An Expanded Class of Parametric Partial Linear Complex Vector Functional Equations

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Abstract

In this paper one expanded class of parametric complex vector partial linear functional equations is solved. The results presented here generalize the results given in [3].

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0 Introduction

First we introduce the following notations. Let $\mathcal{V}$, $\mathcal{V}'$ be finite dimensional complex vector spaces and $\mathbf{Z}_i$, $i \in \mathbf{N}$, be vectors in $\mathcal{V}$. We may assume that $\mathbf{Z}_i = (z_{i1}(t), \cdots, z_{in}(t))^T$, where $z_{ij}(t)$ $(1 \le j \le n)$ are complex functions and $\mathbf{O} = (0, \cdots, 0)^T$ is the zero-vector in $\mathcal{V}$ or $\mathcal{V}'$. We also denote by $\mathcal{V}^0$ the subspace of all real vectors in $\mathcal{V}$ (thus $\mathcal{V} = \mathcal{V}^0 \oplus i\mathcal{V}^0$), and by $\mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ the space of linear mappings $\mathcal{V}^0 \to \mathcal{V}$.

In the present paper we will solve the following simple complex vector functional equation

$$(1) \qquad \sum_{i=1}^{m+n+k} f\Big(\sum_{j=0}^{m-1} a^{m-1-j}\mathbf{Z}_{i+j}, \sum_{j=0}^{n-1} a^{n-1-j}\mathbf{Z}_{i+m+j}, \sum_{j=0}^{k-1} a^{k-1-j}\mathbf{Z}_{i+m+n+j}\Big) = \mathbf{O},$$

$$(\mathbf{Z}_{m+n+k+i} \equiv \mathbf{Z}_i, \quad a \in \mathbf{C}),$$

where $\mathbf{C}$ is the field of complex numbers and $f : \mathcal{V}^3 \to \mathcal{V}'$ is an unknown complex vector function.

The above equation for $k = 0$ was solved in [3]. Also, the functional equation (1) for $a = 1$ was solved in [1] under the hypothesis that the function and variables are real.

The generalized functional equation

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(2) $$\sum_{i=1}^{m+n+k} f_i\Big(\sum_{j=0}^{m-1} a^{m-1-j}\mathbf{Z}_{i+j}, \sum_{j=0}^{n-1} a^{n-1-j}\mathbf{Z}_{i+m+j}, \sum_{j=0}^{k-1} a^{k-1-j}\mathbf{Z}_{i+m+n+j}\Big) = \mathbf{O}$$

$$(f_{m+n+k+i} \equiv f_i, \quad \mathbf{Z}_{m+n+k+i} \equiv \mathbf{Z}_i, \quad a \in \mathbf{C})$$

will be also solved here.

1 Solution of the Simple Functional Equation

Now we will solve the functional equation (1) in the following cases.

I. Let $a = 1$. If we introduce the notations

(3) $$\mathbf{S} = \sum_{j=1}^{m+n+k} \mathbf{Z}_j$$

and

(4) $$f(\mathbf{U}, \mathbf{V}, \mathbf{S}) = g(\mathbf{U}, \mathbf{V},\ \mathbf{S} + \mathbf{U} + \mathbf{V}),$$

then the equation (1) becomes

(5) $$\sum_{j=1}^{m+n+k} g\Big(\sum_{j=0}^{m-1} \mathbf{Z}_{i+j},\ \sum_{j=0}^{n-1} \mathbf{Z}_{m+i+j},\ \mathbf{S}\Big) = \mathbf{O}.$$

If we introduce new variables $\mathbf{T}_i$ $(1 \le i \le m+n+k-1)$ by the equalities

$$\mathbf{T}_i = \mathbf{Z}_i - \frac{\mathbf{S}}{m+n+k} \quad (1 \le i \le m+n+k-1),$$

i.e.

$$\mathbf{Z}_{m+n+k} = \frac{\mathbf{S}}{m+n+k} - \sum_{j=1}^{m+n+k-1} \mathbf{T}_j,$$

then the equation (5) becomes

(6)
$$\sum_{i=1}^{k} g\Big(\frac{m\mathbf{S}}{m+n+k} + \sum_{j=0}^{m-1} \mathbf{T}_{i+j}, \frac{n\mathbf{S}}{m+n+k} + \sum_{j=0}^{n-1} \mathbf{T}_{m+i+j},\ \mathbf{S}\Big) +$$
$$+ \sum_{i=k+1}^{n+k} g\Big(\frac{m\mathbf{S}}{m+n+k} + \sum_{j=0}^{m-1} \mathbf{T}_{i+j},\ \frac{n\mathbf{S}}{m+n+k} - \sum_{j=0}^{m+k-1} \mathbf{T}_{m+n+i+j},\ \mathbf{S}\Big) +$$
$$+ \sum_{i=n+k+1}^{m+n+k} g\Big(\frac{m\mathbf{S}}{m+n+k} - \sum_{j=0}^{n+k-1} \mathbf{T}_{m+i+j},\ \frac{n\mathbf{S}}{m+n+k} + \sum_{j=0}^{n-1} \mathbf{T}_{m+i+j},\ \mathbf{S}\Big) = \mathbf{O}.$$

By putting

(7) $$g\Big(\frac{m\mathbf{S}}{m+n+k} + \mathbf{U},\ \frac{n\mathbf{S}}{m+n+k} + \mathbf{V},\ \mathbf{S}\Big) = H(\mathbf{U}, \mathbf{V}, \mathbf{S}),$$

the equations (6) becomes

$$\tag{8} \begin{aligned} &\sum_{i=1}^{k} H\Big(\sum_{j=0}^{m-1}\mathbf{T}_{i+j},\ \sum_{j=0}^{n-1}\mathbf{T}_{m+i+j},\ \mathbf{S}\Big)+ \\ &+\sum_{i=k+1}^{k+n} H\Big(\sum_{j=0}^{m-1}\mathbf{T}_{i+j},\ -\sum_{j=0}^{m+k-1}\mathbf{T}_{m+n+i+j},\ \mathbf{S}\Big)+ \\ &+\sum_{i=n+k+1}^{m+n+k} H\Big(-\sum_{j=0}^{n+k-1}\mathbf{T}_{m+i+j},\ \sum_{j=0}^{n-1}\mathbf{T}_{m+i+j},\ \mathbf{S}\Big)=\mathbf{O}. \end{aligned}$$

Now, we may suppose that the variable $\mathbf{S}$ is fixed, and we may put

$$\tag{9} H(\mathbf{U},\mathbf{V},\mathbf{S})=h(\mathbf{U},\mathbf{V}).$$

In this case, the functional equation (8) takes the following form

$$\tag{10} \begin{aligned} &\sum_{i=1}^{k} h\Big(\sum_{j=0}^{m-1}\mathbf{T}_{i+j},\ \sum_{j=0}^{n-1}\mathbf{T}_{m+i+j}\Big)+ \\ &+\sum_{i=k+1}^{m+k} h\Big(\sum_{j=0}^{m-1}\mathbf{T}_{i+j},\ -\sum_{j=0}^{m+k-1}\mathbf{T}_{m+n+i+j}\Big)+ \\ &+\sum_{i=n+k+1}^{m+n+k} h\Big(-\sum_{j=0}^{n+k+1}\mathbf{T}_{m+i+j},\ \sum_{j=0}^{n-1}\mathbf{T}_{m+i+j}\Big)=\mathbf{O}. \end{aligned}$$

The complex vector function h has the following properties:

1^o. If $m<n<k$ and $m+n<k$, then the following relations hold

$$\tag{11} h(\mathbf{O},\mathbf{O})=\mathbf{O},$$

$$\tag{12} h(\mathbf{U},-\mathbf{U})+h(\mathbf{U},\mathbf{O})+h(\mathbf{O},-\mathbf{U})=\mathbf{O},$$

$$\tag{13} h(\mathbf{O},\mathbf{U})+h(\mathbf{O},-\mathbf{U})=\mathbf{O},$$

$$\tag{14} h(\mathbf{U},\mathbf{V})=h(\mathbf{U}+\mathbf{V},\mathbf{O})-h(\mathbf{V},\mathbf{O})+h(\mathbf{O},\mathbf{V}),$$

$$\tag{15} h(\mathbf{U},\mathbf{O})+h(-\mathbf{U},\mathbf{O})=\mathbf{O},$$

$$\tag{16} h(\mathbf{U},\mathbf{V})=h(\mathbf{O},\mathbf{U}+\mathbf{V})-h(\mathbf{O},\mathbf{U})+h(\mathbf{U},\mathbf{O}).$$

2^o. If $m<n<k$ and $m+n=k$, then (11), (12), (13), (14), (15) and (16) hold.

3^o. If $m<n<k$ and $m+n>k$, then (11), (12), (13), (14), (15) hold and

$$h(\mathbf{U},\mathbf{V})+h(\mathbf{V},-\mathbf{U}-\mathbf{V})+h(-\mathbf{U}-\mathbf{V},\mathbf{U})=\mathbf{O}. \tag{17}$$

4^o. If $m<n=k$, then (11), (12), (13), (14) and (17) hold.
5^o. If $m=n$ and $2m<k$, then (11), (12) hold and

$$h(\mathbf{U},\mathbf{V})+h(\mathbf{V},\mathbf{O})+h(\mathbf{O},-\mathbf{U}-\mathbf{V})+h(-\mathbf{U}-\mathbf{V},\mathbf{U})=\mathbf{O}. \tag{18}$$

6^o. If $m=n$ and $2m=k$, then (11), (12) and (18) hold.
7^o. If $m=n<k$ and $2m>k$, then (11), (12) and (18) hold.
8^o. If $n<m<k$ and $m+n<k$, then (11), (12), (15), (16), (13) and (14) hold.
9^o. If $n<m<k$ and $m+n=k$, then (11), (12), (15), (16), (13) and (14) hold.
10^o. If $n<m<k$ and $m+n>k$, then (11), (12), (15), (16), (13) and (17) hold.
11^o. If $n<m=k$, then (11), (12), (15), (16) and (17) hold.
12^o. If $m=n=k$, then (11), (12) and (17) hold.

Now, we will prove the theorems which treat the functional equation (1).

Theorem 1. *If $a=1$, $m,n<k$, $m\neq n$ and $m+n\neq k$, then the general continuous solution of the functional equation (1) is*

$$\begin{aligned} f(\mathbf{U},\mathbf{V},\mathbf{W}) &= G_1(\mathbf{U}+\mathbf{V}+\mathbf{W})Re\Big(\mathbf{U}-m\tfrac{\mathbf{U}+\mathbf{V}+\mathbf{W}}{m+n+k}\Big)+\\ &+ G_2(\mathbf{U}+\mathbf{V}+\mathbf{W})Im\Big(\mathbf{U}-m\tfrac{\mathbf{U}+\mathbf{V}+\mathbf{W}}{m+n+k}\Big)+\\ &+ G_3(\mathbf{U}+\mathbf{V}+\mathbf{W})Re\Big(\mathbf{V}-n\tfrac{\mathbf{U}+\mathbf{V}+\mathbf{W}}{m+n+k}\Big)+\\ &+ G_4(\mathbf{U}+\mathbf{V}+\mathbf{W})Im\Big(\mathbf{V}-n\tfrac{\mathbf{U}+\mathbf{V}+\mathbf{W}}{m+n+k}\Big), \end{aligned}$$

where $G_i:\mathcal{V}\to\mathcal{L}(\mathcal{V}^0,\mathcal{V}')$ $(1\le i\le 4)$ are arbitrary continuous complex vector functions and $Re\,\mathbf{U}$ resp. $Im\,\mathbf{U}$ denotes the real resp. imaginary part of $\mathbf{U}$.

Proof. Let $m<n<k$ and $m+n<k$. On the basis of the expressions (14) and (16), we obtain that the complex vector function h satisfies the equation

$$h(\mathbf{U}+\mathbf{V},\mathbf{O})-h(\mathbf{O},\mathbf{U}+\mathbf{V})=h(\mathbf{U},\mathbf{O})-h(\mathbf{O},\mathbf{U})+h(\mathbf{V},\mathbf{O})-h(\mathbf{O},\mathbf{V}),$$

and hence we get

$$h(\mathbf{U},\mathbf{O})-h(\mathbf{O},\mathbf{U})=G'Re\,\mathbf{U}+G''Im\,\mathbf{U},$$

where $G',G''\in\mathcal{L}(\mathcal{V}^0,\mathcal{V}')$ are arbitrary continuous functions of $\mathbf{S}$.

Thus, the equation (16) has the form

$$h(\mathbf{U},\mathbf{V})=h(\mathbf{O},\mathbf{U}+\mathbf{V})+G'Re\,\mathbf{U}+G''Im\,\mathbf{U}. \tag{19}$$

If we substitute the function h determined by (19) into (10), on the basis of the expression (13) we obtain

$$\sum_{i=1}^{k}h\Big(\mathbf{O},\sum_{j=1}^{m+n-1}\mathbf{T}_{i+j}\Big)-\sum_{i=k+1}^{k-1}h\Big(\mathbf{O},\sum_{j=0}^{k-1}\mathbf{T}_{m+n+i+j}\Big)=\mathbf{O}. \tag{20}$$

If we put $\mathbf{T}_1=\mathbf{U}$, $\mathbf{T}_k=\mathbf{V}$ and $\mathbf{T}_j=\mathbf{O}$ $(j\neq 1,k)$ into (20), then we obtain

$$h(\mathbf{O},\mathbf{U}+\mathbf{V})=h(\mathbf{O},\mathbf{U})+h(\mathbf{O},\mathbf{V}),$$

and hence we deduce

$$h(\mathbf{O},\mathbf{U})=G_3 Re\,\mathbf{U}+G_4 Im\,\mathbf{U}, \tag{21}$$

where $G_3, G_4 \in \mathcal{L}(\mathcal{V}^0,\mathcal{V}')$ are arbitrary complex vector functions of $\mathbf{S}$.

On the basis of the expressions (21) and (19), we obtain

$$h(\mathbf{U},\mathbf{V})=G_1 Re\,\mathbf{U}+G_2 Im\,\mathbf{U}+G_3 Re\,\mathbf{V}+G_4 Im\,\mathbf{V}, \tag{22}$$

where $G_1=G'+G_3$ and $G_2=G''+G_4$.

Let $m<n<k$ and $m+n>k$. On the basis of the expressions (14) and (17), we get

$$h(\mathbf{U}+\mathbf{V},\mathbf{O})-h(-\mathbf{U}-\mathbf{V},\mathbf{O})-h(\mathbf{O},\mathbf{U}+\mathbf{V})=$$
$$=h(\mathbf{U},\mathbf{O})-h(-\mathbf{U},\mathbf{O})-h(\mathbf{O},\mathbf{U})+h(\mathbf{V},\mathbf{O})-h(-\mathbf{V},\mathbf{O})-h(\mathbf{O},\mathbf{V}),$$

and hence we can calculate that

$$h(\mathbf{U},\mathbf{O})-h(-\mathbf{U},\mathbf{O})+h(\mathbf{O},\mathbf{U})=G_3 Re\,\mathbf{U}+G_4 Im\,\mathbf{U}, \tag{23}$$

where $G_3, G_4 \in \mathcal{L}(\mathcal{V}^0,\mathcal{V}')$ are arbitrary complex vector functions of $\mathbf{S}$.

On the basis of the identity (23), the equation (14) becomes

$$h(\mathbf{U},\mathbf{V})=h(\mathbf{U}+\mathbf{V},\mathbf{O})-h(-\mathbf{U},\mathbf{O})+G_3 Re\,\mathbf{V}+G_4 Im\,\mathbf{V}. \tag{24}$$

If we substitute the function h determined by (24) into (10), we obtain

$$\begin{aligned}&\sum_{i=1}^{k} h\Big(\sum_{i=1}^{m+n-1}\mathbf{T}_{i+j},\ \mathbf{O}\Big)+\sum_{i=k+1}^{m+n+k} h\Big(-\sum_{j=0}^{k-1}\mathbf{T}_{m+n+i+j},\ \mathbf{O}\Big)-\\ &-\sum_{i=k+1}^{m+n+k} h\Big(\sum_{j=0}^{m+k-1}\mathbf{T}_{m+n+i+j},\ \mathbf{O}\Big)-\sum_{i=1}^{m+k} h\Big(-\sum_{j=0}^{n-1}\mathbf{T}_{i+j},\ \mathbf{O}\Big)=\mathbf{O}.\end{aligned} \tag{25}$$

For $\mathbf{T}_1=\mathbf{U},\ \mathbf{T}_{m+k}=\mathbf{V}$ and $\mathbf{T}_j=\mathbf{O}$ $(j\neq 1, m+k)$ from (25) we have

$$h(\mathbf{U}+\mathbf{U},\mathbf{O})=h(\mathbf{U},\mathbf{O})+h(\mathbf{V},\mathbf{O}),$$

where

$$h(\mathbf{U},\mathbf{O})=G_1 Re\,\mathbf{U}+G_2 Im\,\mathbf{U}.$$

On the basis of this, the equation (24) becomes (25).

Let $n<m<k$ and $m+n<k$. On the basis of the equations (16) and (14) we obtain

$$h(\mathbf{U}+\mathbf{V},\mathbf{O})-h(\mathbf{O},\mathbf{U}+\mathbf{V})=h(\mathbf{U},\mathbf{O})-h(\mathbf{O},\mathbf{U})+h(\mathbf{V},\mathbf{O})-h(\mathbf{O},\mathbf{V}),$$

from where it follows that

$$h(\mathbf{U},\mathbf{O})-h(\mathbf{O},\mathbf{U})=G_1' Re\,\mathbf{U}+G_2'' Im\,\mathbf{U}. \tag{26}$$

On the basis of the equality (26), the equation (16) becomes

$$h(\mathbf{U},\mathbf{V}) = h(\mathbf{O},\mathbf{U}+\mathbf{V}) + G' Re\,\mathbf{U} + G'' Im\,\mathbf{U}. \tag{27}$$

If we substitute the function h determined by (27) into (10), we obtain the following equation

$$\sum_{i=1}^{k} h\Big(\mathbf{O},\ \sum_{j=0}^{m+n+1} \mathbf{T}_{i+j}\Big) = \sum_{i=k+1}^{m+n+k} h\Big(\mathbf{O},\ \sum_{j=0}^{k-1} \mathbf{T}_{m+n+i+j}\Big). \tag{28}$$

By putting $\mathbf{T}_1 = \mathbf{U}$, $\mathbf{T}_{m+n} = \mathbf{V}$ and $\mathbf{T}_j = \mathbf{O}$ $(j \neq 1, m+n)$ in (28) we have

$$h(\mathbf{O},\mathbf{U},\mathbf{V}) = h(\mathbf{O},\mathbf{U}) + h(\mathbf{O},\mathbf{V}),$$

from where we conclude that

$$h(\mathbf{O},\mathbf{U}) = G_3 Re\,\mathbf{U} + G_4 Im\,\mathbf{U}.$$

On the basis of this, the equation (27) becomes (22).

Let $n < m < k$ and $m+n > k$. On the basis of (16) and (17), we obtain that the function h satisfies the functional equation

$$\begin{aligned} &h(\mathbf{U}+\mathbf{V},\mathbf{O}) + h(\mathbf{O},-\mathbf{U}-\mathbf{V}) - h(\mathbf{O},\mathbf{U}+\mathbf{V}) = \\ &= h(\mathbf{U},\mathbf{O}) + h(\mathbf{O},-\mathbf{U}) - h(\mathbf{O},\mathbf{U}) + h(\mathbf{V},\mathbf{O}) + h(\mathbf{O},-\mathbf{V}) - h(\mathbf{O},\mathbf{V}). \end{aligned}$$

Therefore, the function $h(\mathbf{U},\mathbf{O}) + h(\mathbf{O},-\mathbf{U}) - h(\mathbf{O},\mathbf{U})$ is determined by

$$h(\mathbf{U},\mathbf{O}) + h(\mathbf{O},-\mathbf{U}) - h(\mathbf{O},\mathbf{V}) = G_1 Re\,\mathbf{U} + G_2 Im\,\mathbf{U}. \tag{29}$$

The equation (16), on the basis of the equality (29), becomes

$$h(\mathbf{U},\mathbf{V}) = h(\mathbf{O},\mathbf{U}+\mathbf{V}) - h(\mathbf{O},-\mathbf{U}) + G' Re\,\mathbf{U} + G'' Im\,\mathbf{U}. \tag{30}$$

The function h needs to satisfy the equation (10). If we substitute (30) into (10), we get

$$\begin{aligned} &\sum_{i=1}^{k} h\Big(\mathbf{O},\ \sum_{j=0}^{m+n-1} \mathbf{T}_{i+j}\Big) + \sum_{i=k+1}^{m+n+k} h\Big(\mathbf{O},\ -\sum_{j=0}^{k-1} \mathbf{T}_{m+n+i+j}\Big) - \\ &-\sum_{i=1}^{k+1} h\Big(\mathbf{O},\ -\sum_{j=0}^{m-1} \mathbf{T}_{i+j}\Big) - \sum_{i=n+k+1}^{m+n+k} h\Big(\mathbf{O},\ \sum_{j=0}^{m-1} \mathbf{T}_{m+n+i+j}\Big) = \mathbf{O}. \end{aligned} \tag{31}$$

If we put $\mathbf{T}_1 = \mathbf{U}$, $\mathbf{T}_{n+k} = \mathbf{V}$ and $\mathbf{T}_j = \mathbf{O}$ $(j \neq 1, n+k)$ into (31), we obtain

$$h(\mathbf{O},\mathbf{U}+\mathbf{V}) = h(\mathbf{O},\mathbf{U}) + h(\mathbf{O},\mathbf{V}),$$

from where it follows that

$$h(\mathbf{O},\mathbf{U}) = G_3 Re\,\mathbf{U} + G_4 Im\,\mathbf{U}.$$

On the basis of the last equality, the equality (30) has just the form (22). Consequently, in all four cases the function h is determined by (22).

We denote the arbitrary complex functions G_i $(1 \leq i \leq n)$ of $\mathbf{S}$ by $G_i(\mathbf{S})$ $(1 \leq i \leq 4)$. On the basis of that, the equality (22) and the transformations (9), (7), (4) and (3) there follows the proof of the theorem. □

Theorem 2. *If $a = 1$, $m \neq n$ and $m + n = k$, then the general continuous solution of the functional equation (1) is determined by*

$$\begin{aligned} f(\mathbf{U},\mathbf{V},\mathbf{W}) &= F\Big(\mathbf{U}+\mathbf{V}-k\frac{\mathbf{U}+\mathbf{V}+\mathbf{W}}{m+n+k},\ \mathbf{U}+\mathbf{V}+\mathbf{W}\Big)- \\ &- F\Big(-\mathbf{U}-\mathbf{V}+k\frac{\mathbf{U}+\mathbf{V}+\mathbf{W}}{m+n+k},\ \mathbf{U}+\mathbf{V}+\mathbf{W}\Big)+ \\ &+ G_1(\mathbf{U}+\mathbf{V}+\mathbf{W})Re\Big(\mathbf{U}-m\frac{\mathbf{U}+\mathbf{V}+\mathbf{W}}{m+n+k}\Big)+ \\ &+ G_2(\mathbf{U}+\mathbf{V}+\mathbf{W})Im\Big(\mathbf{U}-m\frac{\mathbf{U}+\mathbf{V}+\mathbf{W}}{m+n+k}\Big), \end{aligned}$$

where $F : \mathcal{V}^2 \to \mathcal{V}'$ and $G_i : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ $(i = 1, 2)$ are arbitrary continuous complex vector functions.

Proof. Let $m < n$. As in the proof of the previous theorem, on the basis of the expressions (14) and (16), we may show that the function h has the form determined by (19). If we substitute the function h determined as previously into (1), on the basis of the expression (13), we conclude that the equality is valid and consequently, the function $h(\mathbf{O},\mathbf{U})$ may be arbitrary. According to (13), for $h(\mathbf{O},\mathbf{U})$ we can put

$$h(\mathbf{O},\mathbf{U}) = F(\mathbf{U}) - F(-\mathbf{U}),$$

where F is an arbitrary continuous function.

On the basis of this, the equality (19) becomes

$$h(\mathbf{U},\mathbf{V}) = F(\mathbf{U}+\mathbf{V}) - F(-\mathbf{U}-\mathbf{V}) + G_1 Re\,\mathbf{U} + G_2 Im\,\mathbf{U}. \tag{32}$$

Let $n < m$. As in this case the equalities (14) and (16) hold and the function h has the form determined by (19). If we substitute (19) into (10), we obtain that the equation (10) holds, which means that $h(\mathbf{O},\mathbf{U})$ is an arbitrary function for which (13) holds.

On the basis of this, we obtain that the function h in this case has the form determined by (32).

If we put that the arbitrary continuous complex vector functions G_1 and G_2 are $G_1(\mathbf{S})$ and $G_2(\mathbf{S})$ respectively, on the basis of the expressions (9), (7), (4) and (3), there follows the proof of the theorem. □

Theorem 3. *If $a = 1$ and $m < n = k$, the functional equation (1) has a general continuous solution determined by*

$$\begin{aligned} f(\mathbf{U},\mathbf{V},\mathbf{W}) &= F\Big(\mathbf{U}+\mathbf{V}-(m+n)\frac{\mathbf{U}+\mathbf{V}+\mathbf{W}}{m+n+k},\ \mathbf{U}+\mathbf{V}+\mathbf{W}\Big)- \\ &- F\Big(-\mathbf{V}+n\frac{\mathbf{U}+\mathbf{V}+\mathbf{W}}{m+n+k},\ \mathbf{U}+\mathbf{V}+\mathbf{W}\Big)+ \\ &+ G_1(\mathbf{U}+\mathbf{V}+\mathbf{W})Re\Big(\mathbf{V}-n\frac{\mathbf{U}+\mathbf{V}+\mathbf{W}}{m+n+k}\Big)+ \\ &+ G_2(\mathbf{U}+\mathbf{V}+\mathbf{W})Im\Big(\mathbf{V}-n\frac{\mathbf{U}+\mathbf{V}+\mathbf{W}}{m+n+k}\Big), \end{aligned}$$

where $F:\mathcal{V}^2\to\mathcal{V}'$ *and* $G_i:\mathcal{V}\to\mathcal{L}(\mathcal{V}^0,\mathcal{V}')$ $(i=1,2)$ *are arbitrary continuous complex vector functions.*

Proof. On the basis of the equalities (14) and (17), as in the proof of Theorem 1, we conclude that (23) holds, i.e. (24) holds. If we substitute (24) into (10), since $n=k$, we obtain an identity which means that for $h(\mathbf{U},\mathbf{O})$ we may take an arbitrary continuous function.

On the basis of this, for the function $f(\mathbf{U},\mathbf{V})$ we obtain

$$h(\mathbf{U},\mathbf{V}) = F(\mathbf{U}+\mathbf{V}) - F(-\mathbf{V}) + G_1 Re\,\mathbf{V} + G_2 Im\,\mathbf{V}. \tag{33}$$

From (33), on the basis of the transformations (9), (7), (4) and (3) and putting that G_1 and G_2 are arbitrary continuous functions of $\mathbf{S}$, we obtain that the function f has the form given in the Theorem 3. □

Theorem 4. *If* $a=1$ *and* $n<m=k$*, then the general continuous solution of the functional equation (1) is*

$$\begin{aligned} f(\mathbf{U},\mathbf{V},\mathbf{W}) &= F\Big(\mathbf{U}+\mathbf{V}-(m+n)\frac{\mathbf{U}+\mathbf{V}+\mathbf{W}}{m+n+k},\ \mathbf{U}+\mathbf{V}+\mathbf{W}\Big)- \\ &- F\Big(-\mathbf{U}+m\frac{\mathbf{U}+\mathbf{V}+\mathbf{W}}{m+n+k},\ \mathbf{U}+\mathbf{V}+\mathbf{W}\Big)+ \\ &+ G_1(\mathbf{U}+\mathbf{V}+\mathbf{W})Re\Big(\mathbf{U}-m\frac{\mathbf{U}+\mathbf{V}+\mathbf{W}}{m+n+k}\Big)+ \\ &+ G_2(\mathbf{U}+\mathbf{V}+\mathbf{W})Im\Big(\mathbf{U}-m\frac{\mathbf{U}+\mathbf{V}+\mathbf{W}}{m+n+k}\Big), \end{aligned}$$

where $F:\mathcal{V}^2\to\mathcal{V}'$ *and* $G_i:\mathcal{V}\to\mathcal{L}(\mathcal{V}^0,\mathcal{V}')$ $(i=1,2)$ *are arbitrary continuous complex vector functions.*

Proof. On the basis of (16), (17) and (29), we obtain that the function h satisfies the equation (30).

If the function h determined by (30) is substituted into (10), then the equation (10) becomes an identity, which means that $h(\mathbf{O},\mathbf{U})$ can be substituted by an arbitrary continuous function $F(\mathbf{U})$.

On the basis of this, the equality (30) has the form

$$h(\mathbf{U},\mathbf{V}) = F(\mathbf{U}+\mathbf{V}) - F(-\mathbf{U}) + G_1 Re\,\mathbf{U} + G_2 Im\,\mathbf{U}. \tag{34}$$

According to the transformations (9), (7), (4) and (3), from the equation (34), by putting $G_1=G_1(\mathbf{S})$ and $G_2=G_2(\mathbf{S})$, it follows that the function f has the form which is given in this theorem. □

Theorem 5. *If $a = 1$, $m = n < k$ and $2m \neq k$, then the general continuous solution of the functional equation (1) is determined by*

$$\begin{aligned} f(\mathbf{U},\mathbf{V},\mathbf{W}) &= F\Big(\mathbf{U} - m\frac{\mathbf{U}+\mathbf{V}+\mathbf{W}}{m+n+k},\ \mathbf{U}+\mathbf{V}+\mathbf{W}\Big) - \\ &- F\Big(\mathbf{V} - n\frac{\mathbf{U}+\mathbf{V}+\mathbf{W}}{m+n+k},\ \mathbf{U}+\mathbf{V}+\mathbf{W}\Big) + \\ &+ G_1(\mathbf{U}+\mathbf{V}+\mathbf{W}) Re\Big(\mathbf{U} - m\frac{\mathbf{U}+\mathbf{V}+\mathbf{W}}{m+n+k}\Big) + \\ &+ G_2(\mathbf{U}+\mathbf{V}+\mathbf{W}) Im\Big(\mathbf{U} - m\frac{\mathbf{U}+\mathbf{V}+\mathbf{W}}{m+n+k}\Big), \end{aligned}$$

where $F : \mathcal{V}^2 \to \mathcal{V}'$ and $G_i : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ $(i = 1, 2)$ are arbitrary continuous complex vector functions.

Proof. Let $2m < k$. On the basis of the expression (18), we obtain

$$h(\mathbf{U},\mathbf{V}) + h(\mathbf{V},\mathbf{O}) + h(\mathbf{O},\ -\mathbf{U}-\mathbf{V}) + h(-\mathbf{U}-\mathbf{V},\ \mathbf{U}) = \mathbf{O},$$

$$h(-\mathbf{U}-\mathbf{V},\ \mathbf{U}) + h(\mathbf{U},\mathbf{O}) + h(\mathbf{O},\mathbf{V}) + h(\mathbf{V},\ -\mathbf{U}-\mathbf{V}) = \mathbf{O},$$

$$h(\mathbf{V},\ -\mathbf{U}-\mathbf{V}) + h(-\mathbf{U}-\mathbf{V},\ \mathbf{O}) + h(\mathbf{O},\mathbf{U}) + h(\mathbf{U},\mathbf{V}) = \mathbf{O}.$$

If from these three equalities we eliminate $h(-\mathbf{U}-\mathbf{V},\ \mathbf{U})$ and $h(\mathbf{V},\ -\mathbf{U}-\mathbf{U})$, we obtain

$$2h(\mathbf{U},\mathbf{V}) =$$
$$-h(-\mathbf{U}-\mathbf{V},\ \mathbf{O}) - h(\mathbf{O},\ -\mathbf{U}-\mathbf{V}) + h(\mathbf{U},\mathbf{O}) - h(\mathbf{O},\mathbf{U}) + h(\mathbf{O},\mathbf{V}) - h(\mathbf{V},\mathbf{O}).$$

If we substitute this value of h into (10), we get

$$\begin{aligned} &\sum_{i=1}^{k} h\Big(-\sum_{j=0}^{m+n-1}\mathbf{T}_{i+j},\ \mathbf{O}\Big) + \sum_{i=1}^{k} h\Big(\mathbf{O},\ -\sum_{j=0}^{m+n-1}\mathbf{T}_{i+j}\Big) + \\ &+ \sum_{i=k+1}^{m+n+k} h\Big(\sum_{j=0}^{k-1}\mathbf{T}_{m+n+i+j},\ \mathbf{O}\Big) + \sum_{i=k+1}^{m+n+k} h\Big(\mathbf{O},\ \sum_{j=0}^{k-1}\mathbf{T}_{m+n+i+j}\Big) = \mathbf{O}. \end{aligned} \tag{35}$$

If we put $\mathbf{T}_1 = \mathbf{U}$, $\mathbf{T}_{m+n} = \mathbf{V}$ and $\mathbf{T}_j = \mathbf{O}$ $(j \neq 1, m+n)$ into (35), we have

$$\begin{aligned} &h(\mathbf{U}+\mathbf{V},\ \mathbf{O}) + h(\mathbf{O},\ \mathbf{U}+\mathbf{V}) + (k-1)[h(\mathbf{V},\mathbf{O}) + h(\mathbf{O},\mathbf{V})] + \\ &+ h(-\mathbf{U},\mathbf{O}) + h(\mathbf{O},-\mathbf{U}) + k[h(-\mathbf{V},\mathbf{O}) + h(\mathbf{O},-\mathbf{V})] = \mathbf{O}. \end{aligned} \tag{36}$$

For $\mathbf{V} = \mathbf{O}$, from (36) we obtain

$$h(-\mathbf{U},\mathbf{O}) + h(\mathbf{O},-\mathbf{U}) + h(\mathbf{U},\mathbf{O}) + h(\mathbf{O},\mathbf{U}) = \mathbf{O}, \tag{37}$$

i.e. the equality (36) has the form

$$h(\mathbf{U}+\mathbf{V},\ \mathbf{O}) + h(\mathbf{O},\ \mathbf{U}+\mathbf{V}) = h(\mathbf{U},\mathbf{O}) + h(\mathbf{O},\mathbf{U}) + h(\mathbf{V},\mathbf{O}) + h(\mathbf{O},\mathbf{V}).$$

Hence, it follows that

$$h(\mathbf{U},\mathbf{O})+h(\mathbf{O},\mathbf{U})=G_1 Re\,\mathbf{U}+G_2 Im\,\mathbf{U}.$$

Thus we obtain that the function $h(\mathbf{U},\mathbf{V})$ is determined by

$$h(\mathbf{U},\mathbf{V})=h(\mathbf{U},\mathbf{O})+h(\mathbf{O},\mathbf{V}),$$

or

$$h(\mathbf{U},\mathbf{V})=F(\mathbf{U})-F(\mathbf{V})+G_1 Re\,\mathbf{U}+G_2 Im\,\mathbf{V}, \tag{38}$$

where we have put $F(\mathbf{U})=-h(\mathbf{O},\mathbf{U})$.

Let $2m>k$. If we put $\mathbf{T}_1=\mathbf{U},\ \mathbf{T}_k=\mathbf{V}$ and $\mathbf{T}_j=\mathbf{O}$ $(j\neq 1,k)$ into (35), we get (39)

$$h(\mathbf{U},\mathbf{O})+h(\mathbf{O},\mathbf{U})+(m+n)[h(\mathbf{V},\mathbf{O})+h(\mathbf{O},\mathbf{V})]+$$

$$+h(-\mathbf{U}-\mathbf{V},\ \mathbf{O})+h(\mathbf{O},\ \mathbf{U}-\mathbf{V})+(m+n-1)[h(-\mathbf{V},\mathbf{O})+h(\mathbf{O},-\mathbf{V})]=\mathbf{O},$$

and hence for $\mathbf{V}=\mathbf{O}$ we obtain (37), i.e. the equality (39) has the following form

$$h(\mathbf{U}+\mathbf{V},\ \mathbf{O})+h(\mathbf{O},\ \mathbf{U}+\mathbf{V})=h(\mathbf{U},\mathbf{O})+h(\mathbf{O},\mathbf{U})+h(\mathbf{V},\mathbf{O})+h(\mathbf{O},\mathbf{V}).$$

Hence it follows that the function h has the form (38).

Thus, on the basis of the expressions (38), (9), (7), (4) and (3), we conclude that the function f has the form given in the theorem. □

Theorem 6. *If $a=1$ and $2m=2n=k$, then the general continuous solution of the functional equation (1) is given by*

$$f(\mathbf{U},\mathbf{V},\mathbf{W})=F(\mathbf{U},\ \mathbf{V}+\mathbf{W})-F(\mathbf{V},\ \mathbf{W}+\mathbf{U})+G(\mathbf{U}+\mathbf{V},\ \mathbf{W})-G(\mathbf{W},\ \mathbf{U}+\mathbf{V}),$$

where $F,G:\mathcal{V}^2\to\mathcal{V}'$ are arbitrary complex vector functions.

Proof. From the equation (35), which holds in this case, there immediately follows the equality

$$h(\mathbf{U},\mathbf{O})+h(\mathbf{O},\mathbf{U})+h(-\mathbf{U},\mathbf{O})+h(\mathbf{O},-\mathbf{U})=\mathbf{O},$$

i.e. we may put

$$h(\mathbf{U},\mathbf{O})+h(\mathbf{O},\mathbf{U})=2G(\mathbf{U})-2G(-\mathbf{U}), \tag{40}$$

where G is an arbitrary complex vector function.

Since by virtue of (15) it holds

$$2h(\mathbf{U},\mathbf{V})=$$

$$-h(-\mathbf{U}-\mathbf{V},\ \mathbf{O})-h(\mathbf{O},\ -\mathbf{U}-\mathbf{V})+h(\mathbf{U},\mathbf{O})-h(\mathbf{O},\mathbf{U})+h(\mathbf{O},\mathbf{V})-h(\mathbf{V},\mathbf{O}),$$

on the basis of the expression (40) we have

$$h(\mathbf{U},\mathbf{V})=F(\mathbf{U})-F(\mathbf{V})+G(\mathbf{U}+\mathbf{V})-G(-\mathbf{U}-\mathbf{V}), \tag{41}$$

where we introduced the notation

$$2F(\mathbf{U})=h(\mathbf{U},\mathbf{O})-h(\mathbf{O},\mathbf{U}).$$

From the equalities (41), (9), (7), (4) and (3) there follows the proof of the theorem. □

Theorem 7. *If $a = 1$ and $m = n = k$, then the general continuous solution of the functional equation (1) is given by*

$$f(\mathbf{U},\mathbf{V},\mathbf{W}) = F(\mathbf{U},\mathbf{V},\mathbf{W}) - F(\mathbf{V},\mathbf{W},\mathbf{U}),$$

where $F : \mathcal{V}^3 \to \mathcal{V}'$ is an arbitrary complex vector function.

Proof. If we put $\mathbf{T}_1 = \mathbf{U},\ \mathbf{T}_{m+1} = \mathbf{V}$ and $\mathbf{T}_j = \mathbf{O}$ ($j \neq 1, m+1$) into (10), then we obtain

$$h(\mathbf{U},\mathbf{V}) + h(\mathbf{V},\ -\mathbf{U}-\mathbf{V}) + h(-\mathbf{U}-\mathbf{V},\ \mathbf{U}) = \mathbf{O}.$$

According to [2] the general solution of this equation is given by

$$h(\mathbf{U},\mathbf{V}) = F(\mathbf{U},\mathbf{V}) - F(\mathbf{V},\ -\mathbf{U}-\mathbf{V}), \tag{42}$$

where $F : \mathcal{V}^2 \to \mathcal{V}'$ is an arbitrary complex vector function.

On the basis of the expressions (42), (9), (7), (4) and (3), we conclude that the function f is determined by the form given in the theorem. □

II. Let $a^{m+n+k} \neq 1$. If we put

$$f(\mathbf{U},\mathbf{V},\mathbf{W}) = g(\mathbf{U}+a^{k+m}\mathbf{V}+a^m\mathbf{W},\ \mathbf{V}+a^{m+n}\mathbf{W}+a^n\mathbf{U},\ \mathbf{W}+a^{n+k}\mathbf{U}+a^k\mathbf{V}), \tag{43}$$

then the functional equation (1) becomes

$$\sum_{i=1}^{m+n+k} g\Big(\sum_{j=0}^{m-1} a^{m-1-j}\mathbf{Z}_{i+j} + \sum_{j=0}^{n-1} a^{m+n+k-1-j}\mathbf{Z}_{m+i+j} + \sum_{j=0}^{k-1} a^{m+k-1-j}\mathbf{Z}_{m+n+i+j},$$
$$\sum_{j=0}^{n-1} a^{n-1-j}\mathbf{Z}_{m+i+j} + \sum_{j=0}^{k-1} a^{m+n+k-1-j}\mathbf{Z}_{m+n+i+j} + \sum_{j=0}^{m-1} a^{m+n-1-j}\mathbf{Z}_{i+j},$$
$$\sum_{j=0}^{k-1} a^{k-1-j}\mathbf{Z}_{m+n+i+j} + \sum_{j=0}^{n-1} a^{m+n+k-1-j}\mathbf{Z}_{i+j} + \sum_{j=0}^{n-1} a^{n+k-1-j}\mathbf{Z}_{m+i+j}\Big) = \mathbf{O},$$

i.e.

$$\sum_{i=0}^{m+n+k} g\Big(\sum_{j=0}^{m+n+k-1} a^j\mathbf{Z}_{m-1+i-j}, \sum_{j=0}^{m+n+k-1} a^j\mathbf{Z}_{m+n-1+i-j},$$
$$\sum_{j=0}^{m+n+k-1} a^j\mathbf{Z}_{m+n+k-1+i-j}\Big) = \mathbf{O}.$$

This transformation of the equation (1) is possible since $a^{m+n+k} \neq 1$. If we introduce new variables $\mathbf{T}_i$ by the relations

$$\mathbf{T}_i = \sum_{j=0}^{m+n+k-1} a^j\mathbf{Z}_{m-1+i-j} \quad (1 \leq i \leq m+n+k),$$

then the previous equation becomes

$$\sum_{i=1}^{m+n+k} g(\mathbf{T}_i, \mathbf{T}_{i+n}, \mathbf{T}_{i+n+k}) = \mathbf{O}. \tag{44}$$

Now we will give the following results.

Lemma 1. *The general solution of the functional equation (44) is determined by*

1^0. $g(\mathbf{U},\mathbf{V},\mathbf{W}) = F(\mathbf{U}) - F(\mathbf{W}) - H(\mathbf{V}) - H(\mathbf{W})$

$$(m \neq n \neq k \neq m, \quad m \neq n+k, \quad n \neq m+k, \quad k \neq m+n),$$

2^0. $g(\mathbf{U},\mathbf{V},\mathbf{W}) = F(\mathbf{U}) - F(\mathbf{V}) - G(\mathbf{U},\mathbf{W}) - G(\mathbf{W},\mathbf{U})$

$$(m \neq n \neq k \neq m, \quad m = n+k),$$

3^0. $g(\mathbf{U},\mathbf{V},\mathbf{W}) = F(\mathbf{W}) - F(\mathbf{U}) - G(\mathbf{U},\mathbf{V}) - G(\mathbf{V},\mathbf{U})$

$$(m \neq n \neq k \neq m, \quad n = m+k),$$

4^0. $g(\mathbf{U},\mathbf{V},\mathbf{W}) = F(\mathbf{U}) - F(\mathbf{V}) + G(\mathbf{V},\mathbf{W}) - G(\mathbf{W},\mathbf{V})$

$$(m \neq n \neq k \neq m, \quad k = m+n),$$

5^0. $g(\mathbf{U},\mathbf{V},\mathbf{W}) = G(\mathbf{U},\mathbf{V}) - G(\mathbf{V},\mathbf{W})$

$$(m \neq n = k, \quad m \neq n+k),$$

6^0. $g(\mathbf{U},\mathbf{V},\mathbf{W}) = K(\mathbf{U},\mathbf{V}) - K(\mathbf{V},\mathbf{W}) + G(\mathbf{W},\mathbf{U}) - G(\mathbf{U},\mathbf{W})$

$$(m \neq n = k, \quad m = n+k),$$

7^0. $g(\mathbf{U},\mathbf{V},\mathbf{W}) = G(\mathbf{U},\mathbf{V}) - G(\mathbf{W},\mathbf{U})$

$$(m = n \neq k, \quad k \neq m+n),$$

8^0. $g(\mathbf{U},\mathbf{V},\mathbf{W}) = K(\mathbf{U},\mathbf{V}) - K(\mathbf{W},\mathbf{U}) + G(\mathbf{V},\mathbf{W}) - G(\mathbf{W},\mathbf{V})$

$$(m = n \neq k, \quad k = m+n),$$

9^0. $g(\mathbf{U},\mathbf{V},\mathbf{W}) = G(\mathbf{V},\mathbf{W}) - G(\mathbf{W},\mathbf{U})$

$$(m = k \neq n, \quad n \neq k+m),$$

10^0. $g(\mathbf{U},\mathbf{V},\mathbf{W}) = K(\mathbf{U},\mathbf{V}) - K(\mathbf{V},\mathbf{U}) + G(\mathbf{V},\mathbf{W}) - G(\mathbf{W},\mathbf{U})$

$$(m = k \neq n, \quad n = k+m),$$

11^0. $g(\mathbf{U},\mathbf{V},\mathbf{W}) = L(\mathbf{U},\mathbf{V},\mathbf{W}) - L(\mathbf{V},\mathbf{W},\mathbf{U})$

$$(m = n = k),$$

where $F, H : \mathcal{V} \to \mathcal{V}'$, $G, K : \mathcal{V}^2 \to \mathcal{V}'$ *and* $L : \mathcal{V}^3 \to \mathcal{V}'$ *are arbitrary complex vector functions.*

Proof. Since the proofs of the particular cases are similar or completely the same, we will prove the lemma only in the cases 3^0 and 8^0.

3^0. If all variables in (44), except for $\mathbf{Z}_i$, $\mathbf{Z}_{i+n}$ and $\mathbf{Z}_{i+n+k}$, are equal to some constant, we obtain

$$g(\mathbf{Z}_i, \mathbf{Z}_{i+n}, \mathbf{Z}_{i+n+k}) = F(\mathbf{Z}_{i+n+k}) + K(\mathbf{Z}_i, \mathbf{Z}_{i+n}),$$

i.e.

$$g(\mathbf{U}, \mathbf{V}, \mathbf{W}) = F(\mathbf{W}) + K(\mathbf{U}, \mathbf{V}). \tag{45}$$

Hence, the equation (44) becomes

$$\sum_{i=1}^{m+n+k} F(\mathbf{Z}_i) + \sum_{i=1}^{m+n+k} K(\mathbf{Z}_i, \mathbf{Z}_{i+n}) = \mathbf{O}.$$

If we put $\mathbf{Z}_r = \mathbf{O}$ $(r \neq i, i+n)$ in the above equation, we have

$$F(\mathbf{Z}_i) + F(\mathbf{Z}_{i+n}) + K(\mathbf{Z}_i, \mathbf{Z}_{i+n}) + K(\mathbf{Z}_{i+n}, \mathbf{Z}_i) = \mathbf{O},$$

and thus it follows that

$$K(\mathbf{U}, \mathbf{V}) = G(\mathbf{U}, \mathbf{V}) - G(\mathbf{V}, \mathbf{U}) - F(\mathbf{U}).$$

Now, the equality (45) becomes

$$g(\mathbf{U}, \mathbf{V}, \mathbf{W}) = F(\mathbf{W}) - F(\mathbf{U}) + G(\mathbf{U}, \mathbf{V}) - G(\mathbf{V}, \mathbf{U}).$$

8^0. If we put $\mathbf{Z}_r = \mathbf{O}$ $(r \neq i, i+n, i+n+k)$ into (44), then the equation (44) becomes

$$g(\mathbf{Z}_i, \mathbf{Z}_{i+n}, \mathbf{Z}_{i+n+k}) = K(\mathbf{Z}_i, \mathbf{Z}_{i+n}) + K_1(\mathbf{Z}_{i+n}, \mathbf{Z}_{i+n+k}) + K_2(\mathbf{Z}_{i+n+k}, \mathbf{Z}_i),$$

i.e.

$$g(\mathbf{U}, \mathbf{V}, \mathbf{W}) = K(\mathbf{U}, \mathbf{V}) + K_1(\mathbf{V}, \mathbf{W}) + K_2(\mathbf{W}, \mathbf{U}),$$

where K, K_1, $K_2 : \mathcal{V}^2 \to \mathcal{V}'$ are arbitrary complex vector functions.

By a substitution of the value obtained for g into (44), we get

$$\sum_{i=1}^{m+n+k} K(\mathbf{Z}_i, \mathbf{Z}_{i+n}) + \sum_{i=1}^{m+n+k} K_1(\mathbf{Z}_i, \mathbf{Z}_{i+n}) + \sum_{i=1}^{m+n+k} K_2(\mathbf{Z}_i, \mathbf{Z}_{i+n}) = \mathbf{O}. \tag{46}$$

Hence, for $\mathbf{Z}_r$ $(r \neq i, i+k)$ we obtain

$$K_2(\mathbf{Z}_i, \mathbf{Z}_{i+k}) + K_2(\mathbf{Z}_{i+k}, \mathbf{Z}_i) + H(\mathbf{Z}_i) + H(\mathbf{Z}_{i+k}) = \mathbf{O},$$

where $H : \mathcal{V} \to \mathcal{V}'$ is an arbitrary complex vector function.

From the above equation it follows that the function K_2 has the form

$$K_2(\mathbf{U}, \mathbf{V}) = G(\mathbf{U}, \mathbf{V}) - G(\mathbf{V}, \mathbf{U}) - H(\mathbf{U}), \tag{47}$$

where $G : \mathcal{V}^2 \to \mathcal{V}'$ is an arbitrary complex vector function.

If we put (47) into (46), we obtain

$$\sum_{i=1}^{m+n+k} [K(\mathbf{Z}_i, \mathbf{Z}_{i+n}) + K_1(\mathbf{Z}_i, \mathbf{Z}_{i+n}) + H(\mathbf{Z}_i)] = \mathbf{O}. \tag{48}$$

For $\mathbf{Z}_r = \mathbf{O}$ $(r \neq i, i+n)$ from the above equation we obtain

$$K_1(\mathbf{U}, \mathbf{V}) = -K(\mathbf{U}, \mathbf{V}) - 2H(\mathbf{U}) - 2H(\mathbf{V}).$$

By putting the above obtained value of K_1 into (48), we have

$$\sum_{i=1}^{m+n+k} H(\mathbf{Z}_i) = \mathbf{O},$$

i.e.

$$H(\mathbf{U}) = \mathbf{O}.$$

On the basis of the previous equalities, the function g is determined by

$$g(\mathbf{U}, \mathbf{V}, \mathbf{W}) = K(\mathbf{U}, \mathbf{V}) - K(\mathbf{W}, \mathbf{U}) + G(\mathbf{V}, \mathbf{W}) - G(\mathbf{W}, \mathbf{V}). \qquad \square$$

According to the previous lemma and the transformation (43), we obtain the following theorems.

Theorem 8. *If* $a^{m+n+k} \neq 1$, $m \neq n \neq k \neq m$, $m \neq n+k$, $n \neq m+k$ *and* $k \neq m+n$, *then the general solution of the functional equation (1) is*

$$f(\mathbf{U}, \mathbf{V}, \mathbf{W}) = F(\mathbf{U} + a^{k+m}\mathbf{V} + a^m\mathbf{W}) - F(\mathbf{W} + a^{n+k}\mathbf{U} + a^k\mathbf{V}) +$$
$$+ G(\mathbf{V} + a^{m+n}\mathbf{W} + a^n\mathbf{U}) - G(\mathbf{W} + a^{n+k}\mathbf{U} + a^k\mathbf{V}),$$

where $F, G : \mathcal{V} \to \mathcal{V}'$ *are arbitrary complex vector functions.*

Theorem 9. *If* $a^{m+n+k} \neq 1$, $m \neq n \neq k \neq m$ *and* $m = n + k$, *then the general solution of the functional equation (1) is*

$$f(\mathbf{U}, \mathbf{V}, \mathbf{W}) = F(\mathbf{U} + a^{k+m}\mathbf{V} + a^m\mathbf{W}) - F(\mathbf{V} + a^{m+n}\mathbf{W} + a^n\mathbf{U}) +$$
$$+ G(\mathbf{U} + a^{k+m}\mathbf{V} + a^m\mathbf{W},\ \mathbf{W} + a^{n+k}\mathbf{U} + a^k\mathbf{V}) -$$
$$- G(\mathbf{W} + a^{n+k}\mathbf{U} + a^k\mathbf{V},\ \mathbf{U} + a^{k+n}\mathbf{V} + a^m\mathbf{W}),$$

where $F : \mathcal{V} \to \mathcal{V}'$ *and* $G : \mathcal{V}^2 \to \mathcal{V}'$ *are arbitrary complex vector functions.*

Theorem 10. *If* $a^{m+n+k} \neq 1$, $m \neq n \neq k \neq m$ *and* $n = m + k$, *then the general solution of the functional equation (1) is*

$$f(\mathbf{U}, \mathbf{V}, \mathbf{W}) = F(\mathbf{W} + a^{n+k}\mathbf{U} + a^k\mathbf{V}) - F(\mathbf{U} + a^{k+m}\mathbf{V} + a^m\mathbf{W}) +$$
$$+ G(\mathbf{U} + a^{k+m}\mathbf{V} + a^m\mathbf{W},\ \mathbf{V} + a^{m+n}\mathbf{W} + a^n\mathbf{U}) -$$
$$- G(\mathbf{V} + a^{m+n}\mathbf{W} + a^n\mathbf{U},\ \mathbf{U} + a^{k+m}\mathbf{V} + a^m\mathbf{W}),$$

where $F : \mathcal{V} \to \mathcal{V}'$ *and* $G : \mathcal{V}^2 \to \mathcal{V}'$ *are arbitrary complex vector functions.*

Theorem 11. *If $a^{m+n+k} \neq 1$, $m \neq n \neq k \neq m$ and $k = m+n$, then the functional equation (1) has a general solution*

$$f(\mathbf{U},\mathbf{V},\mathbf{W}) = F(\mathbf{U} + a^{k+m}\mathbf{V} + a^m\mathbf{W}) - F(\mathbf{V} + a^{m+n}\mathbf{W} + a^n\mathbf{U}) +$$
$$+G(\mathbf{V} + a^{m+n}\mathbf{W} + a^n\mathbf{U},\ \mathbf{W} + a^{n+k}\mathbf{U} + a^k\mathbf{V}) -$$
$$-G(\mathbf{W} + a^{n+k}\mathbf{U} + a^k\mathbf{V},\ \mathbf{V} + a^{m+n}\mathbf{W} + a^n\mathbf{U}),$$

where $F : \mathcal{V} \to \mathcal{V}'$ and $G : \mathcal{V}^2 \to \mathcal{V}'$ are arbitrary complex vector functions.

Theorem 12. *If $a^{m+n+k} \neq 1$ and $m \neq n+k$, then the general solution of the functional equation (1) is*

$$f(\mathbf{U},\mathbf{V},\mathbf{W}) = F(\mathbf{U} + a^{k+m}\mathbf{V} + a^m\mathbf{W},\ \mathbf{V} + a^{m+n}\mathbf{W} + a^n\mathbf{U}) -$$
$$-F(\mathbf{V} + a^{m+n}\mathbf{W} + a^n\mathbf{U},\ U + a^{k+m}\mathbf{V} + a^m\mathbf{W}),$$

where $F : \mathcal{V}^2 \to \mathcal{V}'$ is an arbitrary complex vector function.

Theorem 13. *If $a^{m+n+k} \neq 1$, $m \neq n = k$ and $m = 2n$, then the functional equation (1) has a general solution determined by*

$$f(U,\mathbf{V},\mathbf{W}) = F(U + a^{k+m}\mathbf{V} + a^m\mathbf{W},\ \mathbf{V} + a^{m+n}\mathbf{W} + a^nU) -$$
$$-F(\mathbf{V} + a^{m+n}\mathbf{W} + a^nU,\ \mathbf{W} + a^{n+k}U + a^k\mathbf{V}) +$$
$$+G(\mathbf{W} + a^{n+k}U + a^k\mathbf{V},\ U + a^{k+m}\mathbf{V} + a^m\mathbf{W}) -$$
$$-G(U + a^{k+m}\mathbf{V} + a^m\mathbf{W},\ \mathbf{W} + a^{n+k}U + a^k\mathbf{V}),$$

where $F, G : \mathcal{V}^2 \to \mathcal{V}'$ are arbitrary complex vector functions.

Theorem 14. *If $a^{m+n+k} \neq 1$, $m = n \neq k$ and $k \neq m+n$, then the functional equation (1) has a general solution given by*

$$f(U,\mathbf{V},\mathbf{W}) = F(U + a^{k+m}\mathbf{V} + a^m\mathbf{W},\ \mathbf{V} + a^{m+n}\mathbf{W} + a^nU) -$$
$$-F(\mathbf{W} + a^{n+k}U + a^k\mathbf{V},\ U + a^{k+m}\mathbf{V} + a^m\mathbf{W}),$$

where $F : \mathcal{V}^2 \to \mathcal{V}'$ is an arbitrary complex vector function.

Theorem 15. *If $a^{m+n+k} \neq 1$, $m = n \neq k$ and $k = m+n$, then the the functional equation (1) has a general solution*

$$f(U,\mathbf{V},\mathbf{W}) = F(U + a^{k+m}\mathbf{V} + a^m\mathbf{W},\ \mathbf{V} + a^{m+n}\mathbf{W} + a^nU) -$$
$$-F(\mathbf{W} + a^{n+k}U + a^k\mathbf{V},\ U + a^{k+m}\mathbf{V} + a^m\mathbf{W}) +$$
$$+G(\mathbf{V} + a^{m+n}\mathbf{W} + a^nU,\ \mathbf{W} + a^{n+k}U + a^k\mathbf{V}) -$$
$$-G(\mathbf{W} + a^{n+k}U + a^k\mathbf{V},\ \mathbf{V} + a^{m+n}\mathbf{W} + a^nU),$$

where $F, G : \mathcal{V}^2 \to \mathcal{V}'$ are arbitrary complex vector functions.

Theorem 16. *If $a^{m+n+k} \neq 1$, $m = k \neq n$ and $n \neq m+k$, then the functional equation (1) has general solution*

$$f(U,\mathbf{V},\mathbf{W}) = F(\mathbf{V} + a^{m+n}\mathbf{W} + a^nU,\ \mathbf{W} + a^{n+k}U + a^k\mathbf{V}) -$$

$$-F(\mathbf{W}+a^{n+k}U+a^{k}\mathbf{V},\ U+a^{k+m}\mathbf{V}+a^{m}\mathbf{W}),$$

where $F:\mathcal{V}^2\to\mathcal{V}'$ is an arbitrary complex vector function.

Theorem 17. *If $a^{m+n+k}\neq 1$, $m=k\neq n$ and $n=m+k$, then the functional equation (1) has a solution given by*

$$f(U,\mathbf{V},\mathbf{W})=F(U+a^{k+m}\mathbf{V}+a^{m}\mathbf{W},\ \mathbf{V}+a^{m+n}\mathbf{W}+a^{n}U)-$$
$$-F(\mathbf{V}+a^{m+n}\mathbf{W}+a^{n}U,\ U+a^{k+m}\mathbf{V}+a^{m}\mathbf{W})+$$
$$+G(\mathbf{V}+a^{m+n}\mathbf{W}+a^{n}U,\ \mathbf{W}+a^{n+k}U+a^{k}\mathbf{V})-$$
$$-G(\mathbf{W}+a^{n+k}U+a^{k}\mathbf{V},\ U+a^{m+n}\mathbf{V}+a^{n}\mathbf{W}),$$

where $F,G:\mathcal{V}^2\to\mathcal{V}'$ are arbitrary complex vector functions.

Theorem 18. *If $a^{m+n+k}\neq 1$, $m=n=k$, then the general solution of the functional equation (1) is*

$$f(U,\mathbf{V},\mathbf{W})=F(U+a^{k+m}\mathbf{V}+a^{m}\mathbf{W},\ \mathbf{V}+a^{m+n}\mathbf{W}+a^{n}U,\ \mathbf{W}+a^{n+k}U+a^{k}\mathbf{V})-$$
$$-F(\mathbf{V}+a^{m+n}\mathbf{W}+a^{n}U,\ \mathbf{W}+a^{n+k}U+a^{k}\mathbf{V},\ U+a^{k+m}\mathbf{V}+a^{m}\mathbf{W}),$$

where $F:\mathcal{V}^3\to\mathcal{V}'$ is an arbitrary complex vector function.

III. If $a^{m+n+k}=1$, this case is very difficult and up to now we are not able to solve the functional equation (1).

2 Solution of the Generalized Functional Equation

Further, we will consider the solving of the functional equation (2). For the equation (2) we will determine the general solution only if $a^{m+n+k}\neq 1$.

Now, we transform the equation (2). If we put

$$f_i(U,\mathbf{V},\mathbf{W}) = g_i(U+a^{k+m}\mathbf{V}+a^{m}\mathbf{W},\ \mathbf{V}+a^{m+n}\mathbf{W}+a^{n}U,\ \mathbf{W}+a^{n+k}U+a^{k}\mathbf{V}), \tag{49}$$

then the equation (2) becomes

$$\sum_{i=1}^{m+n+k} g_i\Big(\sum_{j=0}^{m-1}a^{m-1-j}\mathbf{Z}_{i+j}+\sum_{j=0}^{n-1}a^{m+n+k-1-j}\mathbf{Z}_{m+i+j}+\sum_{j=0}^{k-1}a^{m+k-1-j}\mathbf{Z}_{m+n+i+j},$$
$$\sum_{j=0}^{n-1}a^{n-1-j}\mathbf{Z}_{m+i+j}+\sum_{j=0}^{k-1}a^{m+n+k-1-j}\mathbf{Z}_{m+n+i+j}+\sum_{j=0}^{m-1}a^{m+n-1-j}\mathbf{Z}_{i+j},$$
$$\sum_{j=0}^{k-1}a^{k-1-j}\mathbf{Z}_{m+n+i+j}+\sum_{j=0}^{m-1}a^{m+n+k-1-j}\mathbf{Z}_{i+j}+\sum_{j=0}^{n-1}a^{n+k-1-j}\mathbf{Z}_{m+i+j}\Big)=\mathbf{O},$$

i.e.

$$\sum_{i=1}^{m+n+k} g_i\Big(\sum_{j=0}^{m+n+k-1}a^{j}\mathbf{Z}_{m-1+i-j},\ \sum_{j=0}^{m+n+k-1}a^{j}\mathbf{Z}_{m+n-1+i-j}, \tag{50}$$

$$\sum_{j=0}^{m+n+k-1} a^j \mathbf{Z}_{m+n+k-1+i-j}\Big) = \mathbf{O}.$$

Since the linear forms

$$\mathbf{T}_i = \sum_{j=0}^{m+n+k-1} a^j \mathbf{Z}_{m-1+i-j} \quad (1 \le i \le m+n+k)$$

are linearly independent, it is possible to introduce new variables $\mathbf{T}_i$ which are determined by the above relations.

Therefore, the functional equation (50) takes the following form

$$\sum_{i=1}^{m+n+k} g_i(\mathbf{T}_i, \mathbf{T}_{i+n}, \mathbf{T}_{i+n+k}) = \mathbf{O}. \tag{51}$$

For the last equation the following lemma holds.

Lemma 2. *The general solution of the functional equation (51) is given by the equalities*

1^0. $g_i(U, \mathbf{V}, \mathbf{W}) = F_i(U) - F_{i+n+k}(\mathbf{W}) - H_i(\mathbf{V}) - H_{i+k}(\mathbf{W}) - A_i,$

$$\sum_{i=1}^{m+n+k} A_i = \mathbf{O}$$

$$(m \ne n \ne k \ne m, \quad m \ne n+k, \quad n \ne m+k, \quad k \ne m+n),$$

2^0. $g_i(U, \mathbf{V}, \mathbf{W}) = F_i(\mathbf{V}) - F_{i+k}(\mathbf{W}) + G_i(U, \mathbf{W}) \quad (1 \le i \le m),$

$$g_i(U, \mathbf{V}, \mathbf{W}) = F_i(\mathbf{V}) - F_{i+k}(\mathbf{W}) - G_{i+m}(\mathbf{W}, U) - A_{i+m} \quad (m+1 \le i \le 2m),$$

$$\sum_{i=1}^{m} A_i = \mathbf{O}$$

$$(m \ne n \ne k \ne m, \quad m \ne n+k),$$

3^0. $g_i(U, \mathbf{V}, \mathbf{W}) = F_i(\mathbf{W}) - F_{i+m}(U) + G_i(U, \mathbf{V}) \quad (1 \le i \le n),$

$$g_i(U, \mathbf{V}, \mathbf{W}) = F_i(\mathbf{W}) - F_{i+m}(U) - G_{i+m}(\mathbf{V}, U) - A_{i+m} \quad (n+1 \le i \le 2n),$$

$$\sum_{i=1}^{n} A_i = \mathbf{O}$$

$$(m \ne n \ne k \ne m, \quad n = m+k),$$

4^0. $g_i(U, \mathbf{V}, \mathbf{W}) = F_i(U) - F_{i+n}(\mathbf{V}) + G_i(\mathbf{V}, \mathbf{W}) \quad (1 \le i \le k),$

$$g_i(U, \mathbf{V}, \mathbf{W}) = F_i(U) - F_{i+n}(\mathbf{V}) - G_{i+k}(\mathbf{W}, \mathbf{V}) - A_{i+k} \quad (k+1 \le i \le 2k),$$

$$\sum_{i=1}^{n} A_i = \mathbf{O}$$

$$(m \neq n \neq k \neq m, \quad k = m + n),$$

5^0. $g_i(U, \mathbf{V}, \mathbf{W}) = G_i(U, \mathbf{V}) - G_{i+n}(\mathbf{V}, \mathbf{W}) + A_i \quad (1 \leq i \leq m + n + k),$

$$\sum_{i=1}^{m+n+k} A_i = \mathbf{O}$$

$$(m \neq n = k, \quad m \neq n + k),$$

6^0. $g_i(U, \mathbf{V}, \mathbf{W}) = K_i(U, \mathbf{V}) - K_{i+n}(\mathbf{V}, \mathbf{W}) + G_i(\mathbf{W}, U) \quad (1 \leq i \leq m),$

$$g_i(U, \mathbf{V}, \mathbf{W}) = K_i(U, \mathbf{V}) - K_{i+n}(\mathbf{V}, \mathbf{W}) - G_{i+m}(U, \mathbf{W}) \quad (m + 1 \leq i \leq 2m),$$

$$(m \neq n = k, \quad m = n + k),$$

7^0. $g_i(U, \mathbf{V}, \mathbf{W}) = G_i(U, \mathbf{V}) - G_{i+n+k}(\mathbf{W}, U) + A_i \ (1 \leq i \leq m + n + k),$

$$\sum_{i=1}^{m+n+k} A_i = \mathbf{O}$$

$$(m = n \neq k, \quad k \neq m + n),$$

8^0. $g_i(U, \mathbf{V}, \mathbf{W}) = K_i(U, \mathbf{V}) - K_{i+n+k}(\mathbf{W}, U) + G_i(\mathbf{V}, \mathbf{W}) + A_i \quad (1 \leq i \leq k),$

$$\sum_{i=1}^{k} A_i = \mathbf{O},$$

$$g_i(U, \mathbf{V}, \mathbf{W}) = K_i(U, \mathbf{V}) - K_{i+n+k}(\mathbf{W}, U) - G_i(\mathbf{V}, \mathbf{W}) \quad (k + 1 \leq i \leq 2k),$$

$$(m = n \neq k, \quad k = m + n),$$

9^0. $g_i(U, \mathbf{V}, \mathbf{W}) = G_i(U, \mathbf{V}) - G_{i+m+n}(\mathbf{W}, \mathbf{V}) + A_i \quad (1 \leq i \leq m + n + k),$

$$\sum_{i=1}^{m+n+k} A_i = \mathbf{O}$$

$$(m = k \neq n, \quad n \neq m + k),$$

10^0. $g_i(U, \mathbf{V}, \mathbf{W}) = K_i(\mathbf{V}, \mathbf{W}) - K_{i+k}(\mathbf{W}, U) + G_i(U, \mathbf{V}) + A_i \quad (1 \leq i \leq n),$

$$\sum_{i=1}^{n} A_i = \mathbf{O},$$

$$g_i(U, \mathbf{V}, \mathbf{W}) = K_i(\mathbf{V}, \mathbf{W}) - K_{i+k}(\mathbf{W}, U) - G_{i+n}(\mathbf{V}, U) \quad (n + 1 \leq i \leq 2n),$$

$$(m = k \neq n, \quad n = m + k),$$

11^0. $g_i(U, \mathbf{V}, \mathbf{W}) = L_i(U, \mathbf{V}, \mathbf{W}) \quad (1 \leq i \leq 2m),$

$$g_i(U, \mathbf{V}, \mathbf{W}) = -L_i(\mathbf{V}, \mathbf{W}, U) - L_{i+2m}(\mathbf{W}, U, \mathbf{V}) \quad (2m + 1 \leq i \leq 3m),$$

$$(m = n = k).$$

In all cases the functions F_i, $H_i : \mathcal{V} \to \mathcal{V}'$, G_i, $K_i : \mathcal{V}^2 \to \mathcal{V}'$ and $L_i : \mathcal{V}^3 \to \mathcal{V}'$ are arbitrary complex vector functions such that $F_{i+n+m+k} = F_i$, $G_{i+n+m+k} = G_i$, $\cdots$. Also, for the arbitrary constant complex vectors A_i the equalities $A_{i+n+m+k} = A_i$ hold.

Proof. Since the statements of the lemma for particular cases may be proven in a similar way, therefore it is sufficient to prove the lemma only for some cases, for example 4^0 and 5^0.

4^0. If in (51) all variables, except $\mathbf{T}_i$, $\mathbf{T}_{i+n}$ and $\mathbf{T}_{i+n+k}$, are equal to some constant, we obtain

$$g_i(\mathbf{T}_i, \mathbf{T}_{i+n}, \mathbf{T}_{i+n+k}) = F_i(\mathbf{T}_i) + K_i(\mathbf{T}_{i+n}, \mathbf{T}_{i+n+k}),$$

i.e.

$$g_i(U, \mathbf{V}, \mathbf{W}) = F_i(U) + K_i(\mathbf{V}, \mathbf{W}), \tag{52}$$

where F_i and K_i $(1 \le i \le m + n + k)$ are arbitrary complex vector functions.

On the basis of the equation (52), the equation (51) becomes

$$\sum_{i=1}^{m+n+k} F_i(\mathbf{T}_i) + \sum_{i=1}^{m+n+k} K_i(\mathbf{T}_i, \mathbf{T}_{i+n+k}) = \mathbf{O},$$

i.e.

$$\sum_{i=1}^{m+n+k} F_{i+n}(\mathbf{T}_{i+n}) + \sum_{i=1}^{m+n+k} K_i(\mathbf{T}_i, \mathbf{T}_{i+n+k}) = \mathbf{O}.$$

If we put $\mathbf{T}_r = \mathbf{O}$ $(r = i + n, i + n + k)$ into the previous equality, we obtain

$$\begin{aligned} F_{i+n}(\mathbf{T}_{i+n}) + F_{i+n+k}(\mathbf{T}_{i+n+k}) + K_i(\mathbf{T}_{i+n}, \mathbf{T}_{i+n+k}) \\ +K_{i+k}(\mathbf{T}_{i+n+k}, \mathbf{T}_{i+n}) + A_{i+k} = \mathbf{O}, \end{aligned} \tag{53}$$

where A_i are arbitrary constant complex vectors.

From (53) we obtain

$$K_i(\mathbf{V}, \mathbf{W}) = -K_{i+m}(\mathbf{W}, \mathbf{V}) - F_{i+n}(\mathbf{V}) - F_{i+n+k}(\mathbf{W}) - A_{i+k},$$

i.e. the equation (52) takes the form

$$g_i(U, \mathbf{V}, \mathbf{W}) = F_i(U) + K_i(\mathbf{V}, \mathbf{W}) \quad (1 \le i \le k),$$

$$g_i(U, \mathbf{V}, \mathbf{W}) = F_i(U) - K_{i+k}(\mathbf{W}, \mathbf{V}) - F_{i+n}(\mathbf{V}) - F_{i+n+k}(\mathbf{W}) - A_{i+k}$$
$$(k + 1 \le i \le 2k),$$

such that

$$\sum_{i=1}^{k} A_i = \mathbf{O}.$$

If we introduce new functions G_i by

$$G_i(\mathbf{V}, \mathbf{W}) = K_i(\mathbf{V}, \mathbf{W}) + F_{i+n}(\mathbf{V}),$$

the previous equations take the following form

$$g_i(U,\mathbf{V},\mathbf{W}) = F_i(U) - F_{i+n}(\mathbf{V}) + G_i(\mathbf{V},\mathbf{W}) \quad (1 \le i \le k),$$
$$g_i(U,\mathbf{V},\mathbf{W}) = F_i(U) - F_{i+n}(\mathbf{V}) - G_{i+k}(\mathbf{W},\mathbf{V}) - A_{i+k} \quad (k+1 \le i \le 2k),$$

where

$$\sum_{i=1}^{k} A_i = \mathbf{O}.$$

5^0. If $\mathbf{T}_j = \mathbf{O}$ $(j \ne i, i+n, i+n+k)$, from the equation (51) we obtain

$$g_i(\mathbf{T}_i, \mathbf{T}_{i+n}, \mathbf{T}_{i+n+k}) = K_i(\mathbf{T}_i, \mathbf{T}_{i+n}) + M_i(\mathbf{T}_{i+n}, \mathbf{T}_{i+n+k}),$$

i.e.

$$g_i(U,\mathbf{V},\mathbf{W}) = K_i(U,\mathbf{V}) + M_i(\mathbf{V},\mathbf{W}), \tag{54}$$

where K_i and M_i are arbitrary complex vector functions.

On the basis of the relation (54), the functional equation (51) becomes

$$\sum_{i=1}^{m+n+k} K_i(\mathbf{T}_i, \mathbf{T}_{i+n}) + \sum_{i=1}^{m+n+k} M_i(\mathbf{T}_{i+n}, \mathbf{T}_{i+n+k}) = \mathbf{O},$$

i.e.

$$\sum_{i=1}^{m+n+k} K_i(\mathbf{T}_i, \mathbf{T}_{i+n}) + \sum_{i=1}^{m+n+k} M_{i+m+n}(\mathbf{T}_i, \mathbf{T}_{i+n}) = \mathbf{O}. \tag{55}$$

If we put $\mathbf{T}_r = \mathbf{O}$ $(r \ne i, i+n)$ into (55), then

$$K_i(\mathbf{T}_i, \mathbf{T}_{i+n}) + M_{i+m+n}(\mathbf{T}_i, \mathbf{T}_{i+n}) + P_i(\mathbf{T}_i) - Q_{i+n}(\mathbf{T}_{i+n}) = \mathbf{O}, \tag{56}$$

where P_i and Q_i are arbitrary complex vector functions.

On the basis of the above equation (65), the equation (55) becomes

$$\sum_{i=1}^{m+n+k} P_i(\mathbf{T}_i) - \sum_{i=1}^{m+n+k} Q_i(\mathbf{T}_i) = \mathbf{O},$$

from which we obtain

$$Q_i(\mathbf{T}_i) = P_i(\mathbf{T}_i) + B_i, \qquad \sum_{i=1}^{m+n+k} B_i = \mathbf{O}.$$

Therefore, the equality (56) takes the form

$$K_i(\mathbf{T}_i, \mathbf{T}_{i+n}) + M_{i+m+k}(\mathbf{T}_i, \mathbf{T}_{i+n}) + P_i(\mathbf{T}_i) - P_{i+n}(\mathbf{T}_{i+n}) - B_{i+n} = \mathbf{O}.$$

On the basis of this equality and (54), we have

$$g_i(U,\mathbf{V},\mathbf{W}) = K_i(U,\mathbf{V}) - K_{i+n}(\mathbf{V},\mathbf{W}) + P_{i+n}(U) - P_{i+2n}(\mathbf{V}) + B_{i+2n},$$

i.e.

$$g_i(U,\mathbf{V},\mathbf{W}) = G_i(U,\mathbf{V}) - G_{i+n}(\mathbf{V},\mathbf{W}) + A_i, \quad \sum_{i=1}^{m+n+k} A_i = \mathbf{O},$$

where we put

$$G_i(U,\mathbf{V}) = K_i(U,\mathbf{V}) + P_{i+n}(U), \quad A_i = B_{i+2n}.$$

On the basis of Lemma 2 and the transformations (49), there follow the theorems which treat the functional equation (2).

Theorem 19. *If* $a^{m+n+k} \neq 1$, $m \neq n \neq k \neq m$, $m \neq n+k$, $n \neq m+k$ *and* $k \neq m+n$, *then the general solution of the functional equation (2) is*

$$\begin{aligned}
&f_i(U,V,W) = F_i(V + a^{m+n}W + a^nU) - F_{i+k}(W + a^{n+k}U + a^kV) + \\
&+G_i(U + a^{k+m}V + a^mW,\ W + a^{n+k}U + a^kV) \quad (1 \leq i \leq m),\\
&f_i(U,V,W) = F_i(V + a^{m+n}W + a^nU) - F_{i+k}(W + a^{n+k}U + a^kV) - \\
&-G_{i+m}(W + a^{n+k}U + a^kV,\ U + a^{k+m}V + a^mW) - A_{i+m} \quad (m+1 \leq i \leq 2m),
\end{aligned}$$

$$\sum_{i=1}^{m} A_i = \mathbf{O},$$

where $F_i : \mathcal{V} \to \mathcal{V}'$ *and* $G_i : \mathcal{V}^2 \to \mathcal{V}'$ *are arbitrary complex vector functions, and* A_i *are arbitrary constant complex vectors such that* $\sum_{i=1}^{m+n+k} A_i = \mathbf{O}$.

Theorem 20. *If* $a^{m+n+k} \neq 1$, $m \neq n \neq k \neq m$ *and* $m = n+k$, *then the general solution of the functional equation (2) is*

$$\begin{aligned}
&f_i(U,V,W) = F_i(V + a^{m+n}W + a^nU) - F_{i+k}(W + a^{n+k}U + a^kV) + \\
&+G_i(U + a^{k+m}V + a^mW,\ W + a^{n+k}U + a^kV) \quad (1 \leq i \leq m),\\
&f_i(U,V,W) = F_i(V + a^{m+n}W + a^nU) - F_{i+k}(W + a^{n+k}U + a^kV) - \\
&-G_{i+m}(W + a^{n+k}U + a^kV,\ U + a^{k+m}V + a^mW) - A_{i+m} \quad (m+1 \leq i \leq 2m),
\end{aligned}$$

where $F_i : \mathcal{V} \to \mathcal{V}'$ *and* $G_i : \mathcal{V}^2 \to \mathcal{V}'$ *are arbitrary complex vector functions, and* A_i *are arbitrary constant complex vectors such that* $\sum_{i=1}^{m} A_i = \mathbf{O}$.

Theorem 21. *If* $a^{m+n+k} \neq 1$, $m \neq n \neq k \neq m$ *and* $n = m+k$, *then the general solution of the functional equation (2) is*

$$\begin{aligned}
&f_i(U,V,W) = F_i(W + a^{n+k}U + a^kV) - F_{i+m}(U + a^{k+m}V + a^mW) + \\
&+G_i(U + a^{k+m}V + a^mW,\ V + a^{m+n}W + a^nU) \quad (1 \leq i \leq n),\\
&f_i(U,V,W) = F_i(W + a^{n+k}U + a^kV) - F_{i+m}(U + a^{k+m}V + a^mW) - \\
&-G_{i+n}(V + a^{m+n}W + a^nU,\ U + a^{k+m}V + a^mW) - A_{i+n} \quad (n+1 \leq i \leq 2n),
\end{aligned}$$

where $F_i : \mathcal{V} \to \mathcal{V}'$ *and* $G_i : \mathcal{V}^2 \to \mathcal{V}'$ *are arbitrary complex vector functions,* A_i *are arbitrary constant complex vectors such that* $\sum_{i=1}^{n} A_i = \mathbf{O}$.

Theorem 22. *If* $a^{m+n+k} \neq 1,\ m \neq n \neq k \neq m$ *and* $k = m + n$, *then the general solution of the functional equation (2) is given by*

$$f_i(U,V,W) = F_i(U + a^{k+m}V + a^m W) - F_{i+n}(V + a^{m+n}W + a^n U)+$$
$$+G_i(V + a^{m+n}W + a^n U,\ W + a^{n+k}U + a^k V) \quad (1 \leq i \leq k),$$
$$f_i(U,V,W) = F_i(U + a^{k+m}V + a^m W) - F_{i+n}(V + a^{m+n}W + a^n U)-$$
$$-G_{i+k}(W + a^{n+k}U + a^k V,\ V + a^{m+n}W + a^n U) - A_{i+k} \quad (k+1 \leq i \leq 2k),$$

where $F_i : \mathcal{V} \to \mathcal{V}'$ *and* $G_i : \mathcal{V}^2 \to \mathcal{V}'$ *are arbitrary complex vector functions,* A_i *are arbitrary constant complex vectors such that* $\sum_{i=1}^{k} A_i = \mathbf{O}$.

Theorem 23. *If* $a^{m+n+k} \neq 1,\ m \neq n = k$ *and* $m \neq n + k$, *then the general solution of the functional equation (2) is*

$$f_i(U,V,W) = F_i(U + a^{k+m}V + a^m W,\ V + a^{m+n}W + a^n U)-$$
$$-F_{i+n}(V + a^{m+n}W + a^n U,\ W + a^{n+k}U + a^k V) + A_i \quad (1 \leq i \leq m+n+k),$$

where $F_i : \mathcal{V}^2 \to \mathcal{V}'$ *are arbitrary complex vector functions,* A_i *are arbitrary constant complex vectors such that* $\sum_{i=1}^{m+n+k} A_i = \mathbf{O}$.

Theorem 24. *If* $a^{m+n+k} \neq 1,\ m \neq n = k$ *and* $m = n + k$, *then the general solution of the functional equation (2) is determined by*

$$f_i(U,V,W) = F_i(U + a^{k+m}V + a^m W,\ V + a^{m+n}W + a^n U)-$$
$$-F_{i+n}(V + a^{m+n}W + a^n U,\ W + a^{n+k}U + a^k V)+$$
$$+G_i(W + a^{n+k}U + a^k V,\ U + a^{k+m}V + a^m W) \quad (1 \leq i \leq m),$$
$$f_i(U,V,W) = F_i(U + a^{k+m}V + a^m W,\ V + a^{m+n}W + a^n U)-$$
$$-F_{i+n}(V + a^{m+n}W + a^n U,\ W + a^{n+k}U + a^k V)-$$
$$-G_{i+m}(U + a^{k+m}V + a^m W,\ W + a^{n+k}U + a^k V) \quad (m+1 \leq i \leq 2m),$$

where $F_i, G_i : \mathcal{V}^2 \to \mathcal{V}'$ *are arbitrary complex vector functions.*

Theorem 25. *If* $a^{m+n+k} \neq 1,\ m = n \neq k$ *and* $k \neq m + n$, *then the general solution of the functional equation (2) is*

$$f_i(U,V,W) = F_i(U + a^{k+m}V + a^m W,\ V + a^{m+n}W + a^n U)-$$
$$-F_{i+n+k}(W + a^{n+k}U + a^k V,\ U + a^{k+m}V + a^m W) + A_i \quad (1 \leq i \leq m+n+k),$$

where $F_i : \mathcal{V}^2 \to \mathcal{V}'$ *are arbitrary complex vector functions,* A_i *are arbitrary constant complex vectors such that* $\sum_{i=1}^{m+n+k} A_i = \mathbf{O}$.

Theorem 26. *If $a^{m+n+k} \neq 1$ and $k = m + n$, then the general solution of the functional equation (2) is*

$$
\begin{aligned}
&f_i(U,V,W) = F_i(U + a^{k+m}V + a^m W,\ V + a^{m+n}W + a^n U)-\\
&-F_{i+n+k}(W + a^{n+k}U + a^k V,\ U + a^{k+m}V + a^m W)+\\
&+G_i(V + a^{m+n}W + a^n U,\ W + a^{n+k}U + a^k V) + A_i \quad (1 \le i \le k),\\
&f_i(U,V,W) = F_i(U + a^{k+m}V + a^m W,\ V + a^{m+n}W + a^n U)-\\
&-F_{i+n+k}(W + a^{n+k}U + a^k V,\ U + a^{k+m}V + a^m W)-\\
&-G_{i+k}(W + a^{n+k}U + a^k V,\ V + a^{m+n}W + a^n U) \quad (k+1 \le i \le 2k),
\end{aligned}
$$

where F_i, $G_i : \mathcal{V}^2 \to \mathcal{V}'$ are arbitrary complex vector functions, and A_i are arbitrary constant complex vectors such that $\sum_{i=1}^{k} A_i = \mathbf{O}$.

Theorem 27. *If $a^{m+n+k} \neq 1$, $m = k \neq n$ and $n \neq m + k$, then the general solution of the functional equation (2) is*

$$
f_i(U,V,W) = F_i(U + a^{k+m}V + a^m W,\ V + a^{m+n}W + a^n U)-
$$

$$
-F_{i+m+n}(W + a^{n+k}U + a^k V,\ V + a^{m+n}W + a^n U) + A_i \quad (1 \le i \le m+n+k),
$$

where $F_i : \mathcal{V}^2 \to \mathcal{V}'$ are arbitrary complex vector functions, and A_i are arbitrary constant complex vectors such that $\sum_{i=1}^{m+n+k} A_i = \mathbf{O}$.

Theorem 28. *If $a^{m+n+k} \neq 1$, $m = k \neq n$ and $n = m + k$, then the general solution of the functional equation (2) is determined by*

$$
\begin{aligned}
&f_i(U,V,W) = F_i(V + a^{m+n}W + a^n U,\ W + a^{n+k}U + a^k V)-\\
&-F_{i+k}(W + a^{n+k}U + a^k V,\ U + a^{k+m}V + a^m W)+\\
&+G_i(U + a^{k+m}V + a^m W,\ V + a^{m+n}W + a^n U) + A_i \quad (1 \le i \le n),\\
&f_i(U,V,W) = F_i(V + a^{m+n}W + a^n U,\ W + a^{n+k}U + a^k V)-\\
&-F_{i+k}(W + a^{n+k}U + a^k V,\ U + a^{k+m}V + a^m W)-\\
&-G_{i+n}(V + a^{m+n}W + a^n U,\ U + a^{k+m}V + a^m W) \quad (n+1 \le i \le 2n),
\end{aligned}
$$

where F_i, $G_i : \mathcal{V}^2 \to \mathcal{V}'$ are arbitrary complex vector functions, and A_i are arbitrary constant complex vectors such that $\sum_{i=1}^{n} A_i = \mathbf{O}$.

Theorem 29. *If $a^{m+n+k} \neq 1$, $m = n = k$, then the general solution of the functional equation (2) is*

$$
f_i(U,V,W) = F_i(U + a^{k+m}V + a^m W,\ V + a^{m+n}W + a^n U,\ W + a^{n+k}U + a^k V)
$$

$$
(1 \le i \le 2m),
$$

$$f_i(U,V,W) = -F_i(V + a^{m+n}W + a^nU,\ W + a^{n+k}U + a^kV,\ U + a^{k+m}V + a^mW) -$$
$$-F_{i+2m}(W + a^{n+k}U + a^kV,\ U + a^{k+m}V + a^mW,\ V + a^{m+n}W + a^nU)$$
$$(2m + 1 \leq i \leq 3m),$$

where $F_i : \mathcal{V}^3 \to \mathcal{V}'$ *are arbitrary complex vector functions.*

If $a = 1$ or $a^{m+n+k} = 1$ $(a \neq 1)$, then the solution of the functional equation (2) is very complicated and up to now we cannot obtain it.

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