On Hamiltonian Submanifolds

Paul Popescu and Marcela Popescu

Abstract

The aim of this paper is to give a positive answer to the following question concerning the Hamiltonian submanifolds: for a given Hamiltonian, does it exist a section of the projection of cotangent bundles, which depends only on the Hamiltonian?

Mathematics Subject Classification: 53C40, 44A15, 53B25
Key words: Lagrangian, Hamiltonian, Legendre transformation, subspace, Hessian

A natural question concerning Hamiltonian submanifolds arises from [13, 14], as follows. Let $i : M' \to M$ be an immersion of a submanifold M' and $i^* : T^*M \to T^*M'$ be the co-differential of i. The study of the induced geometrical objects on T^*M', which arise from a Hamiltonian $H : T^*M \to \mathbf{R}$ on M, is performed in [13, 14] considering a pair $(i, \tilde{i})$, where $\tilde{i}$ is an *arbitrary* section of i^*, i.e., $\tilde{i} : T^*M' \to T^*M$, $i^* \circ \tilde{i} = id_{T^*M'}$. The aim of this paper is to show that a natural distinguished section $\tilde{i}$ exists and it depends only on the Hamiltonian H. It implies that, under natural conditions, the Hamiltonian H on the manifold M induces the Hamiltonian $H' = H \circ \tilde{i}$ on the submanifold M'. This construction is performed in a different way in [8], using the Lagrangian and the Hamiltonian formalisms.

In the first section we recall briefly some classical results used in the paper, concerning Legendre transformations. In the second section we construct explicitly the section $\tilde{i}$, using the Legendre transformation as an essential ingredient.

1 Lagrangians, Hamiltonians and the Legendre transformations in the classical case

This section contains some classical results concerning the Lagrangian and the Hamiltonian formalisms, underlying the role of the Legendre transformations. We refer to [26] for the Legendre transformation in a very general setting and to [8] for examples and further constructions concerning mechanical systems.

Let M be a smooth manifold. A *Lagrangian* (a *Hamiltonian*) on M is a continuous function $L : TM \to \mathbf{R}$ ($H : T^*M \to M$) which is smooth on $TM^* = TM\backslash\{0\}$ (on $T^*M^* = T^*M\backslash\{0\}$), i.e. TM^* (T^*M^*) is the total space of the tangent (cotangent) bundle less the image of the null section.

Balkan Journal of Geometry and Its Applications, Vol.7, No.2, 2002, pp. 79-86.

We consider local coordinates (x^i) on M, (x^i, y^j) on TM and (x^i, p_k) on T^*M $(i, j, k, \ldots = \overline{1,m}$, $m = \dim M)$, which are adapted to the vector bundle structures.

Consider now a Lagrangian L. The *Legendre transformation* associated with L is the function $\mathcal{L} : TM^* \to T^*M^*$, defined by

$$\mathcal{L}\left(X^i \frac{\partial}{\partial x^i}\right) = X^i \frac{\partial L}{\partial y^i}.$$

It can be regarded as well as defined on each fibre of TM^* by $\mathcal{L} = \dfrac{\partial L}{\partial y^j} dx^j$. The local form of the Legendre transformation $\mathcal{L}$ is

$$(x^i, y^j) \to \left(x^i, \frac{\partial L}{\partial y^j}\right). \tag{1}$$

Notice that $\mathcal{L}$ is a fibered manifold morphism, but, in general, it fails to be a vector bundle morphism.

We recall that a Lagrangian L is *regular* if the vertical Hessian

$$\left(\frac{\partial^2 L}{\partial y^i \partial y^j}(x^i, y^j)\right)_{1 \leq i, i \leq m}$$

is non-degenerate in every point on TM^* which has the coordinates (x^i, y^j). Taking into account of the local form of the Legendre transformation, it follows:

Proposition 1.1 *The Lagrangian L is regular iff its Legendre transformation $\mathcal{L}$ is a local diffeomorphism. If it is the case, then $\mathcal{L}$ induces local diffeomorphisms on each fibers.*

Definition 1.1 We say that the Lagrangian L is *L-regular* if its Legendre transformation $\mathcal{L}$ is a global diffeomorphism of TM^* on T^*M^*.

Consider now a Hamiltonian H. The *Legendre* transformation* associated with H is the function $\mathcal{H} : T^*M^* \to TM^*$ defined by

$$\mathcal{H}\left(\omega_i dx^i\right) = \omega_i \frac{\partial H}{\partial p_i}.$$

It can be regarded as well as defined on each fibre of T^*M^* by $\mathcal{H} = \dfrac{\partial L}{\partial p_i} \dfrac{\partial}{\partial x^i}$. The local form of the Legendre* transformation $\mathcal{H}$ is

$$(x^i, p_j) \to \left(x^i, \frac{\partial H}{\partial p_j}\right). \tag{2}$$

Notice that $\mathcal{H}$ is a fibered manifold morphism, but, in general, it fails to be a vector bundle morphism.

In an analogous way, we say that a Hamiltonian H is *regular* if the vertical Hessian $\left(\dfrac{\partial^2 H}{\partial p_i \partial p_j}(x^k, p_l)\right)_{1 \leq i, i \leq m}$ is non-degenerate in every point on T^*M^* which has the coordinates (x^i, p_j). Taking into account of the local form of the Legendre* transformation, it follows:

Proposition 1.2 *The Hamiltonian H is regular iff its Legendre* transformation $\mathcal{H}$ is a local diffeomorphism. If this is the case, then $\mathcal{H}$ induces a local diffeomorphism on each fiber.*

Definition 1.2 We say that the Hamiltonian H is *H-regular* if its Legendre* transformation $\mathcal{H}$ is a global diffeomorphism of T^*M^* on TM^*.

The link between the Lagrange and Hamilton geometry is given by:

Proposition 1.3 *a) If $L : TM \to \boldsymbol{R}$ is an L-regular Lagrangian, then $H = (Z(L) - L) \circ \mathcal{L}^{-1}$ is an H-regular Hamiltonian on T^*M, where $Z \in \mathcal{X}(TM)$ is the Liouville vector field and $\mathcal{L} : TM \to T^*M$ is the Legendre transformation.*

b) If $H : T^*M \to \boldsymbol{R}$ is an H-regular Hamiltonian, then $L = (\Xi(H) - H) \circ \mathcal{H}^{-1}$ is an L-regular Lagrangian on T^*M, where $\Xi \in \mathcal{X}(T^*M)$ is the Liouville vector field and $\mathcal{H} : T^*M \to TM$ is the Legendre* transformation.

Proof. We prove only b), since a) is analogous. We denote $\mathcal{K} = \mathcal{H}^{-1}$. Using local coordinates, it follows that $\mathcal{K}$ has the local form $(x^i, y^j) \to (x^i, K_j(x^k, y^l))$. The condition $\mathcal{K} \circ \mathcal{H} = id_{T^*M^*}$ gives

$$K_j(x^i, H^j(x^k, p_l)) = p_j, \tag{3}$$

where $H^j(x^k, p_l) = \dfrac{\partial H}{\partial p_j}(x^k, p_l)$. Differentiating with respect to p_i the relation (3), it follows that $\dfrac{\partial K_j}{\partial y^k}(x^i, H^j(x^k, p_l)) \cdot \dfrac{\partial H^k}{\partial p_i}(x^k, p_l) = \delta^i_j$, or $\dfrac{\partial K_j}{\partial y^k}(x^i, H^j(x^k, p_l)) \cdot h^{ik} = \delta^i_j$, where $h^{ik} = \dfrac{\partial^2 H}{\partial p_i \partial p_k}$. Denote $(h_{ij}) = (h^{ij})^{-1}$. Then

$$\frac{\partial K_j}{\partial y^i}(x^i, H^j(x^k, p_l)) = h_{ij}(x^k, p_l)\ . \tag{4}$$

The Liouville field has the local form $\Xi = p_i \dfrac{\partial}{\partial p_i}$. Using the definition of L it follows

$$L(x^i, y^j) = K_k(x^i, y^j) \frac{\partial H}{\partial p_k}(x^i, K_j(x^i, y^j)) - H(x^i, K_j(x^i, y^j)).$$

We have $\dfrac{\partial L}{\partial y^i} = \dfrac{\partial K_j}{\partial y^i} \dfrac{\partial H}{\partial p_j} + K_j \dfrac{\partial K_k}{\partial y^i} \dfrac{\partial^2 H}{\partial p_k \partial p_j} - \frac{\partial K_j}{\partial y^i} \frac{\partial H}{\partial p_j} = K_j \frac{\partial K_k}{\partial y^i} \frac{\partial^2 H}{\partial p_k \partial p_j} = K_j h_{ik} h^{kj} = K_i$, thus, using (4), we have $\dfrac{\partial^2 L}{\partial y^i \partial y^j} = \dfrac{\partial K_j}{\partial y^i} = h_{ij}$. It follows

$$\frac{\partial^2 L}{\partial y^i \partial y^j}(x^i, y^j) = h_{ij}(x^i, K_j(x^i, y^j)).$$

□

In the particular case of a Finsler metric on M (i.e. the Lagrangian L is 2-homogeneous), we have $Z(L) = 2L$, thus, in this case, $H = L \circ \mathcal{L}^{-1}$. A similar result is obtained for a 2-homogeneous Hamiltonian on M. In this case $\Xi(H) = 2H$, thus $L = H \circ \mathcal{H}^{-1}$.

2 Subspaces of Hamilton spaces

Besides the theory of Lagrange and Finsler submanifolds, which is studied by many authors, (see the Bibliography), an attempt to study the Hamilton submanifolds is performed in [13, 14], using an arbitrary section of the natural projection of the cotangent bundles. Here we show that there is a distinguished section, which depends only on the Hamiltonian.

Let $M' \subset M$ be a submanifold and $H : T^*M \to \mathbf{R}$ be a regular Hamiltonian. Without loss of generality we can suppose that H is H-regular, thus the Legendre* transformation $\mathcal{H} : T^*M^* \to TM^*$ is a diffeomorphism. We denote by $\mathcal{K} = \mathcal{H}^{-1} : TM^* \to T^*M^*$ the inverse of the Legendre* transformation. Let $i : TM'^* \to TM^*$ be the submanifold inclusion.

We consider the local coordinates $(x^i) = (x^u,\ x^{\bar{u}})$ on M, $(x^i, y^j) = (x^u,\ x^{\bar{u}},\ y^v,\ y^{\bar{v}})$ on TM and $(x^i, p_k) = (x^u,\ x^{\bar{u}},\ p_v,\ p_{\bar{v}})$ on T^*M ($i, j, k, \ldots = \overline{1, m}$, $m = \dim M$, $u, v, \ldots = \overline{1, m'}$, $\bar{u}, \bar{v}, \ldots \in \overline{m' + 1, m}$, $m' = \dim M'$), which are adapted to the vector bundle structures and to the submanifolds structures. Notice that the points in M' have as coordinates $(x^u, x^{\bar{u}} = 0)$, the points in TM' have as coordinates $(x^u, x^{\bar{u}} = 0, y^v, y^{\bar{v}} = 0)$ and the points in T^*M' have as coordinates (x^u, p_u). The local form of the Legendre* transformation $\mathcal{H}$ is $(x^i, p_j) \to (x^i, \dfrac{\partial H}{\partial p_j}(x^k, p_l) = H^j(x^k, p_l))$. We denote by $(x^i, y^j) \to (x^i, K_j(x^k, y^l))$ the local form of $\mathcal{K}$. The local forms of the inclusion i and of the canonical projection $i^* : T^*M^* \to T^*M'^*$ are $(x^u, y^v) \to (x^u, 0, y^v, 0)$ and $(x^u, x^{\bar{u}}, p_v, p_{\bar{v}}) \to (x^u, p_v)$ respectively.

We have that $WM' = \mathcal{K} \circ i(T^*M'^*)$ is a submanifold of T^*M^*.

Proposition 2.1 *The restriction of i^* to WM', $i^*_{|WM'} : WM' \to TM'^*$ is a diffeomorphism.*

Proof. We have: $\mathcal{K}$ is a diffeomorphism, i^* is a surjective submersion and i is an injective immersion. The local form of $i^* \circ \mathcal{K} \circ i$ is $(x^u, y^v) \to (x^u, K_v(x^u, 0, y^v, 0))$, thus it is a local diffeomorphism. In fact $i^* \circ \mathcal{K} \circ i$ is a diffeomorphism, since it sends the fibre $T_xM'^*$ in the fibre $T_xM'^*$ for every $x \in M'$ and $\mathcal{K}$ is a diffeomorphism, thus $i^*_{|WM'}$ is also a diffeomorphism. □

Taking into account of the local form of the Legendre* transformation and of the local coordinates adapted to the submanifold M', it follows that the points of the submanifold WM' have as coordinates $(x^u, 0, p_v, Q_{\bar{v}}(x^u, p_v))$ in TM^*, where

$$\frac{\partial H}{\partial p_{\bar{u}}}(x^u, 0, p_v, Q_{\bar{v}}(x^u, p_v)) = 0, \tag{5}$$

since the Legendre* transformation of the set of these points is a set included in $i(TM^*)$. Differentiating this equation with respect to p_u, we get:

$$\frac{\partial^2 H}{\partial p_u \partial p_{\bar{u}}} + \frac{\partial^2 H}{\partial p_{\bar{v}} \partial p_{\bar{u}}} \cdot \frac{\partial Q_{\bar{v}}}{\partial p_u} = 0.$$

Denoting by $h^{ij} = \dfrac{\partial^2 H}{\partial p_i \partial p_j}$, we suppose that the square matrix $\tilde{h} = (h^{\bar{u}\bar{v}})_{\bar{u}, \bar{v} = \overline{m'+1, m}}$ is non-degenerate; if this condition holds, we say that the *Hamiltonian is non-degenerate along the submanifold* M' (notice that this condition automatically holds when the Hamiltonian defines a positive quadratic form). Considering the inverse $\tilde{h}^{-1} = (\tilde{h}_{\bar{u}\bar{v}})_{\bar{u},\bar{v}=\overline{m'+1,m}}$, it follows that

$$\frac{\partial Q_{\bar{v}}}{\partial p_u} = -h^{u\bar{u}}\tilde{h}_{\bar{u}\bar{v}}. \tag{6}$$

Denote by $\tilde{i} = i^{*-1}_{|WM'} : T^*M'^* \to WM' \subset T^*M^*$. Using the above constructions, we obtain the following result.

Theorem 2.1 *There is a section of i^*, namely $\tilde{i}$, which depends only on H.*

We define $H' = H \circ \tilde{i} : T^*M'^* \to \mathbf{R}$ and we consider the vertical Hessian of H':

$$\left(\frac{\partial^2 H'}{\partial p_u \partial p_v}(x^t, p_w)\right)_{u,v=\overline{1,m'}}$$

at every point of $T^*M'^*$.

Proposition 2.2 *a) If the Hamiltonian H is non-degenerate along the submanifold M', then the vertical Hessian of H' is also non-degenerate in every point of $T^*M'^*$.*

b) If the Hamiltonian has a positive definite metric, then the vertical Hessian of H' is also positive definite.

Proof. We use local coordinates. We have $H'(x^u, p_v) = H(x^u,\ 0,\ p_v,\ Q_{\bar{v}}(x^u, p_v))$. Using formula (5) it follows that

$$\frac{\partial H'}{\partial p_u}(x^u, p_v) = \frac{\partial H}{\partial p_u}(x^u, 0, p_v, Q_{\bar{v}}(x^u, p_v)).$$

Differentiating this formula with respect to p_v, then using formula (6), we get:

$$\frac{\partial^2 H'}{\partial p_v \partial p_u} = \frac{\partial^2 H}{\partial p_v \partial p_u} + \frac{\partial Q_{\bar{v}}}{\partial p_v}\frac{\partial^2 H}{\partial p_{\bar{v}} \partial p_u} = h^{vu} - h^{v\bar{u}}\tilde{h}_{\bar{u}\bar{v}}h^{\bar{v}u}.$$

We use now the following Lemma of linear algebra.

Lemma 2.1 *Let A be a symmetric matrix of dimension p, B a symmetric and non-degenerated matrix of dimension q and C a $p \times q$ matrix such that the symmetric matrix $\begin{pmatrix} A & C \\ C^t & B \end{pmatrix}$ of dimension $p+q$ is non-degenerate. Denote $\begin{pmatrix} A & C \\ C^t & B \end{pmatrix}^{-1} = \begin{pmatrix} X & Z \\ Z^t & Y \end{pmatrix}$, where X, Y and Z have the same dimensions as the matrices A, B and C respectively.*

Then the matrix $A - C \cdot B^{-1}C^t$ is invertible and its inverse is X.

Proof. We have $\begin{pmatrix} A & C \\ C^t & B \end{pmatrix} \cdot \begin{pmatrix} X & Z \\ Z^t & Y \end{pmatrix} = \begin{pmatrix} I_p & 0 \\ 0 & I_q \end{pmatrix}$. Thus $A \cdot X + C \cdot Z^t = I_p$ and $C^t \cdot X + B \cdot Z^t = 0$. The second equality implies $Z^t = B^{-1} \cdot C^t \cdot X$, then introducing in the first equality we get $(A - C \cdot B^{-1} \cdot C^t) \cdot X = I_p$, thus the conclusion follows. □

Turning back to the proof of the Proposition 2.2, consider the matrix $h = (h^{ij}) = \begin{pmatrix} h^{uv} & h^{\bar{u}v} \\ h^{u\bar{v}} & h^{\bar{u}\bar{v}} \end{pmatrix}$. Using the Lemma, it follows that the matrix

$$\left(h^{uv} - h^{u\bar{u}}\tilde{h}_{\bar{u}\bar{v}}h^{\bar{v}u}\right)_{u,v=\overline{1,m'}}$$

is invertible and its inverse is (h_{uv}), where $\begin{pmatrix} h_{uv} & h_{\bar{u}v} \\ h_{u\bar{v}} & h_{\bar{u}\bar{v}} \end{pmatrix} = \begin{pmatrix} h^{uv} & h^{\bar{u}v} \\ h^{u\bar{v}} & h^{\bar{u}\bar{v}} \end{pmatrix}^{-1}$. □

If the inverse $\mathcal{K}$ of the Legendre* transformation of H can be extended to the image of the null section of M', such that $\mathcal{K}$ becomes continuous, it follows that the local functions $(x^u, p_v) \to Q_{\bar{v}}(x^u, p_v)$ becomes continuous, thus H' can be extended continuously on the image of the null section of T^*M'.

Corollary 2.1 *If the inverse $\mathcal{K}$ of the Legendre* transformation of H can be extended to the image of the null section of M' such that it becomes continuous and if the Hamiltonian H is non-degenerate along the submanifold M', then H' defines also a Hamiltonian on M'.*

Finally we remark that H' can be obtained as in [8], in the following way. Consider the Lagrangian $L : TM \to \mathbf{R}$ defined by the Hamiltonian H and the induced Lagrangian $L' : TM' \to \mathbf{R}$ on M'. Let $\mathcal{H}' : T^*M'^* \to TM'^*$ be the inverse of the Legendre transformation determined by L' and $\mathcal{L} : TM^* \to T^*M^*$ be the Legendre transformation determined by L. It can be shown that $\tilde{i} = \mathcal{L} \circ i_* \circ \mathcal{H}'$, thus $H' = H \circ \tilde{i}$ is the same as the induced Hamiltonian obtained in [8]. The condition on H to be non-degenerate along the submanifold M' reads to the condition that $\mathcal{H}'$ exists.

Acknowledgement. The present work was partially supported by the Grant-CNCSIS-MEN 34967 (182)/2001-2002.

References

[1] Anastasiei M., *On subspaces in Finsler spaces*, Proc. of the Romanian-Japanese Coll. on Finsler geometry, Başov, 1984, 17-30.

[2] Anastasiei M., *On Miron 's theory of subspaces in Finsler spaces*, An. St. Univ. *Al. I. Cuza* Iaşi, s.I a, XXX (1984), 11-14.

[3] Bejancu A, *Geometry of Finsler subspaces (I)*, An. St. Univ. *Al. I. Cuza* Iaşi, T.XXXII, s.Ia, Mat., 2(1986), 69-83.

[4] Bejancu A., *Geometry of Finsler subspaces (II)*, Bul. Inst. Pol. Iaşi, T.XXXV, f.1-4, s1 Mec. Mat., (1985), 67-73.

[5] Bejancu A., *Vectorial Finsler connections and theory of Finsler subspaces*, Seminar on Geometry and Topology, Timişoara, 1986, Preprint No. 81.

[6] Bejancu A., *A new viewpoint in geometry of Finsler subspaces*, Bul. Inst. Pol. Iaşi, I, XXXIII, 1-4(1987), 13-19.

[7] Haimovici M., *Les formules fondamentales de la théorie des hypersurfaces d'un espace général*, Ann. St. Univ. Iaşi, 20 (1935), 39-58.

[8] Marle C.-M., *Various approaches to conservative and non-conservative nonholonomic systems*, Reports on mathematical Physics, 42 (1998), 211–229.

[9] Miron R., *Sur la géométrie des espaces d'Hamilton*, C.R. Acad Sci. Paris, 306, I(1988), 195-198.

[10] Miron R., *The geometry of Hamilton spaces*, The Proceedings of the fifth National Seminar of Finsler and Lagrange spaces, Braşov, 1988.

[11] Miron R., *A nonstandard theory of hypersurfaces in Finsler spaces*, An. St. Univ. *Al. I. Cuza* Iaşi, s.I a, XXX, 5(1984), 35-53.

[12] Miron R., Anastasiei M., *Vector bundles. Lagrange spaces. Applications to the theory of relativity*, Ed. Academiei, Bucureşti, 1987.

[13] Miron R., Kikuchi S., Sakaguchi T., *Subspaces in generalized Hamilton spaces endowed with h-metrical connections*, Mem. Secţ. Ştiinţ. ale Academiei R.S.R., Ser. IV, 11, 1 (1988), (1991), 55–71.

[14] Miron R., *Subspaces in Hamilton spaces*, Indian J. Pure Appl. Math. 21, 3 (1990), 211–233.

[15] Miron R., Anastasiei M., *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Acad. Publ., 1994.

[16] Miron R., Bejancu A., *A new method in geometry of Finsler subspaces*, An. St. Univ. *Al. I. Cuza* Iaşi, s.I a, XXX, 4(1984), 56-60.

[17] Miron R., Bejancu A., *A non-standard theory of Finsler subspaces*, Topics in Diff. Geom., Debrecen, (1984), 815-851.

[18] Oproiu V., Papaghiuc N., *Submanifolds in Lagrange geometry*, An. St. Univ. *Al. I. Cuza* Iaşi, s.I a, XXXIV, 2(1988), 153-166.

[19] Popescu Marcela, *Connections on Finsler bundles*, The second international workshop on diff.geom.and appl., 25-28 septembrie 1995, Constanţa, An. St. Univ. *Ovidius* Constanţa, s.mat., vol III, fasc. 2 (1995), 97-101.

[20] Popescu Marcela, *Induced connections on fibred manifolds*, An. Univ. Craiova, ser. mat.-inf., vol.XXII (1995), pag.44-50.

[21] Popescu Marcela, *On induced non-linear connections*, An. Univ *Al. I. Cuza*, Iaşi, seria mat., T.XLII,f.1. (1996), 119-126.

[22] Popescu Marcela, Popescu Paul, *The structure equations of induced non-linear connections*, An. Univ. Timişoara, Seria matem. Vol. XXXIV, fasc. 2 (1996), 255-260.

[23] Popescu Marcela, *On the supplementary vector subbundles*, Proc. of the Workshop on Global Analysis, Diff. Geom., Lie Alg., Thessaloniki 1995, Balkan Press, Bucharest (1997), 65-70.

[24] Popescu Marcela, *On vectorial Finsler connections*, The second international workshop on global Anal. Diff.Geom. Lie Alg., 13-16 dec. 1995, Thessaloniki, Greece, "Balkan Press", Bucharest (1997), 61-64.

[25] Popescu Marcela, *Semisprays induced on submanifolds*, New developements in Diff. Geom., Kluwer Academic Publ., 1998, 317-328.

[26] Tulczyjiew W.M., *The Legendre transformation*, Ann. Inst. H. Poincaré, 17 (1977), 101-114.

University of Craiova
Department of Algebra and Geometry
13, 'Al.I.Cuza' st., Craiova, 1100, Romania
E-mail addresses: paulpopescu@email.com, marcelapopescu@email.com