

Distributions on Spray Spaces

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Abstract

A spray S is a 2-homogeneous second order differential equation (SODE) on a manifold M and the pair (M, S) is called a spray space. The curves solutions of this SODE are called geodesics. A distribution D on M is said to be geodesically invariant (g.i.) if for any geodesics whose initial tangent vector is in D it follows that all its tangent vectors belong to D . In §2 we give a characterisation of g.i. distributions on a spray space (M, S) using the Berwald connection induced by S . In particular, one regains the results of A.D. Lewis for linear connections in ([3]). Applications to certain distributions on tangent manifold TM are also considered.

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1 Spray spaces

Let M be a real, smooth C^∞ i.e., finite dimensional manifold and $\tau : TM \rightarrow M$ its tangent bundle. Let $(U, (x^i))$ be a local chart on M . The indices i, j, k, \dots will run from 1 to $n = \dim M$ and the Einstein convention on summation will be implied. One associates to $v \in \tau^{-1}(U)$ the coordinates $(x^i(\tau(u)))$ and (y^i) provided by

$$v_{\tau(v)} = y^i \partial_i, \quad \partial_i := \frac{\partial}{\partial x^i}$$

and TM becomes a smooth orientable manifold.

A *spray* on M is a vector field

$$S = y^i \partial_i - 2G^i(x, y) \dot{\partial}_i, \quad \dot{\partial}_k = \frac{\partial}{\partial y^k},$$

whitch satisfies the condition

$$(1.1) \quad G^i(x, \lambda y) = \lambda^2 G^i(x, y), \quad \lambda > 0.$$

The functions $G^i(x, y)$ are called the local coefficients of the spray. If the condition (1.1) is not satisfied, then S is called a semispray. The pair (M, S) is called a *spray space*.

Example 1.1. Every manifold endowed with a linear connection ∇ is a spray space. If $\Gamma_{jk}^i(x)$ are the local coefficients of ∇ then

$$G^i(x, y) = \frac{1}{2}\Gamma_{jk}^i(x)y^j y^k.$$

Example 1.2. Let (M, L) be a Lagrange space. Here $L : TM \rightarrow \mathbf{R}$ is a regular Lagrangian, i.e., the matrix $g_{ij} = \frac{1}{2}\dot{\partial}_i \dot{\partial}_j L$ is invertible. Let g^{jk} be its inverse. Then

$$G^i(x, y) = \frac{1}{4}g^{ik}((\dot{\partial}_k \partial_h L)y^h - \partial_k L)$$

is in general a semispray. For a general class of Lagrange spaces for which the above semispray reduces to a spray we refer to [2].

Example 1.3. Let (M, F) be a Finsler space. Taking $L = \frac{1}{2}F^2$ it becomes a Lagrange space and the $G^i(x, y)$ from Example 2 is always a spray. Thus any Finsler space is a spray space. Let $\gamma_{ij}(x, y)$ be the local components of the Finsler metric of (M, F) . Then $F^2 = \gamma_{ij}(x, y)y^i y^j$. If one sets

$$\gamma_{jk}^i(x, y) = \frac{1}{2}\gamma^{ih}(\partial_j \gamma_{hk} + \partial_k \gamma_{jh} - \partial_h \gamma_{jk}),$$

then the local coefficients of the spray are

$$G^i(x, y) = \gamma_{jk}^i(x, y)y^j y^k.$$

Example 1.4. Every Riemannian manifold is a spray space since it is a particular Finsler space. The local coefficients of its spray are given as $G^i(x, y) = \gamma_{jk}^i(x)y^j y^k$, where now $\gamma_{jk}^i(x)$ are the Christoffel symbols.

A curve

$$c : t \rightarrow c(t) = (x^i(t)) \in M, \quad t \in I \subseteq \mathbf{R}$$

is said to be a *geodesic of S* if its lift to TM , given as $(x^i(t), \frac{dx^i}{dt}(t))$ is an integral curve of S . The curve c is a geodesic of S if and only if the functions $(x^i(t))$ are solutions of the equations

$$(1.2) \quad \frac{d^2 x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0.$$

2 Distributions on a spray space

We consider a distribution D of rank r on M . This means a smooth mapping $p \rightarrow D_p \subset T_p M$, for $p \in M$ and D_p a subspace of dimension r in $T_p M$. The union \mathcal{D} of all D_p gives a submanifold of dimension $n + r$ in the tangent manifold TM . We denote by $\mathcal{X}(D)$ the set of sections of \mathcal{D} , i.e., $p \rightarrow X_p \in D_p$.

A distribution D is *geodesically invariant* (g.i) if for every geodesic $c : [a, b] \rightarrow M$ such that $c'(a) \in D_{c(a)}$ it follows that $c'(t) \in D_{c(t)}$ for each $t \in (a, b)$. If D is integrable and g.i. one says that D is *totally geodesic*.

Theorem 2.1. *The distribution \mathcal{D} is g.i. if and only if S is tangent to \mathcal{D} .*

Proof. The spray S is tangent to \mathcal{D} if and only if the integral curves of S , when start from \mathcal{D} , remain in \mathcal{D} . This fact is clearly equivalent with the condition that \mathcal{D} is geodesically invariant. \square

Theorem 2.2. *$X \in \mathcal{X}(\mathcal{D})$ if and only if X^v is tangent to \mathcal{D} . Here X^v denotes the vertical lift of X .*

Proof. Let (U, φ) be a local chart on M and a basis X_1, \dots, X_n for $\mathcal{X}(U)$ such that $(X_1(p), \dots, X_r(p))$ is a basis for \mathcal{D}_p , $p \in U$. Then

$$X_i(p) = A_i^j(p) \left. \frac{\partial}{\partial x^j} \right|_p \quad \text{with} \quad \text{rank}(A_i^j(p)) = n.$$

Let (B_j^k) be the inverse of the matrix (A_i^j) . Thus

$$\left. \frac{\partial}{\partial x^j} \right|_p = B_j^k(p) X_k(p).$$

Let be $X \in \mathcal{X}(U)$. Then

$$X = X^i \frac{\partial}{\partial x^i} = X^i B_i^k X_k$$

and it follows that $X \in \mathcal{X}(\mathcal{D})$ if and only if $B_i^a X^i = 0$, $a = r+1, \dots, n$. We have

$$u = y^i \frac{\partial}{\partial x^i} = y^i B_i^k X_k \quad \text{for} \quad u \in TM.$$

Thus we may use $z^k = y^i B_i^k$ as new coordinates on TM together with (x^i) . It results that

$$\frac{\partial}{\partial y^i} = B_i^k \frac{\partial}{\partial z^k}.$$

The vertical lift of X , that is $X^v = X^i \frac{\partial}{\partial y^i}$ takes the form $X^v = X^i B_i^k \frac{\partial}{\partial z^k}$ which is tangent to $\mathcal{D} \subset TM$ if and only if $B_i^a X^i = 0$, $a = r+1, \dots, n$. \square

It is well-known that the spray S induces the so-called *Berwald connection*. This can be thought as a linear connection in the vertical bundle over TM or as a linear connection on TM . In the former case it is locally given in the form

$$(2.1) \quad \begin{aligned} \widehat{D}_{\partial_i} \dot{\partial}_j &= \dot{\partial}_j (N_i^k) \dot{\partial}_k, \\ \widehat{D}_{\dot{\partial}_i} \dot{\partial}_i &= 0, \end{aligned}$$

where

$$N_i^k = \dot{\partial}_i G^k.$$

Now in the set $\mathcal{X}(M)$ a symmetric product is defined as follows:

$$(2.2) \quad \langle X : Y \rangle = \widehat{D}_X Y^v + \widehat{D}_Y X^v.$$

A direct calculation in local coordinates gives

Theorem 2.3. *For every $X, Y \in \mathcal{X}(M)$ one has*

$$[X^v, [S, Y^v]] = \langle X : Y \rangle .$$

Using the above theorems the following main result is obtained

Theorem 2.4. *A distribution D is g.i. if and only if from $X, Y \in \mathcal{X}(D)$ it follows*

$$\langle X : Y \rangle \in \mathcal{X}(D) \quad \text{for every } X, Y \in \mathcal{X}(M).$$

Let now (M, ∇) be a spray space whose spray is defined by a linear (with torsion) connection ∇ . Then it is easy to see that $D_X Y = (\nabla_X Y)^v$.

This fact in combination with Theorem 2.2 shows

Corollary 2.1. *Let D be a distribution on the manifold M endowed with a linear connection ∇ . Then D is g.i. if and only if from $X, Y \in \mathcal{X}(D)$ it results*

$$\nabla_X Y + \nabla_Y X \in \mathcal{X}(D)$$

for every $X, Y \in \mathcal{X}(M)$.

The result stated in this Corollary was proven by A.D. Lewis in [3].

3 Applications

1. Let (M, S) be a spray space and let the Berwald connection \widehat{D} viewed as a linear connection on TM . In the frame $\left(\delta_i = \partial_i = N_i^k \dot{\partial}_k, \dot{\partial}_k = \frac{\partial}{\partial y^k} \right)$ this is given as follows:

$$\widehat{D}_{\delta_i} \delta_j = \frac{\partial^2 G^k}{\partial y^j \partial y^i} \delta_k,$$

$$\widehat{D}_{\delta_i} \dot{\partial}_j = \frac{\partial^2 G^k}{\partial y^j \partial y^i} \dot{\partial}_k$$

$$\widehat{D}_{\dot{\partial}_i} \delta_j = 0, \quad \widehat{D}_{\dot{\partial}_i} \dot{\partial}_j = 0.$$

On TM there exists the so-called Liouville vector field $C = y^i \dot{\partial}_i$. We consider on TM endowed with the linear connection D the distribution $\{S, C\}$. By a direct calculation one finds

$$[C, S] = S,$$

$$\widehat{D}_C S = S,$$

$$\widehat{D}_S C = 0.$$

Then Corollary 2.1 applies and one gets that the distribution $\{S, C\}$ is g.i. and integrable. Thus it is totally geodesic.

2. Let us consider again the manifold TM endowed with the Berwald connection \widehat{D} and the vertical distribution on TM . This is integrable. From $\widehat{D}_{\dot{\partial}_i} \dot{\partial}_j = 0$ and Corollary 2.1 it follows it is also g.i. The geodesics are straight lines in the leaves $T_p M$. The leaves are totally geodesic. The same fact, except that the geodesics are no more stright lines in the leaves $T_p M$, holds when \widehat{D} is replaced

with a d -connection i.e. with a connection which preserves by parallelism the vertical distribution as well as the horizontal distribution locally spanned by $\delta_i = \partial_i - N_i^k \dot{\partial}_k$. For the definition of d -connection see [4].

3. Let $F^n = (M, F)$ be a Finsler space. Then TM is endowed with a Sasaki type metric

$$G(x, y) = g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^i \otimes \delta y^j,$$

$$\delta y^i = dy^i + N_k^i dx^k.$$

Let ∇ be the Levi-Civita connection of G . We notice only a part of the local form of this connection following the paper [1].

$$\nabla_{\delta_k} \delta_i = F_{jk}^i \delta_i + A_{jk}^a \dot{\partial}_a,$$

$$\nabla_{\dot{\partial}_b} \dot{\partial}_c = C_{cb}^a \dot{\partial}_a + B_{cb}^i \delta_i,$$

with

$$F_{jk}^i = \frac{1}{2} g^{ih} (\delta_j g_{hk} + \delta_k g_{hj} - \delta_h g_{jk}),$$

$$A_{jk}^a = \frac{1}{2} (-R_{jk}^a - g^{ab} \dot{\partial}_b g_{jk}),$$

$$C_{bc}^a = \frac{1}{2} g^{ad} (\dot{\partial}_b g_{cd} + \dot{\partial}_d g_{bd} - \dot{\partial}_d g_{bc}),$$

$$B_{ab}^k = -\frac{1}{2} g^{kj} g_{ab||j},$$

where \parallel denotes the h -covariant derivative with respect to the Berwald connection.

Using these formulae one obtains the following results:

- The vertical distribution is totally geodesic only when F^n reduces to a Riemannian space ;
- The horizontal distribution is g.i. if and only if F^n reduces to a Riemannian space.

We notice that when F^n reduces to a Riemannian space, the projection τ becomes a Riemannian submersion and from the theory of submersions the last two statements are derived, as well. Moreover, the horizontal distribution is integrable if and only if the Riemannian space is flat.

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