R-Separated Spaces

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Abstract

In this paper we have generalized the axioms of the separated spaces T_i, $(i = \overline{0,4})$, by replacing the equality relation on a topological space X, Δ_X, by an arbitrary binary relation, R. Many theorems in general topology may be generalized in this way. It will be interesting to study spaces separated by functions, equivalence relations or order relations. In section 1 are presented axioms and caracterizing theorems of R-separation, in section 2 are presented some properties of spaces separeted by equivalence relations and in section 3 we will obtain some results concerning spaces separated by functions.

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1 Axioms and theorems of R separation

First let us make the following notations: $(X, \mathcal{T})$ is a topological space; $R \subset X \times X$ is a binary relation on X; $\overline{R}$ is the dual of R $\left(\text{i.e.} x\overline{R}y \Leftrightarrow (x,y) \notin R\right)$; R^{-1} is the inverse of R (*i.e.* $xR^{-1}y \Leftrightarrow (y,x) \in R$); $xRA \Leftrightarrow xRy$, $(\forall)$ $y \in A$; $x\overline{R}A \Leftrightarrow x\overline{R}y$ $(\forall)$ $y \in A$; $ARB \Leftrightarrow xRy$, $(\forall)$ $x \in A$, $(\forall)$ $y \in B$; $A\overline{R}B \Leftrightarrow x\overline{R}y$, $(\forall)$ $x \in A$, $(\forall)$ $y \in B$; $R(x) = \{y \mid xRy\}$ and $R^{-1}(x) = \{y \mid yRx\}$; $R(A) = \{y \mid (\exists)\ x \in A \text{ so that } xRy\}$; $R^{-1}(A) = \{y \mid (\exists)\ x \in A \text{ so that } yRx\}$; $\mathcal{V}_x$ is the neighborhood filter of $x \in X$; For any $A \subset X$ we note by $\mathcal{V}_A = \{B \mid (\exists)\ D \in \mathcal{T} \text{ so that } A \subset D \subset B\}$ and we note $CA = X \backslash A$.

We will replace in the classical definitions of the separated spaces $x \neq y$ by $x\overline{R}y; x \notin A$ by $x\overline{R}A$ or $A\overline{R}x$ and $A \cap B$ by $A\overline{R}B$. Replacing in the following considerations the relation R by Δ_X (i.e. $xRy \Leftrightarrow x = y$), we shall find the classical case of T_i spaces $(i = \overline{0,4})$.

Definition 1. X is T_0^R - space iff $(\forall) x, y \in X$ with $x\overline{R}y$, $(\exists)\, V_x \in \mathcal{V}_x$ so that $V_x\overline{R}y$ or $(\exists)\, V_y \in \mathcal{V}_y$ so that $x\overline{R}V_y$.

Definition 2. X is T_1^R - space iff $(\forall)$ $x, y \in X$ with $x\overline{R}y$, $(\exists)\, V_x \in \mathcal{V}_x$ and $(\exists)\, V_y \in \mathcal{V}_y$ so that $x\overline{R}V_y$ and $V_x\overline{R}y$.

Definition 3. X is T_2^R - space iff $(\forall)\, x, y \in X$ with $x\overline{R}y$, $(\exists)\, V_x \in \mathcal{V}_x$, $(\exists)\, V_y \in \mathcal{V}_y$ so that $V_x\overline{R}V_y$.

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Remark 1. $T_2^R \subset T_1^R \subset T_0^R$.
Remark 2. If in the definitions 1,2,3 we take xRy iff $x = y$, then they become those in the classical case.
Theorem 4. *X is T_0^R - space iff for each $x, y \in X$ we have: $y \in \overline{R(x)}$ and $x \in \overline{R^{-1}(y)} \Rightarrow xRy$, where by $\overline{A}$ we note the closure of the subset A of X.*
Proof. $V_x \overline{R} y$ or $x \overline{R} V_y \Leftrightarrow V_x \cap R^{-1}(y) = \emptyset$ or $R(x) \cap V_y = \emptyset \Leftrightarrow x \notin \overline{R^{-1}(y)}$ or $y \notin \overline{R(x)}$ for $x\overline{R}y \Leftrightarrow$if $x \in \overline{R^{-1}(y)}$ and $y \in \overline{R(x)}$, then xRy. □
Theorem 5. *X is T_1^R- space iff $R(x)$ and $R^{-1}(x)$ are closed subsets, for every $x \in X$.*
Proof. "$\Rightarrow$". Suppose that X is T_1^R - space and $x\overline{R}y$. Then exists $V_x \in \mathcal{V}_x$ an $V_y \in \mathcal{V}_x$ so that $x\overline{R}V_y$ and $V_x\overline{R}y$. From here: $R(x) \cap V_y = \emptyset$ and $V_x \cap R^{-1}(y) = \emptyset \Rightarrow y \notin \overline{R(x)}$ and $x \in \overline{R^{-1}(y)}$. So:

(a) X is T_1^R-space $\Rightarrow$ for each $y \notin \overline{R(x)}$ we have $y \notin \overline{R(x)} \Rightarrow \overline{R(x)} \subset R(x) \Rightarrow R(x) = \overline{R(x)}$;

(b) $\overline{R^{-1}(y)} = R^{-1}(y)$, just like in (a).

"$\Leftarrow$". (a) If $R^{-1}(y)$ is a closed subset of X, then for each $x\overline{R}y$, we have $x \notin \overline{R^{-1}(y)}$. From here: $(\exists)$ $V_x \in \mathcal{V}_x$ with $V_x \cap R^{-1}(y) = \emptyset \Rightarrow V_x\overline{R}y$. (b) $(\exists)$ $V_y \in \mathcal{V}_y$ with $x\overline{R}V_y$ just like in (a). □
Theorem 6. *X is T_2^R - space iff R is a closed subset of $X \times X$.*
Proof. X is T_2^R - space iff $(\forall) x\overline{R}y$, $(\exists) V_x \in \mathcal{V}_x$ and $(\exists) V_y \in \mathcal{V}_y$ with $V_x\overline{R}V_y \Leftrightarrow (\forall)(x,y) \notin R, (\exists) V_x \in \mathcal{V}_x, (\exists) V_y \in \mathcal{V}_y$, so that $V_x \times V_y \subset \overline{R} \Leftrightarrow (\forall)(x,y) \in \overline{R}$, then it is interior point of $\overline{R} \Leftrightarrow \overline{R}$ is an open subset of $X \times X$. □
Theorem 7. *X is T_2^R - space iff for each $x \in X$ we have $\bigcap\limits_{V \in \mathcal{V}_x} \overline{R(V)} = R(x)$ and for every $y \in X$ we have $\bigcap\limits_{U \in \mathcal{V}_y} \overline{R^{-1}(U)} = R^{-1}(y)$.*
Proof. "$\Rightarrow$". Suppose X is T_2^R - space and $y \in \bigcap\limits_{V \in \mathcal{V}_x} \overline{R(V)} \Rightarrow y \in \overline{R(V)}$, $(\forall) V \in \mathcal{V}_x$. If $y \notin R(x) \Rightarrow x\overline{R}y \Rightarrow (\exists) V_x \in \mathcal{V}_x$ and $(\exists) V_y \in \mathcal{V}_y$, so that $V_x\overline{R}V_y \Rightarrow R(V_x) \cap V_y = \emptyset \Rightarrow y \notin \overline{R(V_x)}$, contradiction, so $y \in R(x)$ and from here $\bigcap\limits_{V \in \mathcal{V}_x} \overline{R(V)} \subset R(x) \Rightarrow \bigcap\limits_{V \in \mathcal{V}_x} \overline{R(V)} = R(x)$.

In the same way we infer $\bigcap\limits_{U \in \mathcal{V}_y} \overline{R^{-1}(U)} = R^{-1}(y)$.

"$\Leftarrow$". Suppose $x\overline{R}y \Rightarrow y \notin R(x) \Rightarrow y \notin \bigcap\limits_{V \in \mathcal{V}_x} \overline{R(V)} \Rightarrow (\exists)$ $V_x \in \mathcal{V}_x$ so that $y \notin \overline{R(V_x)} \Rightarrow (\exists)$ $V_y \in \mathcal{V}_y$ so that $V_y \cap R(V_x) = \emptyset \Rightarrow V_x\overline{R}V_y \Rightarrow X$ is T_2^R. □
Remark 3. It is enough to replace in Theorem 7, $\mathcal{V}_x$ with a neighborhood basis of x.
Definition 8. (a) X is a R_l - *regular space* iff for each F, closed subset so that $F\overline{R}y$, there exists $V_F \in \mathcal{V}_F$ (neinghorhood of F) and there exists $V_y \in \mathcal{V}_y$ so that $V_F\overline{R}V_y$.

(b) X is a R_r - regular space iff for each F, closed subset with $x\overline{R}F$, there exists $V_F \in \mathcal{V}_F$ and there exists $V_x \in \mathcal{V}_x$ so that $V_x\overline{R}V_F$.

(c) X is a R - regular space iff X is a R_l and R_r space.
Remark 4. If R is a simetric relation (i.e. $R \subset R^{-1}$) then (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c).
Remark 5. If in this definition we take xRy iff $x = y$ then they become those in the classical case.

Theorem 9. (a) *X is a R_l - regular space iff for each $y \in X$ and $U \in \mathcal{V}_{R^{-1}(y)}$ there exists $V \in \mathcal{V}_y$ so that $\overline{R^{-1}(V)} \subset U$.*

(b) *X is a R_r - regular space iff for each $x \in X$ and $U \in \mathcal{V}_{R(x)}$ there exists $V \in \mathcal{V}_x$ so that $\overline{R(V)} \subset U$.*

(c) *X is a R - regular space iff (a) and (b) are both trues.*

Proof. (a) "$\Rightarrow$". Suppose X is R_l regular space. Let $y \in X$ be and $U \in \mathcal{V}_{R^{-1}(y)}$. Suppose that U is an open set. Then $F = CU$ is a closed set $\Rightarrow F \cap R^{-1}(y) = \emptyset \Rightarrow F\overline{R}y \Rightarrow$ there exists $V_F \in \mathcal{V}_F$ so that $V_F\overline{R}V_y \Rightarrow V_F \cap R^{-1}(V_y) = \emptyset$. Without loss the generality we can suppose V_F open set $\Rightarrow CF$ is a closed set $\Rightarrow \overline{R^{-1}(V_y)} \subset \overline{CV_F} = CV_F$. But $F \subset V_F \Rightarrow CV_F \subset CF = U \Rightarrow \overline{R^{-1}(V_y)} \subset U$.

"$\Leftarrow$". Let F be a closed set and $y \in X$ so that $F\overline{R}y \Rightarrow F \cap R^{-1}(y) = \emptyset \Rightarrow R^{-1}(V_y) \subset CF; CF = U$ is an open set $\Rightarrow U \in \mathcal{V}_{R^{-1}(y)} \Rightarrow$ there exists $V \in \mathcal{V}_y$ so that $\overline{R^{-1}(V_y)} \subset U \Rightarrow C\overline{R^{-1}(V_y)} \supset CU = F$. But $C\overline{R^{-1}(V_y)} = V_F$ is an open set $\Rightarrow V_F \in \mathcal{V}_F$. Because $R^{-1}(V_y) \subset \overline{R^{-1}(V_y)}$ we have:

$$CR^{-1}(V_y) \supset C\overline{R^{-1}(V_y)} \Rightarrow CR^{-1}(V_y) \supset V_F \Rightarrow V_F \cap R^{-1}(y) = \emptyset \Rightarrow V_F\overline{R}V_y.$$

(b) In the same way as (a).

(c) Is the consequence of (a) and (b). □

Definition 10. X is a R - *normal space* iff for each F_1, F_2 closed sets so that $F_1\overline{R}F_2$. there exists:

$$V_1 \in \mathcal{V}_{F_1}, V_2 \in \mathcal{V}_{F_2} \text{ so that } V_1\overline{R}V_2.$$

Theorem 11. *X is a R - normal space iff for each F closed set and $U \in \mathcal{V}_{R(F)}$, there exists $V \in \mathcal{V}_F$ so that (a) $\overline{R(V)} \subset U$ and (b) for each $U \in \mathcal{V}_{R^{-1}(y)}$, there exists $V \in \mathcal{V}_F$ so that $\overline{R^{-1}(V)} \subset U$.*

Proof. "$\Rightarrow$". Suppose X is a R - normal space. Let F be a closed set and $U \in \mathcal{V}_{R(F)} \Rightarrow R(F) \subset U$. Suppose U is a open set $\Rightarrow F_1 = CU$ is a closed set. As $CR(F) \supset F_1 \Rightarrow R(F) \cap F_1 = \emptyset \Rightarrow F\overline{R}F_1 \Rightarrow$ exists $V_F \in \mathcal{V}_F$ and $V_1 \in \mathcal{V}_{F_1}$ so that $V_F\overline{R}V_1$. But $V_1 \supset F_1 \Rightarrow CV_1 \subset CF_1$. Suppose V_1 is an open set $\Rightarrow CV_1 = \overline{CV_1}$ is a closed set. But $V_F\overline{R}V_1 \Rightarrow R(V_F) \subset CV_1 = \overline{CV_1} \Rightarrow \overline{R(V_F)} \subset CV_1 \subset CF_1 = U$. So $\overline{R(V_F)} \subset U$. In the same way for each $U \in \mathcal{V}_{R^{-1}(y)}$,there exists $V \in \mathcal{V}_F$ so that $\overline{R^{-1}(V_F)} \subset U$.

"$\Leftarrow$". Let F_1, F_2 be closed sets so that $F_1\overline{R}F_2 \Rightarrow R(F_1) \cap F_2 = \emptyset \Rightarrow R(F_1) \subset CF_2$; $CF_2 = U$ is a open set $\Rightarrow U \in \mathcal{V}_{R(F_1)} \Rightarrow$ there exists $V_1 \in \mathcal{V}_{F_1}$ so that $\overline{R(V_1)} \subset U \Rightarrow C\overline{R(V_1)} = V_2$, is an open set and $V_2 \supset F_2 \Rightarrow V_2 \in \mathcal{V}_{F_2}$. Observe that $CV_2 = \overline{R(V_1)} \supset R(V_1) \Rightarrow R(V_1) \cap V_2 = \emptyset \Rightarrow V_1\overline{R}V_2$. □

Remark 6. Definition 5 $\Leftrightarrow$ condition (a) $\Leftrightarrow$ condition (b), as we can see from the proof. The R - separated spaces can be characterized by using sequences. First we define the T_3^R and T_4^R spaces.

Dfinition 12. (a) X is T_3^R space iff X is a T_1 space and R - regular space.

(b) X is T_4^R space iff X is a T_1 space and an R - normal space.

Theorem 13. $T_4^R \subset T_3^R \subset T_2^R \subset T_1^R \subset T_0^R$.

Proof. Observe that if X is T_1 space then $\{x\} = \overline{\{x\}}$ for each $x \in X$; using this condition results the first and the second inclusion of theorem 7. □

Remark 7. A naturally condition for T_3^R and T_4^R spaces would to be T_1^R space, but it is not good enough to Theorem 7.

Theorem 14. *X is T_0^R space iff for the generalized sequences $(x_\alpha)_{\alpha\in I}$ and $(y_\beta)_{\beta\in J}$ we have*

$$\left.\begin{array}{l} x_\alpha \to x \\ y_\beta \to y \\ x_\alpha R y \\ x R y_\beta \end{array}\right\} \Rightarrow xRy.$$

Proof. We will use Theorem 1.

" $\Rightarrow$ ". Suppose $y \in \overline{R(x)}$ and $x \in \overline{R^{-1}(y)} \Rightarrow xRy$. See that:

$$\left.\begin{array}{l} x_\alpha \to x \\ x_\alpha R y \Rightarrow x \in R^{-1}(y) \end{array}\right\} \Rightarrow x \in \overline{R^{-1}(y)}.$$

Also:

$$\left.\begin{array}{l} y_\beta \to y \\ x R y_\beta \Rightarrow y \in R(x) \end{array}\right\} \Rightarrow y \in \overline{R(x)}.$$

By using Theorem 4 we have xRy.

"$\Leftarrow$". $y \in \overline{R(x)} \Rightarrow$ there exists $(y_\beta)_{\beta\in J}$ so that $y_\beta \to y$ and $y_\beta \in R(x) \Rightarrow xRy_\beta$, $x \in \overline{R^{-1}(y)} \Rightarrow$ there exists $(x_\alpha)_{\alpha\in I}$ so that $x_\alpha \to x$ and $x \in R^{-1}(y) \Rightarrow x_\alpha R y$. From here it follows xRy. □

Theorem 15. *X is T_1^R space iff for the generalized sequences $(x_\alpha)_{\alpha\in I}$ and $(y_\beta)_{\beta\in J}$ we have*

$$\left.\begin{array}{l} x_\alpha \to x \\ x_\alpha R y \end{array}\right\} \Rightarrow xRy \quad \text{and} \quad \left.\begin{array}{l} y_\beta \to y \\ x R y_\beta \end{array}\right\} \Rightarrow xRy.$$

Proof. We use Theorem 5.

"$\Rightarrow$". $x_\alpha \to x; x_\alpha R y \Rightarrow x_\alpha \in R^{-1}(y) = \overline{R^{-1}(y)} \Rightarrow x \in R^{-1}(y) \Rightarrow xRy$, $y_\beta \to y; xRy_\beta \Rightarrow y_\beta \in R(x) = \overline{R(x)} \Rightarrow y \in \overline{R(x)} \Rightarrow xRy$

"$\Leftarrow$". $y \in \overline{R(x)} \Rightarrow$ there exists $(y_\beta)_{\beta\in J}$ a generalized sequence so that $y_\beta \in R(x)$ and $y_\beta \to y$. Observe that $xRy_\beta \Rightarrow xRy \Rightarrow y \in R(x) \Rightarrow R(x) = \overline{R(x)}$. $x \in \overline{R^{-1}(y)} \Rightarrow$ there exists $(x_\alpha)_{\alpha\in I}$ a generalized sequence so that $x_\alpha \in R^{-1}(y)$ and $x_\alpha \to x$. Observe that $x_\alpha R y \Rightarrow xRy \Rightarrow x \in R^{-1}(y) \Rightarrow xRy$. □

Theorem 16. *X is T_2^R space iff for each generalized sequences $(x_\alpha)_{\alpha\in I}$ and $(y_\alpha)_{\alpha\in I}$ so that $x_\alpha R y_\alpha, x_\alpha \to x$ and $y_\alpha \to y$, we have xRy.*

Proof. We use Theorem 6.

X is T_2^R space iff $R \subset X \times X$ is a closed set $\Leftrightarrow$ for each $(x_\alpha, y_\alpha) \in R$ so that $(x_\alpha, y_\alpha) \to (x,y)$ we have $(x,y) \in R \Leftrightarrow$ for each $x_\alpha \to x$; $y_\alpha \to y$, $x_\alpha R y_\alpha$ it follows xRy. □

Example 17. If " $\leq$ " is an order relation on X and if X is a T_1^R space then $x_\alpha \leq y;\ x_\alpha \to x \Rightarrow x \leq y;\ x \leq y_\beta;\ \ y_\beta \to y \Rightarrow x \leq y$.

If X is a T_2^R space, then from $x_\alpha \leq y_\alpha$; $x_\alpha \to x$; $y_\alpha \to y$ results $x \leq y$.

Example 18. Let " $\prec$ " be an order relation on $\mathbf{R} : x \prec y$ iff $y - x \in \mathbf{N}$. $\mathbf{R}$ is $T_i^\prec$ space for $i \in \{0,1,2\}$ but it is not $T_i^\prec$ space for $i \in \{3,4\}$.

Proof. (a) It is obious that " $\prec$ " is an order relation on $\mathbf{R}$ and more: $\prec (x) = x + \mathbf{N}$, $\prec^{-1} (y) = y - \mathbf{N}$, for each $x, y \in \mathbf{R}$. $x_n \prec y_n \Rightarrow y_n - x_n \in \mathbf{N}$, where $(x_n)_{n\in\mathbf{N}}$ and $(y_n)_{n\in\mathbf{N}}$ are real sequences. $x_n \to x$, $y_n \to y \Rightarrow y_n - x_n \to y - x$. From here exists $n_0 \in \mathbf{N}$ so that $y_n - x_n \in (y - x - 1/2, y - x + 1/2)$ for each $n \geq n_0$. Because $y_n - x_n \in \mathbf{N}$ it follows $y_n - x_n = m \in \mathbf{N}$, so $y_n - x_n$ is constant for

$n \geq n_0 \Rightarrow y_n - x_n = y - x \in \mathbf{N} \Rightarrow x \prec y$. Using Theorem 16 results $\mathbf{R}$ is $T_2^{\prec}$ space, so $T_i^{\prec}$ space for every $i \in \{0, 1, 2\}$.

(b) $\mathbf{R}$ is not $\prec_r$ regular space. From Theorem 9 we have that for $x = 0$, if $V = \bigcup_{n=0}^{\infty} \left(n - \frac{1}{2^n}; n + \frac{1}{2^n} \right) \in \mathcal{V}_{\prec(0)}$, because $\prec (0) = \mathbf{N}$. $\mathbf{R}$ is $\prec_r$ regular space $\Rightarrow$ exists $U \in \mathcal{V}_0$ so that $\overline{\prec (U)} \subset V$. Let $\varepsilon > 0$ be so that $U_0 = (-\varepsilon, \varepsilon) \subset U$. We have $\prec (U_0) \subset \overline{\prec (U_0)} \subset \overline{\prec (U)} \subset V$ and then $U_0 + \mathbf{N} \subset V$. So $\bigcup_{n=0}^{\infty} (n - \varepsilon;\ n + \varepsilon) \subset \bigcup_{n=0}^{\infty} \left(n - \frac{1}{2^n}; n + \frac{1}{2^n} \right)$. It follows $0 < \varepsilon \leq \frac{1}{2^n}$ for each $n \in bn \Rightarrow \varepsilon = 0$, contradiction. Therefore $\mathbf{R}$ is not $T_i^{\prec}$ space for $i \in \{3, 4\}$. □

2 Spaces separated by equivalence relations

In this section we will consider the case of an equivalence relation $R = \rho$. First we define a notion of continuity of a binary relation.

Definition 19. (a) A binary relation R on a topological space $(X, \mathcal{T})$ is continuous iff for every $D \in \mathcal{T}$ an open set, $R^{-1}(D) \in \mathcal{T}$ is an open set.

(b) R is an open relation iff for each $D \in \mathcal{T}$, $R(D) \in \mathcal{T}$ is an open set.

Remark 8. If $R = \rho$ is an equivalence relation, then $\rho = \rho^{-1}$ so (a)$\Leftrightarrow$(b).

Theorem 20. *Let ρ be an equivalence relation on X and $(\widehat{X}, \widehat{\mathcal{T}})$ be the quotient space. If $\widehat{X}$ is T_i space then X is T_i^R space for each $i = \overline{0, 4}$.*

Proof. Let $\widehat{x} = \rho(x)$ be the equivalence class of each $x \in X$, and $p : X \to \widehat{X}, p(x) = \widehat{x}$ be the canonical projection. Suppose $\widehat{X}$ is T_2 space and let $x, y \in X$ be so that $x\overline{\rho}y \Leftrightarrow \widehat{x} \neq \widehat{y}$, more, $\rho(x) \cap \rho(y) = \emptyset$. There exists $V_{\widehat{x}} \in \mathcal{V}_{\widehat{x}}$ and $V_{\widehat{y}} \in \mathcal{V}_{\widehat{y}}$ neighborhoods of $\widehat{x}$, $\widehat{y}$ so that $V_{\widehat{x}} \cap V_{\widehat{y}} = \emptyset$.

If $V_x = p^{-1}(V_{\widehat{x}})$ and $V_y = p^{-1}(V_{\widehat{y}})$, then, as p is continuous we have $V_x \in \mathcal{V}_x$ and $V_y \in V_y$ and more because $V_{\widehat{x}} \cap V_{\widehat{y}} = \emptyset \Rightarrow V_x \overline{\rho} V_y \Rightarrow X$ is T_2^{ρ} space. The cases of $i \in \{0, 1, 2, 3, 4\}$ can be proved analogously. □

This theorem has a converse given by:

Theorem 21. *If ρ is an equivalence relation on X and if ρ is continous then X is T_i^{ρ} space $\Rightarrow \widehat{X}$ is T_i space, for each $i = \overline{0, 4}$.*

Proof. We will prove this result only in the case of $i = 2$. Suppose X is T_2^{ρ} space. Fisrt, observe that p is an open map $\Leftrightarrow \rho$ is an open relation. Let be $\widehat{x}, \widehat{y} \in \widehat{X}$ so that $\widehat{x} \neq \widehat{y} \Rightarrow x\overline{\rho}y \Rightarrow$there exists $V_x \in \mathcal{V}_x$ and $V_y \in \mathcal{V}_y$ so that $V_x \overline{\rho}\ V_y$. Because p is open map we have that $V_{\widehat{x}} = p(V_x)$ and $V_{\widehat{y}} = p(V_y)$ are neighborhoods of $\widehat{x}$ and $\widehat{y}$ and more $V_x \overline{\rho}\ V_y \Rightarrow V_{\widehat{x}} \cap V_{\widehat{y}} = \emptyset$ so $\widehat{X}$ is T_2 space. □

Example 22. 1) Let $X = \mathbf{R}$ be with usualy topology and ρ the equivalence on $\mathbf{R}$ defined by: $x\rho y \Leftrightarrow x - y \in \mathbf{Q}$. Observe that for each D open set, $\rho(D) = D + \mathbf{Q} = \mathbf{R}$, so ρ is a continuous relation. From here we can see that R is not T_0^{ρ}space because for each $x\overline{\rho}y, V_x \in \mathcal{V}_x$, $V_y \in \mathcal{V}_y$ results $\rho(V_x) = \rho(V_y) = \mathbf{R}$ so $y \in \rho(V_x)$ and $x \in \rho(V_y)$.

If $\widehat{\mathbf{R}} = \mathbf{R}/_{\rho}$ would be T_0 space, then $\mathbf{R}$ would be T_0^{ρ} space, so $\mathbf{R}/_{\rho}$ is not T_i space, $i = \overline{0, 4}$.

2) Some surfaces can be obtained as quotient spaces by identifying points of the border of a plane quadrate $P \subset \mathbf{R}^2$. For example the 2-sphere, S^2. Construction of

S^2: We note by Int P the interior of P and by bP the border of P. We consider P as a topological subspace of $\mathbf{R}^2$. See that if D is an open set of P, then $D \cup bP$ is also an open set of P. We define on P the relation $\rho : x\rho y \Leftrightarrow x = y$ or $x, y \in bP$. We see that ρ is a continuous equivalence relation, since if D is an open subset of P :

a) If $D \cap bP = \emptyset$ then $\rho(D) = D$ is an open subset of P.

b) If $D \cap bP \neq \emptyset$ then $\rho(D) = D \cup bP$ is also an open subset of P.

We define $S^2 = P/_\rho$ the quotient space of P. It is not difficult to see that S^2 is homeomorphe with any sphere of $\mathbf{R}^3$. P is a T_2^ρ space, using Theorem 16 of Section 1.

3 Function relations

Let $f : X \to X$ be a function and $(X, \mathcal{T})$ be a topological space. We will establish some properties of T_i^f spaces, $i = \overline{0,4}$.

Theorem 23. *If X is T_0 space and f is continuous, then X is T_0^f space.*

Proof. Observe that iff is continuous then $f(\overline{A}) \subset \overline{f(A)}$ and $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ for each A, B subsets of X. X is T_0 space iff:

(1) $$x \in \overline{\{y\}} \textit{ and } y \in \overline{\{x\}} \Rightarrow x = y.$$

We shall prove:

(2) $$y \in \{\overline{f(x)}\} \textit{ and } x \in \{\overline{f^{-1}(y)}\} \Rightarrow f(x) = y.$$

$x \in \{\overline{f^{-1}(y)}\}$ and $\{\overline{f^{-1}(y)}\} \subset f^{-1}(\overline{\{y\}}) \Rightarrow x \in f^{-1}(\overline{\{y\}})$, because f is continuous. From here $f(x) \in \overline{\{y\}}$ and $\left.\begin{array}{c} f(x) \in \overline{\{y\}} \\ y \in \overline{\{f(x)\}} \end{array}\right\} \overset{(1)}{\Rightarrow} y = f(x)$, so X is T_0^f space (2). □

Theorem 24. *If X is T_0^f space and if f is bijection, having the inverse f^{-1} continuous, then X is T_0 space.*

Proof. Let $x, y \in X$ be. We shall prove that: $x \in \overline{\{y\}}$ and $y \in \overline{\{x\}} \Rightarrow x = y$, i.e. X is T_0^f space. Since f is bijection $\Rightarrow$exists $z \in X$ so that $y = f(z)$. Suppose that $x \in \overline{\{f(z)\}}$ and $f(z) \in \overline{\{x\}} \Leftrightarrow x \in \overline{\{y\}}$ and $y \in \overline{\{x\}}$. Because $f^{-1}$is continuous we have: $z \in f^{-1}(\overline{\{x\}}) \subset \overline{\{f^{-1}(x)\}}$ and so $z \in \overline{\{f^{-1}(x)\}}$. From here $x \in \overline{\{f(z)\}}$ and $z \in \overline{\{f^{-1}(x)\}} \Rightarrow x = f(z)$, because X is T_0^f space. So we have $x = f(z) = y$, and then X is T_0 space. □

Theorem 25. *If X is T_1 space and f is continuous, then X is T_1^f space.*

Proof. Observe that X is T_1^f space iff $\{f(x)\}$ and $f^{-1}(y)$ are closed sets, for each $x, y \in X$. X is T_1 space $\Rightarrow \{f(x)\}$ is closed set (1). Because $\{y\}$ is a closed set and f is continuous, we have that $f^{-1}(y)$ is a closed set (2). From (1) and (2) we have that X is T_1^f space. □

Theorem 26. *If X is T_1^f space and f is onto, then X is T_1 space.*

Proof. X is T_1^f space $\Rightarrow \{f(x)\}$ is a closed set for each $x \in X$. Because for each $y \in X$,exists $x \in X$ so that $y = f(x)$, we have that $\{y\} = \{f(x)\}$ is a closed set so X is T_1 space. □

Theorem 27. *If X is T_2 space and f is continuous, then X is T_2^f space.*

Proof. Note G_f the graph of f. We shall prove that CG_f is an open subset of $X \times X$. Let be $(x, y) \in CG_f$. Then $f(x) \neq y$. Because X is T_2 space, there exists $V \in \mathcal{V}_y$

and $V_1 \in \mathcal{V}_{f(x)}$ so that $V \cap V_1 = \emptyset$. But f is continuous and from here1 there exists $U \in \mathcal{V}_x$ so that $f(U) \subset V_1 \Rightarrow f(U) \cap V = \emptyset \Rightarrow U \times V \subset CG_f \Rightarrow CG_f$ is a open set $\Rightarrow G_f$ is a closed set $\Rightarrow X$ is T_2^f (see Theorem 4 Section 1). □

Remark 9. Theorem 27 states that each continuous function on a T_2 space has a closed graphical.

Theorem 28. *If X is T_2^f space and f is bijection having the inverse f^{-1} continuous, then X is T_2 space.*

Proof. Let $y \in X$ be $\Rightarrow$ there exists $x \in X$ so that $y = f(x) \Rightarrow x = f^{-1}(y)$. Because f^{-1} is continuous we have: For each $U \in \mathcal{V}_x$ there exists $V \in \mathcal{V}_y$ so that $f^{-1}(V) \subset U \Rightarrow V \subset f(U) \Rightarrow \overline{V} \subset \overline{f(U)}$. From here we infer

$$y \in \bigcap_{V \in \mathcal{V}_y} \overline{V} = \bigcap_{V \in \mathcal{V}_{f(x)}} \overline{V} \subset \bigcap_{V \in \mathcal{V}_y} \overline{f(U)} = \{f(x)\} = \{y\},$$

because X is T_2^f (Theorem 4 Section 1). So we have: $\bigcap\limits_{V \in \mathcal{V}_y} \bigcap \overline{V} = \{y\}$ for each $y \in X$, and then X is T_2^f space. □

Theorem 29. *If X is T_2^f space and compact, then f is continuous.*

Proof. X is T_2^f space $\Rightarrow \bigcap\limits_{U \in \mathcal{V}_x} \overline{f(U)} = \{f(x)\}$ for each $x \in X$. Hence $\bigcup\limits_{U \in \mathcal{V}_x} C\overline{f(U)} = X \setminus \{(f(x)\}$, for an arbitrary point x of X. Let $V \in \mathcal{V}_{f(x)}$; then $X = \bigcup\limits_{U \in \mathcal{V}_x} C\overline{f(U)} \bigcup V$. Because X is a compact space there exist $U_1, U_2, ..., U_n \in \mathcal{V}_x$ such that

$$X = C\overline{f(U_1)} \cup C\overline{f(U_2)} \cup, ..., \cup C\overline{f(U_n)} \cup V = C\left[\overline{f(U_1)} \cup \overline{f(U_2)} \cup, ..., \cup \overline{f(U_n)}\right] \cup V.$$

Let $U \in \mathcal{V}_x$ be so that $U \subset \bigcap\limits_{i=1}^{n} U_i$. We have $\overline{f(U)} \subset \bigcap\limits_{i=1}^{n} \overline{f(U_i)}$. Hence

$$C\overline{f(u)} \supset C\left[\bigcap_{i=1}^{n} \overline{f(U_i)}\right] \Rightarrow C\overline{f(U)} \cup V = X \Rightarrow C\overline{f(U)} \cap CV = \emptyset \Rightarrow \overline{f(U)} \subset V.$$

We have proved that f is continuous in x. So f is continuous on X. □

Remark 10. This theorem is in fact an alternative of the principle of the closed graphical.

Theorem 30. *If X is f_r-regular space, then f is continuous.*

Proof. It is a consequence of Theorem 9 Section 1. □

Theorem 31. *If X is $f-$regular space and f is bijection, then f is homeomorphism.*

Proof. It is a consequence of Theorem 9 Section 1. □

Theorem 32. *If X is a regular space and f is continuous, then X is f_r-regular space.*

Proof. Let $x \in X$ be and $V \in \mathcal{V}_{f(x)}$ be $\Rightarrow$ there exists $V_1 = \overline{V_1} \in \mathcal{V}_{f(x)}$ so that $V_1 \subset V$, because X is regular space. As f is continuous there exists $U \in \mathcal{V}_x$ so that

$$f(U) \subset V_1 \Rightarrow \overline{f(U)} \subset V_1 = \overline{V_1} \Rightarrow \overline{f(U)} \subset V.$$

Using now Theorem 9 Section 1, it follows that X is f_r-regular space. □

Theorem 33. *If X is regular space and f is homeomorphism, then X is f regular space.*

Proof. It is a consequence of Theorem 9 Section 1. □

Theorem 34. *If X is T_4^f space(i.e. f $-$normal and T_1) then f is continuous. If f is bijection, then it is a homeomorphism.*

Proof. It is a consequence of Theorem 11 Section 1, if we remark that X is T_1 space $\Rightarrow \{x\}$ is a closed set for each $x \in X$. □

Theorem 35. (a) *If f is continuous and X is regular space, then X is f_r-regular.*

(b) *If f is homeomorphism and X is regular space, then X is f $-$ regular.*

Proof. (a) Let $x \in X$ be; for each $V \in \mathcal{V}_{f(x)}$, there exists $U \in \mathcal{V}_x$ so that $f(U) \subset V$. Suppose $V = \overline{V}$, because X is regular space. From here $\overline{f(U)} \subset \overline{V}$; because $V = \overline{V} \Rightarrow$ X is f_r-regular space.

(b) It is a consequence of (a), observing that f^{-1} is continuous. □

Theorem 36. *If f is homeomorfism and X is normal space, then X is f-normal.*

Proof. Let F be a closed subset of $X \Rightarrow f(F)$ is a closed set, because f is homeomorfism. If $V \in \mathcal{V}_{f(F)}$ then $V \in \mathcal{V}_{f(x)}$ for each $x \in F$. But f is continuous $\Rightarrow$ there exists $U_x \in \mathcal{V}_x$ so that $f(U_x) \subset V$. Let $U = \bigcup\limits_{x \in F} U_x$ be; notice U is neighborhood of F it follows $f(U) = \bigcup\limits_{x \in F} f(U_x) \subset V$. Because X is normal space, suppose $V = \overline{V} \Rightarrow \overline{f(U)} \subset V$ (1).

In the same way replacing f with f^{-1} that is continuous too, we will have : for each F a closed subset of X, and for each $U \in \mathcal{V}_{f^{-1}(x)}$ there exists $V \in \mathcal{V}_F$ so that $f^{-1}(V) \subset U$, supposing $U = \overline{U}$ (2) From (1), and (2), using Theorem 11 Section 1 we have that X is f-normal space. □

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