. $H_{\bar{i}jk\bar{m}n} = 0$ or $W_{\bar{i}jk\bar{m}n} = 0$). If $Z$ (resp. $H$ or $W$) is parallel, then $M$ is said to be $Z$(resp. $H$ or $W$)-*symmetric.* Simply $M$ is said to be *symmetric* when $M$ is $H$-symmetric, $Z$-symmetric or $W$-symmetric.

Then we can prove **Proposition 4.1.** *Let $M$ be an $n(\geqq 2)$-dimensional semi-definite Kähler manifold. Then it is $H$-symmetric if and only if it is locally symmetric.*
**Proof.** By the definition (3.1) of $H$ and the assumption $\nabla H = 0$, we have

$$K_{\bar{i}jk\bar{l}m} - \frac{1}{2(n+1)}\{\epsilon_j(\delta_{ji}S_{k\bar{l}m} + \delta_{jl}S_{k\bar{i}m}) + \epsilon_k(\delta_{kl}S_{j\bar{i}m} + \delta_{ki}S_{j\bar{l}m})\} = 0. \tag{4.1}$$

Putting $i = m$ in (4.1), and summing up with respect to $i$, we obtain

$$S_{j\bar{l}k} - \frac{1}{4(n+1)}\{(2\epsilon_j + 2\epsilon_k)S_{j\bar{l}k} + r_j\epsilon_k\delta_{kl} + r_k\epsilon_j\delta_{jl}\} = 0,$$

where we have used (2.12), (2.13) and (2.15). From this it follows that

$$4nS_{j\bar{l}k} = r_j\epsilon_k\delta_{kl} + r_k\epsilon_j\delta_{jl}, \tag{4.2}$$

and hence we have $(n-1)r_j = 0$. Since $n \geqq 2$, we get $r_j = 0$. It means that the scalar curvature $r$ is constant and by the above equation (4.2) we have $S_{j\bar{l}k} = 0$, from which, together with (4.1), we have $K_{\bar{i}jk\bar{l}m} = 0$. Thus we get $\nabla R = 0$, that is, $M$ is locally symmetric.

Conversely, differentiating (3.1) covariantly and using the assumption $\nabla R = 0$, we find

$$\begin{aligned} H_{\bar{i}jk\bar{l}m} &= K_{\bar{i}jk\bar{l}m} - \\ &- \frac{1}{2(n+1)}\{\epsilon_j(\delta_{ji}S_{k\bar{l}m} + \delta_{jl}S_{k\bar{i}m}) + \epsilon_k(\delta_{kl}S_{j\bar{i}m} + \delta_{ki}S_{j\bar{l}m})\} = 0. \end{aligned}$$

This implies $\nabla H = 0$.

It completes the proof. □

**Proposition 4.2.** *Let $M$ be an $n(\geqq 2)$-dimensional semi-definite Kähler manifold. Then it is $Z$-symmetric if and only if it is locally symmetric.*

**Proof.** By the definition (3.2) of $Z$ and the assumption $\nabla Z = 0$, we have

$$K_{\bar{i}jk\bar{l}m} - \frac{r_m}{n(n+1)}\epsilon_{jk}(\delta_{ji}\delta_{kl} + \delta_{jl}\delta_{ki}) = 0. \tag{4.3}$$

Putting $i = j$ in (4.3), and summing up with respect to $i$, we obtain

$$S_{k\bar{l}m} = \frac{r_m}{n(n+1)}(n+1)\epsilon_k\delta_{kl}.$$

From this it follows that

$$S_{k\bar{l}m} = \frac{r_m}{n}\epsilon_k\delta_{kl},$$

and hence we have $r_m = 0$. It means that the scalar curvature $r$ is constant and by the above equation (4.3) we have $K_{\bar{i}jk\bar{l}m} = 0$. Thus we get $\nabla R = 0$, that is, $M$ is locally symmetric.

Conversely, differentiating (3.2) covariantly, we have

$$Z_{\bar{i}jk\bar{l}m} = K_{\bar{i}jk\bar{l}m} - \frac{r_m}{n(n+1)}\epsilon_{jk}(\delta_{ji}\delta_{kl} + \delta_{jl}\delta_{ki}).$$

Using the assumption $\nabla R = 0$, we have $r_m = 0$, and hence the above equation yields $\nabla Z = 0$.

It completes the proof. □

**Proposition 4.3.** *Let $M$ be an $n(\geqq 2)$-dimensional semi-definite Kähler manifold. Then it is $W$-symmetric if and only if it is locally symmetric.*

**Proof.** By the definition (3.3) of $W$ and the assumption $\nabla W = 0$, we have

$$K_{\bar{i}jk\bar{l}m} - \frac{1}{2(n+1)}\epsilon_j(\delta_{ji}S_{k\bar{l}m} + \delta_{jl}S_{k\bar{i}m}) = 0. \tag{4.4}$$

Putting $i = m$ in (4.4), and summing up with respect to $i$, we obtain

$$S_{j\bar{l}k} - \frac{1}{2(n+1)}(S_{j\bar{l}k} + r_k\epsilon_j\delta_{jl}) = 0,$$

where we have used (2.12), (2.13) and (2.15). From this it follows that

$$S_{j\bar{l}k} = \frac{1}{2(2n+1)}r_k\epsilon_j\delta_{jl}, \tag{4.5}$$

and hence we have $r_j = 0$. It means that the scalar curvature $r$ is constant and by the above equation (4.5) we have $S_{j\bar{l}k} = 0$, from which together with (4.4) we have $K_{\bar{i}jk\bar{l}m} = 0$. Thus we get $\nabla R = 0$, that is, $M$ is locally symmetric.

Conversely, differentiating (3.3) covariantly and using the assumption $\nabla R = 0$, we have

$$W_{\bar{i}jk\bar{l}m} = K_{\bar{i}jk\bar{l}m} - \frac{1}{2(n+1)}\epsilon_j(\delta_{ji}S_{k\bar{l}m} + \delta_{jl}S_{k\bar{i}m}) = 0.$$

Thus this means $\nabla H = 0$.

It completes the proof. □

**Remark 4.1.** From the proof of Propositions 4.1∼ 4.3, we can see that an $n(\geqq 2)$-dimensional symmetric semi-definite Kähler manifolds have the costant scalar curvature, parallel Ricci tensor $S$ and parallel Riemannian curvature tensor $R$.

Owing to Propositions 4.1 ∼ 4.3, we have the following theorem.

**Theorem 4.4.** *Let $M$ be an $n(\geqq 2)$-dimensional semi-definite Kähler manifold. Then the following are equivalent:*

*(1) $H$-symmetric,*

*(2) $Z$-symmetric,*

*(3) $W$-symmetric,*

*(4) locally symmetric.*

Consequently, from Theorem 4.4, we get the following fact:

Let $M$ be an $n(\geqq 2)$-dimensional semi-definite Kähler manifold. Then it is symmetric if and only if it is locally symmetric.

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