Nonholonomic Frames in Finsler Geometry

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Abstract

We determine a nonholonomic Finsler frame for a class of Generalized Lagrange spaces, for a class of Lagrange spaces with (α, β)-metric and for Finsler spaces with (α, β)-metric. Then, a special Finsler connection induced by such a nonholonomic frame is determined. Finally we study the integrability conditions for Cartan's structure equations of a Finsler connection.

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Introduction

In [8,9] P.R. Holland studies a unified formalism that uses a nonholonomic Finsler frame on space-time arising from consideration of a charged particle moving in an external electromagnetic field. In fact, R.S. Ingarden in [10] was first to point out that the Lorentz force law, in this case, could be written as geodesic equations on a Finsler space called Randers space ([16]). In [5,6] a gauge transformation is viewed as a nonholonomic frame on the tangent bundle of a four dimensional base manifold. The geometry that follows from these considerations gives a more unified approach to gravitation and gauge symmetries. In the above mentioned papers, the common Finsler idea used by the physicists R.G. Beil and P.R. Holland is the existence of a nonholonomic frame on the vertical subbundle VTM of the tangent bundle of a base manifold M. This nonholonomic frame relates a semi-Riemannian metric (the Minkowski or the Lorentz metric) with an induced Finsler metric. In [2,3], with P.L.Antonelli we found such a nonholonomic frame for two important classes of Finsler spaces that are dual in the sense of [7]: Randers and Kropina spaces. As Randers and Kropina spaces are members of a bigger class of Finsler spaces, namely the Finsler spaces with (α, β)-metric, it appears a natural question: does a Finsler space with (α, β)-metric have such a nonholonomic frame? As the fundamental tensor of a Finsler space with (α, β)-metric is not so easy to handle with, we didn't find so far, a direct method to determine a nonholonomic frame for these spaces.

In this paper we find a nonholonomic Finsler frame for a class of Generalized Lagrange spaces introduced and studied by M.Anastasiei and H.Shimada. In [1], the

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metric tensor of such a Generalized Lagrange space has been called the Beil metric. The Beil metric can be viewed also as a deformation of a Riemannian metric. In this work we consider the most general case of Beil's metric and we find a nonholonomic frame for it. This frame reduces in a particular case to that considered by R.G.Beil in [5,6]. Then we can use these ideas to find a nonholonomic frame for a class of Lagrange spaces proposed by R.G. Beil, the so-called Lagrange spaces with (α, β)-metric. We prove that the fundamental metric tensor of a Finsler space with (α, β)-metric can be derived from a Riemannian metric using two Beil deformations (1.5). Using these ideas we can find a nonholonomic frame for a Finsler space with (α, β)-metric. As Randers and Kropina spaces are Finsler spaces with (α, β)-metric we may use these techniques to find nonholonomic Finsler frames for these Finsler spaces.

We prove that every nonholonomic frame induces a special linear connection on the total space of the tangent bundle of the base manifold M. This linear connection has no curvature and the frame is parallel with respect to it. Using the Cartan's structure equations we show that a special linear connection, called a Finsler connection, has no curvature if and only if it is induced by a nonholonomic Finsler frame. The frame is holonomic if and only if a set of two forms of torsions vanishes.

R.Miron have been studied nonholonomic Finsler frames and the induced Finsler connection in [15] for the so-called strongly non-Riemannian Finsler spaces. M. Matsumoto studied these nonholonomic frames also, in [11], where he called such frames the Miron frames of a strongly non-Riemannian Finsler space. The Miron frame is a natural generalization of the Berwald frame for a two dimensional Finsler space or the Moor frame for a Finsler space of dimension three.

1 Finsler spaces and related Finsler objects

As the Finsler geometry is a part of the geometry of the tangent bundle of a manifold M, we present first some natural geometric objects that live on TM as the vertical distribution, the almost tangent structure. An important tool in the geometry of the tangent bundle is the nonlinear connection. Metric structures on TM are defined and we prove that in some conditions, Lagrange spaces with (α, β)-metric are generalized Lagrange spaces with Beil metric.

We start with a real n-dimensional manifold M of C^∞-class. Denote by (TM, π, M) the tangent bundle of the base manifold M and by $(\widetilde{TM}, \pi, M)$, the tangent bundle with the null cross-section removed. For every point $p \in M$, there exist local charts $(U, \varphi = (x^i))$ on $p \in M$ and $(\pi^{-1}(U), \phi = (x^i, y^i))$ on $u \in \pi^{-1}(p) \subset TM$ such that with respect to these the canonical submersion π has the equations $\pi : (x^i, y^i) \in \pi^{-1}(U) \mapsto (x^i) \in U$. The local charts on TM of the form $(\pi^{-1}(U), \phi = (x^i, y^i))$ are called induced local charts, (y^i) are coordinates of vectors $y^i \frac{\partial}{\partial x^i}|_p$ from T_pM, and $\frac{\partial}{\partial x^i}|_p$ is the natural basis of T_pM.

Denote by π_* the linear map induced by the canonical submersion $\pi : TM \to M$. As for every $u \in TM$, $\pi_{*,u} : T_uTM \to T_{\pi(u)}M$ is an epimorphism, then its kernel determines a n-dimensional distribution $V : u \in TM \mapsto V_uTM = Ker\pi_{*,u} \subset T_uTM$. We call it the *vertical distribution* of the tangent bundle. This is the tangent space to the natural foliation induced by the submersion π and consequently we have that the vertical distribution is integrable. If the natural basis of T_uTM induced by a local chart $(\pi^{-1}(U), \phi = (x^i, y^i))$ at u is denoted by $\{\frac{\partial}{\partial x^i}|_u, \frac{\partial}{\partial y^i}|_u\}$, then $\{\frac{\partial}{\partial y^i}|_u\}$ is a basis of V_uTM.

For every $u \in TM$ we consider the linear map $J_u : T_uTM \to T_uTM$, $J_u = \frac{\partial}{\partial y^i}|_u \otimes dx^i|_u$[1]. It is called the *almost tangent structure* of the tangent bundle (or the vertical endomorphism) and it has the properties: $J_u^2 = 0$ and $Ker J_u = Im J_u = V_uTM$.

We denote by $\mathcal{F}(TM)$ the ring of C^∞-functions over TM and by $\mathcal{X}(TM)$ the $\mathcal{F}(TM)$-module of vector fields over TM. With respect to the Poisson bracket, $\mathcal{X}(TM)$ is a real Lie algebra. Then the almost tangent structure J may be taught as an $\mathcal{F}(TM)$-linear map $J : \mathcal{X}(TM) \to \mathcal{X}(TM)$ with the local expression $J = \frac{\partial}{\partial y^i} \otimes dx^i$.

1.1. Definition We call a *nonlinear connection* on TM a n-dimensional distribution $HTM : u \in TM \mapsto H_uTM \subset T_uTM$ that is supplementary to the vertical distribution, which means that we have the direct sum:

$$T_uTM = H_uTM \oplus V_uTM, \quad \forall u \in TM. \tag{1.1}$$

As $\pi_{*,u} : T_uTM \to T_{\pi(u)}M$ is an epimorphism, $\forall u \in TM$, then the restriction of it to H_uTM gives us an isomorphism between H_uTM and $T_{\pi(u)}M$. The inverse map of this isomorphism is denoted by $l_{h,u} : T_{\pi(u)}M \to H_uTM$ and it is called the *horizontal lift* induced by the given nonlinear connection HTM. If we fix an induced local chart $(\pi^{-1}(U), \phi = (x^i, y^i))$ at $u \in TM$, because $\pi_{*,u} \circ l_{h,u} = Id_{H_uTM}$ we have that

$$l_{h,u}\left(\left.\frac{\partial}{\partial x^i}\right|_{\pi(u)}\right) = \left.\frac{\partial}{\partial x^i}\right|_u - N_i^j(u)\left.\frac{\partial}{\partial y^j}\right|_u =: \left.\frac{\delta}{\delta x^i}\right|_u .$$

The functions N_j^i are defined over $\pi^{-1}(U)$ and are called the *local coefficients* of the nonlinear connection HTM. For every $u \in TM$ and a local chart $(\pi^{-1}(U), \phi = (x^i, y^i))$ at u we have now a basis $\{\frac{\delta}{\delta x^i}|_u, \frac{\partial}{\partial y^i}|_u\}$ of T_uTM adapted to the decomposition (1.1). We call it the *Berwald basis* of the given nonlinear connection. We may remark here that if we change induced local charts from $(\pi^{-1}(U), \phi = (x^i, y^i))$ to $(\pi^{-1}(V), \psi = (\tilde{x}^i, \tilde{y}^i))$ then the corresponding Berwald base and the local coefficients of the nonlinear connection are related as follows:

$$\frac{\delta}{\delta x^i} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\delta}{\delta \tilde{x}^j}, \ \frac{\partial}{\partial y^i} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial}{\partial \tilde{y}^j}, \ rank(\frac{\partial \tilde{x}^j}{\partial x^i}) = n;$$

$$N_i^k \frac{\partial \tilde{x}^j}{\partial x^k} = \frac{\partial \tilde{x}^k}{\partial x^i}\widetilde{N}_k^j + \frac{\partial \tilde{y}^j}{\partial x^i}.$$

At every point $u \in TM$ we denote by T_u^*TM the cotangent space at u to TM, that is the dual space of T_uTM over $I\!R$. Then $\{dx^i|_u, \delta y^i|_u = dy^i|_u + N_j^i(u)dx^j|_u\}$ is a basis of T_u^*TM, that is called the Berwald cobasis of the nonlinear connection (it is the dual basis of the Berwald basis).

For a nonlinear connection HTM we define the map $\theta : \mathcal{X}(TM) \to \mathcal{X}(TM)$ locally given by

$$\theta = \frac{\delta}{\delta x^i} \otimes \delta y^i. \tag{1.2}$$

[1] In this paper the summation convention on upper and lower repeated indices is implied

We have that θ is globally defined and it has the properties: $\theta^2 = 0$, $Ker\theta = Im\theta = HTM$. The maps $h_u = \theta_u \circ J_u$ and $v_u = J_u \circ \theta_u$ are the horizontal and the vertical projectors that correspond to the decomposition (1.1).

1.2. Definition *A generalized Lagrange metric* (or a GL-metric for short) is a metric g on the vertical subbundle VTM of the tangent space TM. This means that for every $u \in TM$, $g_u : V_uTM \times V_uTM \to I\!R$ is bilinear, symmetric, of rank n and of constant signature. A pair $GL^n = (M, g)$, with g a GL-metric is called a *generalized Lagrange space*, or a GL-space for short.

If $(\pi^{-1}(U), \phi = (x^i, y^i))$ is an induced local chart at $u = (x, y) \in TM$, we denote by $g_{ij}(u) = g_u(\frac{\partial}{\partial y^i}|_u, \frac{\partial}{\partial y^j}|_u)$. Then a GL-metric may be given by a collection of functions $g_{ij}(x, y)$ such that we have:

1^o $rank(g_{ij}) = n$, $g_{ij}(x, y) = g_{ji}(x, y)$;

2^o the quadratic form $g_{ij}(x, y)\xi^i\xi^j$ has constant signature on TM;

3^o if another local chart $(\pi^{-1}(V), \psi = (\tilde{x}^i, \tilde{y}^i))$ at $u \in TM$ is given and $\widetilde{g}_{kl}(x, y) = g_u(\frac{\partial}{\partial \tilde{y}^k}|_u, \frac{\partial}{\partial \tilde{y}^l}|_u)$ then g_{ij} and $\widetilde{g}_{kl}$ are related by

$$g_{ij} = \frac{\partial \tilde{x}^k}{\partial x^i}\frac{\partial \tilde{x}^l}{\partial x^j}\widetilde{g}_{kl}. \tag{1.3}$$

A tensor field of (r, s)-type on TM whose components transform under a change of local coordinates on TM like the components of a tensor field of (r, s)-type on the base manifold is called a *Finsler tensor field*. From (1.3) we can see that a GL-metric is a Finsler tensor field of (0,2)-type.

If a nonlinear connection is given on a GL-space, then we may extend the metric g to the whole TM by taking:

$$G_u(X_u, Y_u) = g_u(J_uX_u, J_uY_u) + g_u(J_u\theta_uX_u, J_u\theta_uY_u), \forall X_u, Y_u \in T_uTM. \tag{1.4}$$

With respect to this metric, the vertical and horizontal distributions are orthogonal. In general, a GL-space doesn't have a canonical nonlinear connection.

1.3. Example Consider $a_{ij}(x)$ the components of a Riemannian metric on the base manifold M, $a(x, y) > 0$ and $b(x, y) \geq 0$ two Finsler scalars and $B(x, y) = B_i(x, y)dx^i$ a Finsler 1-form. Then:

$$g_{ij}(x, y) = a(x, y)a_{ij}(x) + b(x, y)B_i(x, y)B_j(x, y) \tag{1.5}$$

is a generalized Lagrange metric ([1]), called the *Beil metric*. We say also that the metric tensor g_{ij} is a *Beil deformation* of the Riemannian metric a_{ij}. It has been studied and applied by R.Miron and R.K.Tavakol in General Relativity for $a(x, y) = exp(2\sigma(x, y))$ and $b = 0$. The case $a(x, y) = 1$ with various choices of b and B_i was introduced and studied by R.G.Beil for constructing a new unified field theory in [5].

1.4. Definition *A Finsler metric* on TM is a function $F : TM \to I\!R$ with the properties:

1^o F is a positive function of C^∞-class on $\widetilde{TM}$ and only continuous on the null cross-section of the tangent bundle;

2^o F is positively homogeneous of degree one on $\widetilde{TM}$ with respect to y^i;

3^o The matrix with the entries:

$$g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j} \tag{1.6}$$

has rank n on $\widetilde{TM}$ and the quadratic form $g_{ij}(x,y)\xi^i\xi^j$ has constant signature on $\widetilde{TM}$.

A *Finsler space* is a pair $F^n = (M, F)$ with F a Finsler metric. The tensor field with the components given by (1.6) is called the *metric tensor* of the Finsler space. We denote by g^{ij} the components of the inverse matrix of g_{ij}, that is $g_{ij}g^{jk} = \delta_i^k$.

If we do not ask for the homogeneity condition 2^o, then F is called a *Lagrange metric*. The pair (M, F) is called a *Lagrange space*. The geometry of these spaces was intensively studied by R.Miron in [14].

For a Lagrange space F^n, the metric tensor (1.6) determine a GL-metric. The converse of this is not true and the Beil metric (1.5) is an example of GL-metric that is not reducible to a Finsler or Lagrange metric.

It is well known that every Lagrange space induces a canonical nonlinear connection, namely the Cartan nonlinear connection ([14]). This has the local coefficients given by:

$$N_j^i = \frac{\partial G^i}{\partial y^j},\ with$$

$$4G^i = g^{ik}\left(\frac{\partial^2 F^2}{\partial y^k \partial x^m} y^m - \frac{\partial F^2}{\partial x^k}\right).$$

Then a Lagrange space F^n has a canonical metric G given by formula (1.4).

An important class of Finsler spaces is the class of Finsler spaces with (α,β)-metrics ([12]). The first Finsler spaces with (α,β)-metric were introduced in forties by the physicist G.Randers and them are called the Randers spaces, [16]. Recently, R.G. Beil suggested to consider a more general case, the class of Lagrange spaces with (α,β)-metric.

1.5. Definition A Finsler space $F^n = (M, F(x,y))$ is called with (α,β)-*metric* if there exists a 2-homogeneous function L of two variables such that the Finsler metric $F : TM \to I\!\!R$ is given by:

$$F^2(x,y) = L(\alpha(x,y), \beta(x,y)),\ where \tag{1.7}$$

$$\begin{array}{ll} \alpha^2(x,y) = a_{ij}(x)y^iy^j, & a_{ij}(x)\ is\ a\ Riemannian\ metric\ on\ M; \\ \beta(x,y) = b_i(x)y^i, & b_i(x)dx^i\ is\ a\ 1-form\ on\ M. \end{array}$$

If we do not ask for the function L to be homogeneous of order two with respect to (α,β) variables, then we have a *Lagrange space with* (α,β)*-metric*.

1.6. Example

1^o If $L(\alpha,\beta) = (\alpha+\beta)^2$, then the Finsler space with Finsler metric $F(x,y) = (a_{ij}(x)y^iy^j)^{\frac{1}{2}} + b_i(x)y^i$ is called a *Randers space*.

2^o If $L(\alpha,\beta) = \dfrac{\alpha^4}{\beta^2}$, then the Finsler space with Finsler metric $F(x,y) = \dfrac{a_{ij}(x)y^iy^j}{|b_i(x)y^i|}$ is called a *Kropina space*.

These classes of Finsler spaces play an important role in Finsler geometry and they are dual in the sense of [7].

3^o If $L(\alpha,\beta) = \alpha^n\beta^m$, then we have a Lagrange space with (α,β)-metric, where the Lagrange metric is $F(x,y) = (a_{ij}(x)y^iy^j)^{\frac{n}{2}}(b_i(x)y^i)^m$. This Lagrange spaces reduces to a Finsler spaces with (α,β)-metric if and only if $n+m=2$.

Throughout this paper we shall rise and lower indices only with the Riemannian metric $a_{ij}(x)$, that is $y_i = a_{ij}y^j$, $b^i = a^{ij}b_j$, and so on.

For a Lagrange space with (α, β)-metric $F^2(x, y) = L(\alpha(x, y), \beta(x, y))$ it is usual to denote ([11]):

$$\rho = \frac{1}{2\alpha}\frac{\partial L}{\partial \alpha}; \qquad \rho_0 = \frac{1}{2}\frac{\partial^2 L}{\partial \beta^2};$$
$$\rho_{-1} = \frac{1}{2\alpha}\frac{\partial^2 L}{\partial \alpha \partial \beta}; \quad \rho_{-2} = \frac{1}{2\alpha^2}\left(\frac{\partial^2 L}{\partial \alpha^2} - \frac{1}{\alpha}\frac{\partial L}{\partial \alpha}\right). \tag{1.8}$$

For a Finsler space with (α, β)-metric, that is L is homogeneous of degree two with respect to α and β we have:

$$\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0. \tag{1.8$'$}$$

With respect to these notations we have that the metric tensor g_{ij} of a Lagrange space with (α, β)-metric is given by ([12]):

$$g_{ij}(x, y) = \rho a_{ij}(x) + \rho_0 b_i(x)b_j(x) + \rho_{-1}(b_i(x)y_j + b_j(x)y_i) + \rho_{-2}y_iy_j. \tag{1.9}$$

We may remark here that the formula (1.9) was determined in [12] for Finsler spaces with (α, β)-metric but it works more generally for Lagrange spaces with (α, β)-metric. The metric tensor g_{ij} of a Lagrange space with (α, β)-metric can be arranged into the form:

$$g_{ij} = \rho a_{ij} + \frac{1}{\rho_{-2}}(\rho_{-1}b_i + \rho_{-2}y_i)(\rho_{-1}b_j + \rho_{-2}y_j) + \frac{1}{\rho_{-2}}(\rho_0\rho_{-2} - \rho_{-1}^2)b_ib_j. \tag{1.9$'$}$$

If the b_ib_j coefficient vanishes we have:

1.7. Proposition *If for a Lagrange space with (α, β)-metric the condition:*

$$\rho_{-1}^2 = \rho_0\rho_{-2} \tag{1.10}$$

holds true, then the metric tensor g_{ij} can be written in the equivalent form:

$$g_{ij}(x, y) = \rho(x, y)a_{ij}(x) + \frac{1}{\rho_{-2}}B_i(x, y)B_j(x, y), \ \textit{where} \tag{1.11}$$

$$B_i(x, y) = \rho_{-1}(x, y)b_i(x) + \rho_{-2}(x, y)y_i.$$

If we compare (1.11) to (1.5) we have the following result:

1.8. Corollary *If for a Lagrange space with (α, β)-metric the condition (1.10) holds true, then its fundamental metric tensor is a Beil metric.*

1.9. Remark For the Lagrange space with (α, β)-metric suggested by R.G.Beil, $L(\alpha, \beta) = \alpha^n\beta^m$, the condition (1.10) is true if and only if $m^2n^2 = mn(m-1)(n-2)$. An example of Lagrange space with (α, β)-metric that satisfies the condition (1.10) has the Lagrange metric $L(\alpha, \beta) = \frac{\alpha^4}{\beta}$.

2 Nonholonomic Finsler frames for special metrics

The physicists R.G.Beil in [5,6] and P.R. Holland in [8,9] are using nonholonomic Finsler frames to develop unified field theories. In this section, we determine a nonholonomic Finsler frame for a Beil metric (1.5). In the particular case when $a(x,y)=1$ and $b(x,y)$ is a constant k we get the frame used by R.G. Beil in [5]. In the previous section, we found conditions in which the fundamental metric of a Lagrange space with (α,β)-metric is a Beil metric. Then we can determine a nonholonomic Finsler frame for a Lagrange space with (α,β)-metric from the nonholonomic Finsler frame of a Beil metric. From (1.9)' we can see that the fundamental metric tensor of a Finsler space with (α,β)-metric can be derived from a Riemannian metric a_{ij} using the Beil deformation (1.5) in two steps. Using this idea we can determine a nonholonomic frame for a Finsler space with (α,β)-metric as a product of two nonholonomic frames, each of these being determined by a Beil deformation.
Let U be an open set of TM and

$$V_i : u \in U \mapsto V_i(u) \in V_uTM,\ i \in \{1,...,n\}$$

be a vertical frame over U. If $V_i(u) = V_i^j(u)\frac{\partial}{\partial y^j}|_u$, then $V_i^j(u)$ are the entries of a invertible matrix for all $u \in U$. Denote by $\widetilde{V}_k^j(u)$ the inverse of this matrix. This means that:

$$V_j^i\widetilde{V}_k^j = \delta_k^i,\ \widetilde{V}_j^i V_k^j = \delta_k^i.$$

We call V_j^i a *nonholonomic Finsler frame.*
2.1. Theorem *Consider a GL-space with Beil metric (1.5) and denote by* $B^2(x,y) = a_{ij}(x)B^i(x,y)B^j(x,y)$*. Then:*

$$V_j^i = \sqrt{a}\delta_j^i - \frac{1}{B^2}(\sqrt{a} \pm \sqrt{a+bB^2})B^iB_j \tag{2.1}$$

is a nonholonomic Finsler frame. The Beil metric (1.5) and the Riemannian metric $a_{ij}(x)$ *are related by:*

$$g_{ij}(x,y) = V_i^k(x,y)V_j^l(x,y)a_{kl}(x). \tag{2.2}$$

Proof. Consider also:

$$\widetilde{V}_k^j = \frac{1}{\sqrt{a}}\delta_k^j - \frac{1}{B^2}\left(\frac{1}{\sqrt{a}} \pm \frac{1}{\sqrt{a+bB^2}}\right)B^jB_k. \tag{2.1$'$}$$

It is a direct calculation to check that $\widetilde{V}_k^j$ is the inverse of V_j^i, that is V_j^i is a nonholonomic frame. Next we have that $V_i^kV_j^la_{kl} = aa_{ij} + bB_iB_j = g_{ij}$ so the formula (2.2) holds true.
2.2. Corollary *The Beil metric (1.5) is positive definite on* $\widetilde{TM}$.
Proof. As the Finsler scalars $a(x,y)$ and $b(x,y)$ that define the metric (1.5) are positive and the metric a_{ij} is positive definite from (2.1)' we can see that $\widetilde{V}_k^i$ is well defined on $\widetilde{TM}$. Then V_j^i from (2.1) is a nonholonomic Finsler frame on $\widetilde{TM}$. From (2.2) we have that g_{ij} and a_{ij} have the same signature, so g_{ij} is positive definite on $\widetilde{TM}$.

2.3. Remark If we take $a(x,y)=1$ and $b(x,y)=k$, the nonholonomic Finsler frame (2.1) is the frame used by R.G.Beil in [5], formula (5.1).
2.4. Theorem *Let $F^2(x,y)=L(\alpha(x,y),\beta(x,y))$ be the metric function of a Lagrange space with (α,β)-metric for which the condition $\rho_{-1}^2=\rho_0\rho_{-2}$ is true. Then:*

$$V_j^i=\sqrt{\rho}\delta_j^i-\frac{1}{B^2}\left(\sqrt{\rho}\pm\sqrt{\rho+\frac{B^2}{\rho_{-2}}}\right)(\rho_{-1}b^i+\rho_{-2}y^i)(\rho_{-1}b_j+\rho_{-2}y_j) \tag{2.3}$$

is a nonholonomic Finsler frame, where $B^2=\rho_{-1}^2b^2+\rho_{-2}^2\alpha^2+2\beta\rho_{-1}\rho_{-2}$, ρ, ρ_0, ρ_{-1} and ρ_{-2} are the invariants of the Lagrange space with (α,β)-metric defined in (1.8).

For a Lagrange space with (α,β)-metric $L=\frac{\alpha^4}{\beta}$ we have:

$$\rho=\frac{2\alpha^2}{\beta},\ \rho_0=\frac{\alpha^4}{\beta^3},\ \rho_{-1}=\frac{-2\alpha^2}{\beta^2},\ \rho_{-2}=\frac{4}{\beta}.$$

We have then that the condition (1.10) is true and $B^2=\frac{4\alpha^4b^2}{\beta^4}$. Consequently a nonholonomic frame for the given Lagrange space with (α,β)-metric is given by:

$$V_j^i=\alpha\sqrt{\frac{2}{\beta}}\delta_j^i-\frac{1}{\alpha^3b^2}\left(\sqrt{\frac{2}{\beta}}\pm\sqrt{\frac{2}{\beta}+\frac{\alpha^2b^2}{\beta^3}}\right)(2\beta y^i-\alpha^2b^i)(2\beta y_j-\alpha^2b_j).$$

Consider now a Finsler space with (α,β)-metric. From (1.9)' we can see that g_{ij} is the result of two Beil deformations:

$$\begin{aligned} &a_{ij}\mapsto h_{ij}=\rho a_{ij}+\tfrac{1}{\rho_{-2}}(\rho_{-1}b_i+\rho_{-2}y_i)(\rho_{-1}b_j+\rho_{-2}y_j)\ \ \textit{and}\\ &h_{ij}\mapsto g_{ij}=h_{ij}+\tfrac{1}{\rho_{-2}}(\rho_0\rho_{-2}-\rho_{-1}^2)b_ib_j. \end{aligned} \tag{2.4}$$

The nonholonomic Finsler frame that corresponds to the first deformation (2.4) is, according to the Theorem 2.1, given by:

$$X_j^i=\sqrt{\rho}\delta_j^i-\frac{1}{B^2}\left(\sqrt{\rho}\pm\sqrt{\rho+\frac{B^2}{\rho_{-2}}}\right)(\rho_{-1}b^i+\rho_{-2}y^i)(\rho_{-1}b_j+\rho_{-2}y_j), \tag{2.5}$$

where $B^2=a_{ij}(\rho_{-1}b^i+\rho_{-2}y^i)(\rho_{-1}b^j+\rho_{-2}y^j)=\rho_{-1}^2b^2+\beta\rho_{-1}\rho_{-2}$. The metric tensors a_{ij} and h_{ij} are related by:

$$h_{ij}=X_i^kX_j^la_{kl}. \tag{2.6}$$

According to the Theorem 2.1, the nonholonomic Finsler frame that corresponds to the second deformation (2.4) is given by:

$$Y_j^i=\delta_j^i-\frac{1}{C^2}\left(1\pm\sqrt{1+\frac{\rho_{-2}C^2}{\rho_0\rho_{-2}-\rho_{-1}^2}}\right)b^ib_j, \tag{2.5$'$}$$

where $C^2=h_{ij}b^ib^j=\rho b^2+\frac{1}{\rho_{-2}}(\rho_{-1}b^2+\rho_{-2}\beta)^2$. The metric tensors h_{ij} and g_{ij} are related by the formula:

$$(2.6)' \qquad g_{mn} = Y^i_m Y^j_n h_{ij}.$$

From (2.6) and (2.6)' we have that $V^k_m = X^k_i Y^i_m$, with X^k_i given by (2.5) and Y^i_m given by (2.5)', is a nonholonomic Finsler frame of the Finsler space with (α, β)-metric.

For a Randers space with the fundamental function $L = (\alpha+\beta)^2 = F^2$, the Finsler invariants (1.8) are given by:

$$\rho = \frac{\alpha+\beta}{\alpha} = \frac{F}{\alpha}, \quad \rho_0 = 1, \quad \rho_{-1} = \frac{1}{\alpha}, \quad \rho_{-2} = \frac{-\beta}{\alpha^3},$$
$$B^2 = \frac{b^2\alpha^2 - \beta^2}{\alpha^4}.$$

We have then that the condition (1.10) is not satisfied. If we use the previous idea, then $V^k_m = X^k_i Y^i_m$ is a nonholonomic Finsler frame of a Randers space, where:

$$X^i_j = \sqrt{\frac{\alpha+\beta}{\alpha}}\delta^i_j - \frac{\alpha^2}{\alpha^2 b^2 - \beta^2}\left[\sqrt{\frac{\alpha+\beta}{\alpha}} \pm \sqrt{\frac{\alpha\beta + 2\beta^2 - b^2\alpha^2}{\alpha\beta}}\right](b^i - \frac{\beta y^i}{\alpha^2})(b_j - \frac{\beta y_j}{\alpha^2}),$$

$$Y^i_j = \delta^i_j - \frac{1}{C^2}\left(1 \pm \sqrt{1 + \frac{\beta C^2}{\alpha+\beta}}\right) b^i b_j, \ \textit{and}$$

$$C^2 = \frac{(\alpha+\beta)b^2}{\alpha} - \frac{\alpha}{\beta}\left(b^2 - \frac{\beta^2}{\alpha^2}\right)^2.$$

In a similar way we may find a nonholonomic Finsler frame for a Kropina space with the fundamental function $L = \dfrac{\alpha^4}{\beta^2} = F^2$. In this case, the Finsler invariants are given by:

$$\rho = \frac{2\alpha^2}{\beta^2}, \ \rho_0 = 3\frac{\alpha^4}{\beta^4}, \ \rho_{-1} = \frac{-4\alpha^2}{\beta^3}, \ \rho_{-2} = \frac{4}{\beta^2},$$
$$B^2 = 16\frac{\alpha^2}{\beta^4}\left(\frac{\alpha^2 b^2}{\beta^2} - 1\right).$$

2.5. Remark One may use also the two steps deformations (2.4) to determine the contravariant tensor (g^{ij}) of a Finsler space with (α, β)-metric.

3 Finsler connections induced by a nonholonomic Finsler Frame

Consider now that on the tangent bundle of a manifold M we have a nonlinear connection HTM. Then we consider a special linear connection on TM that preserves by parallelism the horizontal and the vertical distributions and we call it a Finsler connection. We prove that a nonholonomic Finsler frame determine a Finsler connection with no curvature. We study the integrability conditions of the Cartan's structure equations of a Finsler connection. Using these, we can prove that if a Finsler connection has no curvature then it is induced by a nonholonomic Finsler frame.

3.1. Definition A linear connection D on TM is called a *Finsler connection* if:
1° D preserves by parallelism the horizontal distribution HTM;
2° The almost tangent structure J is absolutely parallel with respect to D.

For a Finsler connection D it is immediate that D preserves also the vertical distribution. With respect to the Berwald basis $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$ of the nonlinear connection a Finsler connection can be expressed as:

$$\text{(3.1)} \qquad \begin{cases} D_{\frac{\delta}{\delta x^i}} \dfrac{\delta}{\delta x^j} = F^k_{ji} \dfrac{\delta}{\delta x^k}; & D_{\frac{\delta}{\delta x^i}} \dfrac{\partial}{\partial y^j} = F^k_{ji} \dfrac{\partial}{\partial y^k}; \\ D_{\frac{\partial}{\partial y^i}} \dfrac{\delta}{\delta x^j} = C^k_{ji} \dfrac{\delta}{\delta x^k}; & D_{\frac{\partial}{\partial y^i}} \dfrac{\partial}{\partial y^j} = C^k_{ji} \dfrac{\partial}{\partial y^k}. \end{cases}$$

Observe that under a change of induced coordinates on TM the functions F^k_{ji} transform like the coefficients of a linear connection on the base manifold M and C^k_{ji} are the components of a Finsler tensor field of (1,2)-type.

If $(T^{i_1\cdots i_r}_{j_1\cdots j_s})$ are the components of a (r,s)-type Finsler tensor field T, then the absolute differential of T with respect to the Finsler connection D is given by:

$$DT^{i_1\cdots i_r}_{j_1\cdots j_s} = dT^{i_1\cdots i_r}_{j_1\cdots j_s} + \omega^{i_1}_p T^{pi_2\cdots i_r}_{j_1\cdots j_s} + \cdots + \omega^{i_r}_p T^{i_1\cdots i_{r-1}p}_{j_1\cdots j_s} - \omega^p_{j_1} T^{i_1\cdots i_r}_{pj_1\cdots j_s} - \cdots - \omega^p_{j_s} T^{i_1\cdots i_r}_{j_1\cdots j_{s-1}p},$$

where $\omega^i_j = F^i_{jk} dx^k + C^i_{jk} \delta y^k$ are the connection 1-forms of D.

We can write the previous formula in an equivalent form:

$$DT^{i_1\cdots i_r}_{j_1\cdots j_s} = T^{i_1\cdots i_r}_{j_1\cdots j_s|k} dx^k + T^{i_1\cdots i_r}_{j_1\cdots j_s}|_k \delta y^k.$$

Here $T^{i_1\cdots i_r}_{j_1\cdots j_s|k}$ and $T^{i_1\cdots i_r}_{j_1\cdots j_s}|_k$ stand for *horizontal and vertical covariant derivatives* of $T^{i_1\cdots i_r}_{j_1\cdots j_s}$, ([14]).

For a Finsler connection D one considers typically:

$$T(X,Y) = D_X Y - D_Y X - [X,Y],$$

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z$$

the *torsion* and the *curvature*. It is well known ([4], [14]) that with respect to the Berwald basis $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$ there are only five nonzero components of torsion and three components of curvature. The five nonzero components of torsion are:

$$\text{(3.2)} \qquad \begin{cases} hT\left(\dfrac{\delta}{\delta x^i}, \dfrac{\delta}{\delta x^j}\right) =: T^k_{ij} \dfrac{\delta}{\delta x^k} = (F^k_{ji} - F^k_{ij}) \dfrac{\delta}{\delta x^k}; & (h)h\text{-}torsion \\ vT\left(\dfrac{\delta}{\delta x^i}, \dfrac{\delta}{\delta x^j}\right) =: R^k_{ij} \dfrac{\partial}{\partial y^k} = \left(\dfrac{\delta N^k_i}{\delta x^j} - \dfrac{\delta N^k_j}{\delta x^i}\right) \dfrac{\partial}{\partial y^k}; & (v)h\text{-}torsion \\ hT\left(\dfrac{\partial}{\partial y^i}, \dfrac{\delta}{\delta x^j}\right) = C^k_{ji} \frac{\delta}{\delta x^k}; & (h)hv\text{-}torsion \\ vT\left(\dfrac{\partial}{\partial y^i}, \dfrac{\delta}{\delta x^j}\right) =: P^k_{ij} \dfrac{\partial}{\partial y^k} = \left(\dfrac{\partial N^k_j}{\partial y^i} - F^k_{ij}\right) \dfrac{\partial}{\partial y^k}; & (v)hv\text{-}torsion \\ vT\left(\dfrac{\partial}{\partial y^i}, \dfrac{\partial}{\partial y^j}\right) =: S^k_{ij} \dfrac{\partial}{\partial y^k} = (C^k_{ji} - C^k_{ij}) \dfrac{\partial}{\partial y^k}; & (v)v\text{-}torsion. \end{cases}$$

The three components of curvature are given by:

$$(3.3)\qquad \begin{aligned} R_{j\ kh}^{\ i} &= \tfrac{\delta F^i_{jk}}{\delta x^h} - \tfrac{\delta F^i_{jh}}{\delta x^k} + F^m_{jk}F^i_{mh} - F^m_{jh}F^i_{mk} + C^i_{jm}R^m_{kh};\\ P_{j\ kh}^{\ i} &= \tfrac{\partial F^i_{jk}}{\partial y^h} - C^i_{jk|h} + C^i_{jm}P^m_{kh};\\ S_{j\ kh}^{\ i} &= \tfrac{\partial C^i_{jk}}{\partial y^h} - \tfrac{\partial C^i_{jh}}{\partial y^k} + C^m_{jk}C^i_{mh} - C^m_{jh}C^i_{mk}. \end{aligned}$$

For a Finsler connection D we have the following Ricci identities:

$$(3.4)\qquad \begin{cases} X^i_{|k|r} - X^i_{|r|k} = X^m R_{m\ kr}^{\ i} - X^i_{|m}T^m_{kr} - X^i|_m R^m_{kr};\\ X^i_{|k}|_r - X^i|_{r|k} = X^m P_{m\ kr}^{\ i} - X^i_{|m}C^m_{kr} - X^i|_m P^m_{kr};\\ X^i|_k|_r - X^i|_r|_k = X^m S_{m\ kr}^{\ i} - X^i|_m S^m_{kr}. \end{cases}$$

Consider now a nonholonomic Finsler frame $V_j = V^i_j \frac{\partial}{\partial y^i}$ on a open set U of TM. That is $V^i_j(u)$ are the entries of a nonsingular matrices over U. We denote by $\widetilde{V}^j_k$ the inverse matrix of V^i_j.

3.2. Theorem *There exists a unique Finsler connection D on TM such that the absolute differential of the given nonholonomic frame $V_j = V^i_j \frac{\partial}{\partial y^i}$ with respect to D, is zero. For this Finsler connection D all components of curvature are zero.*

Proof. The absolute differential of the given nonholonomic frame V_j with respect to D is given by $DV^i_j = V^i_{j|k}dx^k + V^i_j|_k \delta y^k$ for every fixed $j \in \{1, 2, ..., n\}$. So, $DV^i_j = 0$ if and only if the frame is $h-$ and $v-$covariant constant with respect to D.

The nonholonomic frame $V_j = V^i_j \frac{\partial}{\partial y^i}$ is h-covariant constant if for all $j \in \{1, ..., n\}$ we have $V^i_{j|k} = 0$. This is equivalent to $\frac{\delta V^i_j}{\delta x^k} + F^i_{mk}V^m_j = 0$. If we solve this for F^i_{mk} we have

$$F^i_{mk} = -\frac{\delta V^i_j}{\delta x^k}\widetilde{V}^j_m = V^i_j \frac{\delta \widetilde{V}^j_m}{\delta x^k}.$$

Similarly, the nonholonomic frame V_j is v-covariant constant if for all $j \in \{1, ..., n\}$ we have $V^i_j|_k = 0$. This is equivalent to $\frac{\partial V^i_j}{\partial y^k} + C^i_{mk}V^m_j = 0$. If we solve this for C^i_{mk} we have

$$C^i_{mk} = -\frac{\partial V^i_j}{\partial y^k}\widetilde{V}^j_m = V^i_j \frac{\partial \widetilde{V}^j_m}{\partial y^k}.$$

If we use the Ricci identities (3.4) for V_j, we have: $R_{m\ kj}^{\ i}V^m_j = 0$, $P_{m\ kj}^{\ i}V^m_j = 0$, and $S_{m\ kj}^{\ i}V^m_j = 0$, $\forall j \in \{1, ..., n\}$. As V^m_j is invertible one obtain: $R_{m\ kj}^{\ i} = P_{m\ kj}^{\ i} = S_{m\ kj}^{\ i} = 0$.

The Finsler connection we have defined in Theorem 3.1 is called the *Crystallographic connection* of the nonholonomic frame V^i_j ([2]).

Next we denote by $\{X_a\}_{a=\overline{1,2n}}$ the vector fields of the Berwald basis $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$ induced by a nonlinear connection HTM and by $\{\theta^a\}_{a=\overline{1,2n}}$ the dual basis $\{dx^i, \delta y^i\}$. For a Finsler connection D, the connection 1-forms (ω^a_b) corresponding to these base are defined as follows:

$$\omega^a_b(X) = \theta^a(D_X X_b), \quad \forall X \in \chi(TM).$$

It is a straightforward calculation to check that the connection 1-forms are given by $\omega^a_b = \begin{pmatrix} \omega^i_j & 0 \\ 0 & \omega^i_j \end{pmatrix}$, where $\omega^i_j = F^i_{jk}dx^k + C^i_{jk}\delta y^k$. For a vector field $W = W^a X_a \in \chi(TM)$ we have that

$$D_V W = (V(W^a) + W^b\omega^a_b(V))X_a, \ \textit{that is}$$
$$\theta^a(D_V W) = V(\theta^a(W)) + \theta^b(W)\omega^a_b(V).$$

3.3. Theorem *The Cartan's first structure equations of a Finsler connection D are given by:*

$$\text{(3.5)} \qquad \begin{cases} \qquad\qquad -dx^h \wedge \omega^i_h = -\Theta^i, \\ d(\delta y^i) \quad -\delta y^h \wedge \omega^i_h = -\widetilde{\Theta}^i, \end{cases}$$

where the 2-forms of torsions $\Theta^a = (\Theta^i, \widetilde{\Theta}^i)$ are defined by:

$$\Theta^a(X,Y) = \theta^a(T(X,Y)), \ \textit{and are given by}:$$

$$\text{(3.6)} \qquad \begin{cases} \Theta^i = \frac{1}{2}T^i_{jk}dx^j \wedge dx^k + C^i_{jk}dx^j \wedge \delta y^k, \\ \widetilde{\Theta}^i = \frac{1}{2}R^i_{jk}dx^j \wedge dx^k + P^i_{jk}dx^j \wedge \delta y^k + \frac{1}{2}S^i_{jk}\delta y^j \wedge \delta y^k. \end{cases}$$

The Cartan's second structure equations of a Finsler connection D are given by:

$$\text{(3.7)} \qquad d\omega^i_j - \omega^h_j \wedge \omega^i_h = -\Omega^i_j,$$

where the curvature 2-forms $(\Omega^a_b) = \begin{pmatrix} \Omega^i_j & 0 \\ 0 & \Omega^i_j \end{pmatrix}$, are defined by:

$$\Omega^a_b(X,Y) = \theta^a(R(X,Y)X_b), \ \textit{and are given by}:$$

$$\text{(3.8)} \qquad \Omega^i_j = \frac{1}{2}R_j{}^i{}_{kh}dx^k \wedge dx^h + P_j{}^i{}_{kh}dx^k \wedge \delta y^h + \frac{1}{2}S_j{}^i{}_{kh}\delta y^k \wedge \delta y^h.$$

Proof. We have that

$$\begin{aligned} \Theta^a(X,Y) &= \theta^a(T(X,Y)) = \theta^a(D_X Y) - \theta^a(D_Y X) - \theta^a([X,Y]) = \\ &= X(\theta^a(Y)) + \theta^b(Y)\omega^a_b(X) - Y(\theta^a(X)) - \theta^b(X)\omega^a_b(Y) - \theta^a([X,Y]) = \\ &= d\theta^a(X,Y) + (\omega^a_b \wedge \theta^b)(X,Y). \end{aligned}$$

If we take θ^a to be dx^i and δy^i, respectively, then we get the Cartan's first structure equations (3.5).

From $\Omega^a_b(X,Y) = \theta^a(R(X,Y)X_b) = d\omega^a_b(X,Y) + (\omega^a_c \wedge \omega^c_b)(X,Y)$ we have the Cartan's second structure equations (3.7).

3.4. Theorem *If for a Finsler connection D on TM the curvature 2-forms Ω^i_j vanish, then there exists a nonholonomic Finsler frame V^i_j such that the local coefficients of the connection D are given by:*

$$(3.9)\qquad \begin{cases} F^i_{jk} = -\dfrac{\delta V^i_m}{\delta x^k}\widetilde{V}^m_j = V^i_m \dfrac{\delta \widetilde{V}^m_j}{\delta x^k} \\[2ex] C^i_{jk} = -\dfrac{\partial V^i_m}{\partial y^k}\widetilde{V}^m_j = V^i_m \dfrac{\partial \widetilde{V}^m_j}{\partial y^k}. \end{cases}$$

Proof. If the curvature two-forms of D vanish, then the Cartan's second structure equations are:

$$d\omega^a_b + \omega^a_c \wedge \omega^c_b = 0.$$

Then there exists a frame $V^a_b(x, y)$ on the tangent space TM such that

$$(3.10)\qquad dV^a_b + \omega^a_c V^c_b = 0.$$

As $\omega^a_b = \begin{pmatrix} \omega^i_j & 0 \\ 0 & \omega^i_j \end{pmatrix}$ and if we denote $V^a_b = \begin{pmatrix} V^i_j & V^i_{\bar{j}} \\ V^{\bar{i}}_j & V^{\bar{i}}_{\bar{j}} \end{pmatrix}$ then, the equations (3.10) are equivalent to:

$$(3.10)'\qquad \begin{cases} dV^i_j + \omega^i_k V^k_j = 0, \\ dV^i_{\bar{j}} + \omega^i_k V^k_{\bar{j}} = 0, \\ dV^{\bar{i}}_j + \omega^i_k V^{\bar{k}}_j = 0, \\ dV^{\bar{i}}_{\bar{j}} + \omega^i_k V^{\bar{k}}_{\bar{j}} = 0. \end{cases}$$

As (V^a_b) are the entries of a non-singular matrix of order $2n$, whose blocks are solutions of (3.10)' we have that at least two of these blocks are invertible. Suppose (V^i_j) is one of them and $\widetilde{V}^i_j := (V^i_j)^{-1}$. Then $\omega^i_j = -\widetilde{V}^i_k dV^k_j = d\widetilde{V}^i_k V^k_j$ and consequently we have that the local coefficients of D are given by (3.9).

The Theorems 3.2 and 3.4 say that the only Finsler connections that have zero curvature are induced by nonholonomic Finsler frames.

The frame $\{H_j = V^i_j \frac{\delta}{\delta x^i}, V_j = V^i_j \frac{\partial}{\partial y^i}\}$ is said to be *holonomic* if there exist n functions ϕ^j on the base manifold M such that $\widetilde{V}^j_i = \frac{\partial \phi^j}{\partial x^i}$, that is equivalent to say that the one-forms $\eta^j = V^j_i dx^i$ are exact.

3.5. Proposition *A frame V^i_j is holonomic if and only if the torsion two-forms Θ^i, defined by $(3.6)_1$ of the Crystallographic connection induced by V^i_j, vanish.*

Proof. From $(3.6)_1$ we have that $\Theta^i = 0$ if and only if $T^i_{jk} = 0$ and $C^i_{jk} = 0$, where $T^i_{jk} = F^i_{kj} - F^i_{jk}$, and F^i_{jk} and C^i_{jk} are given by (3.9). However $C^i_{jk} = 0$ if and only if V^i_j are functions of (x) only. Then $T^i_{jk} = 0$ if and only if $\frac{\partial \widetilde{V}^i_j}{\partial x^k} = \frac{\partial \widetilde{V}^i_k}{\partial x^j}$ and this is equivalent to the fact that $\widetilde{V}^j_i$ are the gradient of n functions ϕ^j on the base manifold M.

3.6. Proposition *If for a Finsler connection D on TM the torsion two-forms Θ^i and the curvature two-forms Ω^i_j vanish, then local coordinates may be found on the base manifold M such that with respect to the induced coordinates on TM we have $F^i_{jk} = C^i_{jk} = 0$.*

Proof. If the curvature two-forms Ω^i_j of the Finsler connection D vanish then according to the Theorem 3.4 there is a frame V^i_j such that the local coefficients of the Finsler connection D are given by (3.9). From Proposition 3.5 we have that the frame V^j_i is holonomic, that is there exist n functions ϕ^j such that $\widetilde{V}^j_i = \dfrac{\partial \phi^j}{\partial x^i}$. Then, ϕ^j are coordinate functions on M and with respect to the induced coordinates on TM, the local coefficients of the Finsler connection D, vanish.

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