Homogeneous Lorentzian Structures on Some Gödel-Levichev's Spacetimes, and Associated Reductive Decompositions

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Dedicated to the Memory of Grigorios TSAGAS (1935-2003), President of Balkan Society of Geometers (1997-2003)

Abstract

For the Levichev homogeneous spacetimes of type $2a$ on the Gödel group, the homogeneous Lorentzian structures and the associated reductive decompositions are determined.

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1 Introduction and preliminaries

É. Cartan gave in [2] the classical characterization of Riemannian symmetric spaces as the spaces of parallel curvature. This was extended by Ambrose and Singer, who gave in [1] a characterization for a connected, simply connected and complete Riemannian manifold to be homogeneous, in terms of a $(1,2)$ tensor field S, called by Tricerri and Vanhecke in [7] a *homogeneous Riemannian structure*, which satisfies certain equations (see (1.1) below). In [3] it is defined a *homogeneous pseudo-Riemannian structure* on a pseudo-Riemannian manifold (M,g) as a tensor field S of type $(1,2)$ such that ∇ being the Levi-Civita connection and R its curvature tensor, the connection $\widetilde{\nabla} = \nabla - S$ satisfies the Ambrose-Singer equations

$$\widetilde{\nabla} g = 0, \quad \widetilde{\nabla} R = 0, \quad \widetilde{\nabla} S = 0. \tag{1.1}$$

In [3] it is proved that *if the pseudo-Riemannian manifold (M,g) is connected, simply connected and geodesically complete then it admits a homogeneous pseudo-Riemannian structure if and only if it is a reductive homogeneous pseudo-Riemannian manifold*. This means that $M = G/H$, where G is a connected Lie group acting transitively and effectively on M as a group of isometries, H is the isotropy group at a point $o \in M$, and the Lie algebra $\mathbf{g}$ of G may be decomposed into a vector space

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direct sum of the Lie algebra $\mathbf{h}$ oh H and an Ad (H)-invariant subspace $\mathbf{m}$, that is $\mathbf{g} = \mathbf{h} \oplus \mathbf{m}$, Ad (H)$\mathbf{m} \subset \mathbf{m}$. (If G is connected and M is simply connected then H is connected, and the latter condition is equivalent to $[\mathbf{h}, \mathbf{m}] \subset \mathbf{m}$.)

Let (M, g) be a connected, simply connected, and geodesically complete pseudo-Riemannian manifold, and suppose that S is a homogeneous pseudo-Riemannian structure on (M, g). We fix a point $o \in M$ and put $\mathbf{m} = T_o(M)$. If $\widetilde{R}$ is the curvature tensor of the connection $\widetilde{\nabla} = \nabla - S$, we can consider the holonomy algebra $\tilde{\mathbf{h}}$ of $\widetilde{\nabla}$ as the Lie subalgebra of "skew-symmetric" endomorphisms of $(\mathbf{m}, g_o)$ generated by the operators $\widetilde{R}_{ZW}$, where $Z, W \in \mathbf{m}$. Then, according to the Ambrose-Singer construction [1, 7], a Lie bracket is defined in the vector space direct sum $\tilde{\mathbf{g}} = \tilde{\mathbf{h}} \oplus \mathbf{m}$ by

$$\begin{aligned} &[U, V] = UV - VU, && U, V \in \tilde{\mathbf{h}}, \\ &[U, Z] = U(Z), && U \in \tilde{\mathbf{h}},\ Z \in \mathbf{m}, \\ &[Z, W] = \widetilde{R}_{ZW} + S_Z W - S_W Z, && Z, W \in \mathbf{m}\,, \end{aligned} \tag{1.2}$$

and we say that $(\tilde{\mathbf{g}}, \tilde{\mathbf{h}})$ is the *reductive pair* associated to the homogeneous pseudo-Riemannian structure S.

Tricerri and Vanhecke [7] have classified the homogeneous Riemannian structures into eight classes, which are defined by the invariant subspaces of certain space $\mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \mathcal{S}_3$. In [4] a similar classification for the pseudo-Riemannian case is given. For more details see below.

On the other hand, Levichev consider in [5] the usual Gödel metric

$$g = -\frac{e^{-2x_4}}{2}\, dx_1^2 - 2\, e^{-2x_4}\, dx_1\, dx_2 - dx_2^2 + dx_3^2 + dx_4^2,$$

as a left-invariant metric on the Gödel group G, and defines several families of metrics on G, thus obtaining several types of homogeneous Lorentz spaces. The ones of type $2a$ are connected, simply connected, and geodesically complete. In the present note we determine the homogeneous Lorentzian structures on these homogeneous space-times and their type in Tricerri-Vanhecke's classification, and the associated reductive decompositions.

2 Homogeneous Lorentzian structures

The Gödel group is the simply connected Lie group G whose Lie algebra $\mathbf{g}$ has four generators e_1, e_2, e_3, e_4, with the only nonvanishing bracket

$$[e_4, e_1] = e_1.$$

The group G admits a realization as $\mathbf{R}^4 = \{(x_1, x_2, x_3, x_4)\}$ with multiplication $z = x \cdot y$ obtained from the matrix expression

$$x \equiv \begin{pmatrix} e^{x_4} & 0 & 0 & x_1 \\ 0 & 1 & 0 & x_2 \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The commutation relations of its Lie algebra in the system of coordinates chosen on G coincide with the brackets above.

Consider the subspaces L_1, L_2, L_3 of $\mathbf{g}$ generated respectively by e_1; e_2, e_3; and e_1, e_2, e_3. Then the homogeneous Lorentz group of type $2a$ is defined by the conditions: L_2, L_3 are timelike, and L_1 is spacelike (for more details see [5]). Then, for each couple of real numbers p, q with $0 \leq p < 1$, $q > 0$, the left-invariant Lorentzian metric $g_{p,q}$ on G obtained by left translations from the scalar product at the origin with matrix given, with respect to the above basis of $\mathbf{g}$, by

$$\langle\ ,\ \rangle_{p,q} = \begin{pmatrix} 1 & p & 0 & 0 \\ p & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \tag{2.1}$$

is given by

$$g_{p,q} = \begin{pmatrix} e^{-2x_4} & e^{-2x_4}p & 0 & 0 \\ e^{-2x_4}p & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.$$

As causal spacetimes, the Lorentz Lie groups corresponding to the Gödel group with the metric of type $2a$ are homogeneously globally hyperbolic, which is a strong causality condition. We recall that: A causal curve in a Lorentz manifold M is a curve whose velocity vectors are all nonspacelike; if M is globally hyperbolic then any pair of points that can be joined by a causal curve can be joined by a (longest) causal geodesic; a solvable Lorentz Lie group G is said to be homogeneously globally hyperbolic if it is globally hyperbolic and has a Cauchy surface S passing through the identity element $e \in G$ and containing the center of G (for more details see [5, 6]); a Cauchy surface of a spacetime is a subset that is met exactly once by every inextendible timelike curve in the spacetime.

On account of Koszul's formula for the Levi-Civita connection for a left-invariant metric g on a Lie group,

$$2g(\nabla_{e_i} e_j, e_k) = g([e_i, e_j], e_k) - g([e_j, e_k], e_i) + g([e_k, e_i], e_j),$$

we obtain that the non-null covariant derivatives between generators are

$$\nabla_{e_1} e_1 = \frac{1}{q} e_4, \qquad \nabla_{e_1} e_2 = \nabla_{e_2} e_1 = \frac{p}{2q} e_4,$$

$$\nabla_{e_1} e_4 = \frac{p^2 - 2}{2(1-p^2)} e_1 + \frac{p}{2(1-p^2)} e_2,$$

$$\nabla_{e_2} e_4 = \nabla_{e_4} e_2 = -\frac{p}{2(1-p^2)} e_1 + \frac{p^2}{2(1-p^2)} e_2,$$

$$\nabla_{e_4} e_1 = -\frac{p^2}{2(1-p^2)} e_1 + \frac{p}{2(1-p^2)} e_2.$$

So, the nonvanishing components of the curvature tensor, with the convention $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, are, putting $R_{e_ie_j}e_k$ for $R(e_i,e_j)e_k$,

$$R_{e_1e_2}e_1 = \frac{p^3}{4q(1-p^2)}e_1 - \frac{p^2}{4q(1-p^2)}e_2,$$

$$R_{e_1e_2}e_2 = \frac{p^2}{4q(1-p^2)}e_1 - \frac{p^3}{4q(1-p^2)}e_2,$$

$$R_{e_1e_4}e_1 = \frac{p(2-p)}{4q(1-p^2)}e_4, \qquad R_{e_1e_4}e_2 = -\frac{p^3}{4q(1-p^2)}e_4,$$

$$R_{e_1e_4}e_4 = \frac{p^2-4}{4(1-p^2)}e_1 + \frac{p}{1-p^2}e_2, \qquad R_{e_2e_4}e_1 = -\frac{p^3}{4q(1-p^2)}e_4,$$

$$R_{e_2e_4}e_2 = -\frac{p^2}{4q(1-p^2)}e_4, \qquad R_{e_2e_4}e_4 = \frac{p^2}{4(1-p^2)}e_2,$$

and the nonvanishing components of the Riemann-Christoffel curvature tensor, with the convention $R(X,Y,Z,W) = g(R(Z,W)Y,X)$, putting $R_{e_ie_je_ke_l}$ for $g(R(e_k,e_l)e_j,e_i)$, are

$$R_{e_1e_2e_1e_2} = \frac{p^2}{4q}, \qquad R_{e_1e_4e_1e_4} = \frac{5p^2-4}{4(1-p^2)},$$

$$R_{e_1e_4e_2e_4} = \frac{p^3}{4(1-p^2)}, \quad R_{e_2e_4e_2e_4} = \frac{p^2}{4(1-p^2)}.$$

We shall now determine the homogeneous Lorentzian structures on these spaces. For this, we must solve the Ambrose-Singer equations 1.1. The first Ambrose-Singer equation amounts to $S_{XYZ} = -S_{XZY}$ for any homogeneous pseudo-Riemannian structure S. One can write the second Ambrose-Singer equation $\widetilde{\nabla}R = 0$ as

$$\begin{aligned} &R_{\nabla_U XYZW} + R_{X\nabla_U YZW} + R_{XY\nabla_U ZW} + R_{XYZ\nabla_U W} \\ &= S_{UXR(Z,W)Y} - S_{UYR(Z,W)X} + S_{UZR(X,Y)W} - S_{UWR(X,Y)Z}. \end{aligned}$$

Solving, we obtain that the nonvanishing components of S are

$$S_{e_1e_2e_4} = 1-p^2, \qquad S_{e_4e_1e_2} = \frac{p}{2},$$

except for $S_{e_ie_1e_4}$, $i = 1,\dots,4$, for which we must use the third Ambrose-Singer equation. In our case, since we are considering left-invariant differential forms, the forms involved in this equation are linear combinations with constant coefficients of the basis $\{\theta^1,\theta^2,\theta^3,\theta^4\}$ of left-invariant forms on G dual to the basis $\{e_1,e_2,e_3,e_4\}$. Moreover, since for a constant function f, one has $\nabla_X f = 0$ and $\widetilde{\nabla}_X f = 0$, we also have $S_X f = 0$. Thus, the third Ambrose-Singer equation $\widetilde{S} = 0$ can be written as

$$S_{\nabla_X YZW} + S_{Y\nabla_X ZW} + S_{YZ\nabla_X W} = S_{S_X YZW} + S_{YS_X ZW} + S_{YZS_X W},$$

for $X,Y,Z,W \in \mathbf{g}$.

Solving, we obtain the nonzero components

$$S_{e_1e_1e_4} = 1, \qquad S_{e_2e_1e_4} = \frac{p}{2}.$$

Consequently, the non-null components $S_{e_i}e_j$ are

$$S_{e_1}e_1 = \frac{1}{q}e_4, \quad S_{e_1}e_2 = \frac{1-p^2}{q}e_4,$$

$$S_{e_1}e_4 = \frac{-p^3+p-1}{1-p^2}e_1 + \frac{p^2+p-1}{1-p^2}e_2, \quad S_{e_2}e_1 = \frac{p}{2q}e_4,$$

$$S_{e_4}e_1 = -\frac{p^2}{2(1-p^2)}e_1 + \frac{p}{2(1-p^2)}e_2, \quad S_{e_4}e_2 = -\frac{p}{2(1-p^2)}e_1 + \frac{p^2}{2(1-p^2)}e_2,$$

Then, with the convention $v \wedge w = v \otimes w - w \otimes v$ for the exterior product, we have proved the following

Theorem 1 *The homogeneous Lorentzian structures on the Gödel-Levichev space* $(G, g_{p,q})$ *of type* 2a *are given by*

$$\theta^1 \otimes \theta^1 \wedge \theta^4 + (1-p^2)\,\theta^1 \otimes \theta^2 \wedge \theta^4 + \frac{p}{2}\,(\theta^2 \otimes \theta^1 \wedge \theta^4 + \theta^4 \otimes \theta^1 \wedge \theta^2).$$

We recall some definitions and a result from Tricerri and Vanhecke [7] (see also [4]). Let E be a real vector space of dimension n endowed with an inner product $\langle\ ,\ \rangle$ of signature $(k, n-k)$. The space $(E, \langle\ ,\ \rangle)$ will be the model for each tangent space T_xM, $x \in M$, of a reductive homogeneous pseudo-Riemannian manifold of signature $(k, n-k)$. Consider the vector space $\mathcal{S}(E)$ of tensors of type $(0,3)$ on $(E, \langle\ ,\ \rangle)$ satisfying the same symmetries as those of a homogeneous pseudo-Riemannian structure S, that is, $\mathcal{S}(E) = \{S \in \otimes^3 E^* : S_{XYZ} = -S_{XZY},\ X, Y, Z \in E\}$, where $S_{XYZ} = \langle S_X Y, Z\rangle$. Let $c_{12}\colon \mathcal{S}(E) \to V^*$ be the map defined by $c_{12}(S)(Z) = \sum_{i=1}^{n} \varepsilon_i S_{e_i e_i Z}$, $Z \in E$, where $\{e_i\}$ is an orthonormal basis of E, $\langle e_i, e_i\rangle = \varepsilon_i = \pm 1$. Then we have that *if* $\dim E \geq 3$, *then* $\mathcal{S}(E)$ *decomposes into the orthogonal direct sum of subspaces which are invariant and irreducible under the action of the pseudo-orthogonal group* $O(k, n-k) : \mathcal{S}(E) = \mathcal{S}_1(E) \oplus \mathcal{S}_2(E) \oplus \mathcal{S}_3(E)$, *where*

$$\begin{array}{lcl}
\mathcal{S}_1(E) & = & \{\, S \in \mathcal{S}(E) : S_{XYZ} = \langle X, Y\rangle \omega(Z) - \langle X, Z\rangle \omega(Y),\ \omega \in E^* \,\}, \\
\mathcal{S}_2(E) & = & \{\, S \in \mathcal{S}(E) : \underset{XYZ}{\mathfrak{S}}\, S_{XYZ} = 0,\ c_{12}(S) = 0 \,\}, \\
\mathcal{S}_3(E) & = & \{\, S \in \mathcal{S}(E) : S_{XYZ} + S_{YXZ} = 0 \,\}. \\
\mathcal{S}_1(E) \oplus \mathcal{S}_2(E) & = & \{\, S \in \mathcal{S}(E) : \underset{XYZ}{\mathfrak{S}}\, S_{XYZ} = 0 \,\}, \\
\mathcal{S}_2(E) \oplus \mathcal{S}_3(E) & = & \{\, S \in \mathcal{S}(E) : c_{12}(S) = 0 \,\}, \\
\mathcal{S}_1(E) \oplus \mathcal{S}_3(E) & = & \{\, S \in \mathcal{S}(E) : S_{XYZ} + S_{YXZ} = 2\langle X, Y\rangle \omega(Z) \\
 & - & \langle X, Z\rangle \omega(Y) - \langle Y, Z\rangle \omega(X),\ \omega \in E^* \,\}.
\end{array}$$

In the present case we deduce

Corollary 1 *The homogeneous Lorentzian structures on* $(G, g_{p,q})$ *belong to*

$$\mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \mathcal{S}_3 - \{(\mathcal{S}_1 \oplus \mathcal{S}_2) \cup (\mathcal{S}_1 \oplus \mathcal{S}_3) \cup (\mathcal{S}_2 \oplus \mathcal{S}_3)\}.$$

In particular none of the associated reductive homogeneous spaces is either Lorentzian symmetric, or naturally reductive or cotorsionless.

Proof. Take the orthonormal basis

$$\tilde{e}_1 = \frac{1}{\sqrt{2(1+p)}}(e_1+e_2), \quad \tilde{e}_2 = \frac{1}{\sqrt{2(1-p)}}(e_1-e_2), \quad \tilde{e}_3 = e_3, \quad \tilde{e}_1 = \frac{1}{\sqrt{q}}e_4.$$

As a calculation with respect to this basis shows, the condition $c_{12}(S) = 0$ is not satisfied. On the other hand, since for instance $S_{e_1e_2e_4} + S_{e_2e_4e_1} + S_{e_4e_1e_2} \neq 0$, no structure belong to $\mathcal{S}_1 \oplus \mathcal{S}_2$. Moreover, since for instance $S_{e_1e_2e_4} \neq -S_{e_2e_1e_4}$, no structure belong to $\mathcal{S}_3$; not even to $\mathcal{S}_2 \oplus \mathcal{S}_3$, as the sum $S_{e_1e_2e_4} + S_{e_2e_1e_4}$ shows. The Lorentzian symmetric spaces correspond to the class $\{0\}$, and in [4] it has been proved the equivalence of the third class with the naturally reductive spaces, and of the class $\mathcal{S}_1 \oplus \mathcal{S}_2$ with the cotorsionless spaces. For more details see [4].

3 Associated reductive decompositions

Consider now the Ambrose-Singer connection $\widetilde{\nabla} = \nabla - S$. Then, the non-null covariant derivatives between generators are

$$\widetilde{\nabla}_{e_1}e_2 = \frac{2p^2+p-2}{2q}, \qquad \widetilde{\nabla}_{e_1}e_4 = \frac{p(2p^2+p-2)}{2(1-p^2)}e_1 - \frac{2p^2+p-2}{2(1-p^2)}e_2,$$

and, as a calculation shows, the only nonvanishing curvature operator is

$$\widetilde{R}_{e_1e_4} \equiv \begin{pmatrix} & 0 & 0 & 0 & \frac{p}{2(1-p^2)} \\ (2p^2+p-2) & 0 & 0 & 0 & -\frac{1}{2(1-p^2)} \\ & 0 & 0 & 0 & 0 \\ & 0 & \frac{1}{2q} & 0 & 0 \end{pmatrix},$$

According to Ambrose-Singer's Theorem on holonomy, the algebra of holonomy of a connection is generated by the curvature operators. In the present case, the holonomy algebra $\tilde{\mathbf{h}}$ has the only generator $V = \widetilde{R}_{e_1e_4}$. Putting $\mathbf{m}$ for $\mathbf{g}$, and taking $T = V + e_1$ we have

Theorem 2 *The reductive pairs* $(\tilde{\mathbf{g}}, \tilde{\mathbf{h}})$ *associated to the reductive decompositions* $\tilde{\mathbf{g}} = \tilde{\mathbf{h}} \oplus \mathbf{m}$ *corresponding to the homogeneous Lorentzian structures on* $(G, g_{p,q})$ *given in* Theorem 1*, are given in terms of the basis* $\{e_1, e_2, e_3, e_4, T\}$ *by the* (*nonvanishing*) *Lie brackets*

$$[T, e_4] = 2e_1 - T, \qquad [e_1, e_2] = -\frac{2p^2+p-2}{2q}e_4,$$

$$[e_1, e_4] = T - \frac{2p^3-3p^2-2p+4}{2(1-p^2)}e_1 + \frac{2p^2+p-2}{2(1-p^2)}e_2.$$

Proof. On account of the expressions (1.2), we obtain that

$$[V, e_2] = \frac{2p^2 + p - 2}{2q} e_4, \qquad [V, e_4] = \frac{p(2p^2 + p - 2)}{2(1 - p^2)} e_1 - \frac{2p^2 + p - 2}{2(1 - p^2)} e_2,$$

$$[e_1, e_2] = -\frac{2p^2 + p - 2}{2q} e_4, \; [e_1, e_4] = V - \frac{2p^3 - p^2 - 2p + 2}{2(1 - p^2)} e_1 + \frac{2p^2 + p - 2}{2(1 - p^2)} e_2.$$

Then, making the change $T = V + e_1$ we conclude.

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