Connections and Applications of Some Tangency Relation of Sets

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Dedicated to the Memory of Grigorios TSAGAS (1935-2003), President of Balkan Society of Geometers (1997-2003)

Abstract

In this paper are presented certain connections between the tangency relations of sets given by W. Waliszewski and considered earlier definitions of the tangency of sets in metric spaces. Some applications of results of my monographic paper for further investigations of the tangency of sets in metric spaces are discussed in the present paper. In Section 2 of this paper is shown that the W. Waliszewski's definition of the tangency of regular arcs is strictly related to the Alexandrov's and Riemann's angles between these arcs in a smooth Riemannian manifold.

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1 Introduction

Let E be any non-empty set. By E_0 we shall denote a family of all non-empty subsets of the set E. Let l be a non-negative real function defined on the Cartesian product $E_0 \times E_0$ of the family E_0 and let by the definition

$$l_0(x,y) = l(\{x\},\{y\}) \text{ for } x, y \in E. \tag{1.1}$$

If we put suitable conditions on the function l, then the function l_0 defined by (1.1) will be the metric of the set E. For this reason the pair (E,l) can be treated as a certain generalization of the metric space and we call it the generalized metric space (see [9]).

Similarly as in a metric space, using (1.1), we may define in the generalized metric space (E,l) the open ball $K_{l_0}(p,r)$ with the centre at the point $p \in E$ and the radius $r \geq$:

$$K_{l_0}(p,r) = \{x \in E\colon\ l_0(p,x) < r\}. \tag{1.2}$$

Assuming the family of all open balls $K_{l_0}(p,r)$ with positive radiuses for the complete system of neighbourhoods, we give to the set E the character of a topological

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space. Then any generalized metric space (E, l) determines a certain topological space (E, τ_l).

The sets of the family τ_l are unions of open balls. The family of all open balls $K_{l_0}(p, r)$ constitutes the base of topological space (E, τ_l). We consider the set $A \subset E$ as open in the topology τ_l iff for any point $p \in A$ there exists a number $r > 0$ such that $K_{l_0}(p, r) \subset A$.

By $S_{l_0}(p, r)_u$ we denote the so-called u-neighbourhood of the sphere $S_{l_0}(p, r)$ (with the centre at the point $p \in E$ and the radius r) in the generalized metric space (E, l) of the form

$$S_{l_0}(p,r)_u = \begin{cases} \bigcup q \in S_{l_0}(p,r) \bigcup K_{l_0}(q,u) & \text{for} \quad u > 0, \\ S_{l_0}(p,r) & \text{for} \quad u = 0. \end{cases} \tag{1.3}$$

Let a, b be arbitrary non-negative real functions defined in a certain right-hand side neighbourhood of 0 such that

$$a(r) \xrightarrow[r \to 0^+]{} 0 \quad \text{and} \quad b(r) \xrightarrow[r \to 0^+]{} 0. \tag{1.4}$$

We say that the pair (A, B) of sets A, B of the family E_0 is (a, b)-clustered at the point p of the space (E, l), if 0 is the cluster point of the set of all real numbers $r > 0$, such that the sets $A \cap S_{l_0}(p, r)_{a(r)}$ and $B \cap S_{l_0}(p, r)_{b(r)}$ are non-empty.

According to the definition given by W. Waliszewski in the paper [9], the set $A \in E_0$ is (a, b)-tangent of order $k > 0$ to the set $B \in E_0$ at the point p of the generalized metric space (E, l), if the pair of sets (A, B) is (a, b)-clustered at the point $p \in E$ and

$$\frac{1}{r^k} l(A \cap S_{l_0}(p,r)_{a(r)}, B \cap S_{l_0}(p,r)_{b(r)}) \xrightarrow[r \to 0^+]{} 0. \tag{1.5}$$

If the set A is (a, b)-tangent of order $k > 0$ to the set B at the point $p \in E$, then we shall write: $(A, B) \in T_l(a, b, k, p)$.

The set $T_l(a, b, k, p)$ we call the relation of (a, b)-tangency of order k at the point p (shortly: the tangency relation of sets) in the generalized metric space (E, l).

Let ρ be a metric of the set E and let ρ_0 be the function defined by the formula

$$\rho_0(A, B) = \sup\{\rho(x, B) : \ x \in A\} \quad \text{for } A, B \in E_0, \tag{1.6}$$

where $\rho(x, B)$ is the distance from the point $x \in A$ to the set B in the metric space (E, ρ).

If in the condition (1.5) we suppose $k = 1$ and in place of the function l we put the function ρ_0 defined by (1.6), then we get

$$\frac{1}{r} \rho_0(A \cap S_\rho(p,r)_{a(r)}, B \cap S_\rho(p,r)_{b(r)}) \xrightarrow[r \to 0^+]{} 0. \tag{1.7}$$

Setting $a(r) = 0$ and $b(r) = r$ for $r > 0$, the condition (1.7) we can write in the form

$$\frac{1}{r} \rho_0(A \cap S_\rho(p,r), B) \xrightarrow[r \to 0^+]{} 0,$$

i.e.

$$\frac{1}{r}\sup\{\rho(x,B):\ x\in A \ \text{ and } \ \rho(p,x)=r\} \xrightarrow[r\to 0^+]{} 0. \tag{1.8}$$

The condition (1.8) is equivalent to the condition

$$\frac{\rho(x,B)}{\rho(p,x)} \xrightarrow[A\ni x\to p]{} 0. \tag{1.9}$$

If $p \in A'$, where A' is the set of all cluster points of the set A, then the formula (1.9) presents the well-known definition of the tangency of sets, in particular of simple arcs, in the metric space (E,ρ).

Because here $k=1$ and the function ρ_0 is the special case of the function l, then the W. Waliszewski's definition essentially generalizes the above mentioned definition of the tangency of sets in the metric space (E,ρ).

The tangency relation of sets $T_l(a,b,k,p)$ given by W. Waliszewski we call the relation of equivalence in the set E, if is reflexive, symmetric and transitive in this set.

Two tangency relations of sets $T_{l_1}(a_1,b_1,k,p)$ and $T_{l_2}(a_2,b_2,k,p)$ are called compatible (equivalent) in the set E, if $(A,B)\in T_{l_1}(a_1,b_1,k,p)$ iff $(A,B)\in T_{l_2}(a_2,b_2,k,p)$ for $A,B\in E_0$.

We say that the tangency relation of sets $T_l(a,b,k,p)$ is homogeneous of the order 0 in some class of functions $\mathbf{F}$, if $(A,B)\in T_{ml}(a,b,k,p) \Leftrightarrow (A,B)\in T_l(a,b,k,p)$ for $m>0$, $l\in\mathbf{F}$ and $A,B\in E_0$.

In my monographic paper [4] I gave many theorems which are necessary and sufficient conditions for the equivalence, compatibility and homogeneity of the tangency relation of sets $T_l(a,b,k,p)$.

2 On some connections and applications

Let ρ be a metric of the set E and let A be any set of the family E_0. Let k be a fixed positive real number. We put by the definition (see [4, 6]).

$\widetilde{M}_{p,k} = \{A\in E_0:\ p\in A'$, and there exists a number $\mu>0$ such that for an arbitrary $\varepsilon>0$ there exists $\delta>0$ such that for any pair of points $(x,y)\in[A,p;\mu,k]$ if

$$\rho(p,x)<\delta \ \text{ and } \ \frac{\rho(x,A)}{\rho^k(p,x)}<\delta, \ \text{ then } \ \frac{\rho(x,y)}{\rho^k(p,x)}<\varepsilon\}, \tag{2.1}$$

where

$$[A,p;\mu,k]=\{(x,y):\ x\in E,\ y\in A \ \text{ and } \ \mu\rho(x,A)<\rho^k(p,x)=\rho^k(p,y)\}. \tag{2.2}$$

For $k=1$ the class of sets $\widetilde{M}_{p,k}$ contains the classes of sets H_p, A_p^* and the class A_p of rectifiable arcs (see [4]). Moreover $\widetilde{M}_{p,k}\supset A^*_{p,k}$ for any $k>0$ and $p\in E$ (see [5, 7]).

Let f be subadditive increasing and continuous real function defined in a certain right-hand side neighbourhood of 0 such that $f(0)=0$. By $\mathbf{F}_{f,\rho}$ we shall denote the class of all functions l fulfilling the conditions:

1^0 $\quad l: E_0\times E_0 \longmapsto [0,\infty)$,

2^0 $f(\rho(A,B)) \leq l(A,B) \leq f(d_\rho(A\cup B))$ for $A,B\in E_0$,
where $\rho(A,B)$ is the distance of sets A,B and $d_\rho(A\cup B)$ is the diameter of the union of sets A,B in the metric space (E,ρ).

Because

$$f(\rho(x,y)) = f(\rho(\{x\},\{y\})) \leq l(\{x\},\{y\}) \leq f(d_\rho(\{x\}\cup\{y\})) = f(\rho(x,y)),$$

then from the above and from (1.1) it follows that

$$l_0(x,y) = f(\rho(x,y)) \quad \text{for} \quad l\in \mathbf{F}_{f,\rho} \quad \text{and} \quad x,y\in E. \tag{2.3}$$

It is easy to check that the function l_0 defined by the formula (2.3) is the metric of the set E.

We say that the set $A\in E_0$ has the Darboux property at the point p of the metric space (E,l_0), which we write: $A\in D_p(E,l_0)$, if there exists a number $\sigma>0$ such that the set $A\cap S_{l_0}(p,r)$ is non-empty for $r\in(0,\sigma)$.

In the monographic paper [4] I proved, among others, the following theorems concerning the compatibility and homogeneity of the tangency relations of sets:

Theorem 2.1 *If the functions a,b fulfil the condition*

$$\frac{a(r)}{r^k}\xrightarrow[r\to 0^+]{} 0 \quad \text{and} \quad \frac{b(r)}{r^k}\xrightarrow[r\to 0^+]{} 0, \tag{2.4}$$

then for arbitrary functions $l_1,l_2\in\mathbf{F}_{f,\rho}$ the tangency relations $T_{l_1}(a,b,k,p)$ and $T_{l_2}(a,b,k,p)$ are compatible in the classes of sets $\widetilde{M}_{p,k}\cap D_p(E,l_0)$.

Theorem 2.2 *If the functions a_i,b_i $(i=1,2)$ fulfil the condition*

$$\frac{a_i(r)}{r^k}\xrightarrow[r\to 0^+]{} 0 \quad \text{and} \quad \frac{b_i(r)}{r^k}\xrightarrow[r\to 0^+]{} 0, \tag{2.5}$$

then for any function $l\in\mathbf{F}_{f,\rho}$ the tangency relations $T_l(a_1,b_1,k,p)$ and $T_l(a_2,b_2,k,p)$ are compatible in the classes of sets $\widetilde{M}_{p,k}\cap D_p(E,l_0)$.

Theorem 2.3 *If the non-decreasing functions a,b fulfil the condition* (2.4), *then for arbitrary sets of the classes $\widetilde{M}_{p,k}\cap D_p(E,l_0)$ the tangency relation $T_l(a,b,k,p)$ is homogeneous of the order* 0 *in the class of functions $\mathbf{F}_{f,\rho}$.*

The mentioned above W. Waliszewski's definition of the tangency of sets and the theorems proved in the paper [4] may have the essential meaning for further investigations connected with the tangency of sets in metric spaces. Some connections and applications of the above I show below.

A. From the Theorem 2.3 and Theorem 2.1 on the homogeneity and compatibility of the tangency relations of sets and from the definition of the class of the functions $\mathbf{F}_{f,\rho}$ $(\mathbf{F}_{mf,\rho})$ the corollaries follow:

Corollary 2.1 *For arbitrary functions l_1,l_2 such that $l_1\in\mathbf{F}_{f,\rho}$, $l_2\in\mathbf{F}_{mf,\rho}$ the tangency relations $T_{l_1}(a,b,k,p)$, $T_{l_2}(a,b,k,p)$ are compatible in the classes of sets $\widetilde{M}_{p,k}\cap D_p(E,l_0)$, if the non-decreasing functions a,b fulfil the condition* (2.4).

Corollary 2.2 *If the non-decreasing functions a, b fulfil the condition* (2.4)*, then $(A,B) \in T_{ml_1}(a,b,k,p)$ iff $(A,B) \in T_{Ml_2}(a,b,k,p)$ for arbitrary sets $A, B \in \widetilde{M}_{p,k} \cap D_p(E,l_0)$, the functions $l_1, l_2 \in \mathbf{F}_{f,\rho}$ and numbers $m, M > 0$.*

From the above corollaries it follows that the Theorem 2.3 on the homogeneity of the tangency relations of sets may have essential meaning for the investigation of the tangency of sets in the generalized metric spaces (E,l_1), (E,l_2), in which the functions l_1 and l_2 do not generate the same metric on the set E.

This theorem is a certain criterion, which allows to compare the tangency of sets of some classes in two different (although not completely arbitrary) metric spaces.

The problem of the compatibility of the tangency relations of sets for the functions belonging to the class F_ρ and generating different metrics was considered in some of my earlier papers. At that time it was assumed that any real function $l \in F_\rho$ defined on the Cartesian product $E_0 \times E_0$ of the family E_0 of all non-empty subsets of the set E generates on the set E a metric and fulfils the inequality

$$m\,\rho(A,B) \le l(A,B) \le M\,d_\rho(A \cup B) \quad \text{for} \quad A,B \in E_0, \tag{2.6}$$

where m, M are numbers such that $0 < m \le M < \infty$.

The results concerning this problem were obtained there by putting enough strong restriction on the functions $l_1, l_2 \in F_\rho$. Namely, it was assumed that these functions fulfil in any set $A \in E_0$ of the considered class of sets, the so-called condition of the proximity of the spheres of order $k > 0$ at the point $p \in E$ with regard to the metric ρ:

$$\frac{1}{r^k}\,\rho(A \cap S_{l_1}(p,r),\, A \cap S_{l_2}(p,r)) \xrightarrow[r \to 0^+]{} 0. \tag{2.7}$$

Let f_1, f_2 be functions fulfilling the same assumptions just as the function f. Hence and from the Corollary 2.1 it follows that, if the functions f_1, f_2 fulfil the equality

$$f_2 = m f_1 \quad \text{for} \quad m > 0, \tag{2.8}$$

then for $l_1 \in \mathbf{F}_{f_1,\rho}$ and $l_2 \in \mathbf{F}_{f_2,\rho}$ the tangency relations $T_{l_1}(a,b,k,p)$, $T_{l_2}(a,b,k,p)$ are compatible in the classes of sets $\widetilde{M}_{p,k} \cap D_p(E,l_0)$, when the non-decreasing functions a, b fulfil the condition (2.4).

In connection with this the following question arises: possibly with what other assumptions relating to the functions f_1, f_2 are the tangency relations $T_{l_1}(a,b,k,p)$ and $T_{l_2}(a,b,k,p)$ compatible in the classes of sets $\widetilde{M}_{p,k} \cap D_p(E,l_0)$?

I believe that, similarly as in case of the class of the functions F_ρ, we may get certain results concerning the compatibility of the tangency of sets of the classes $\widetilde{M}_{p,k} \cap D_p(E,l_0)$, if we put on the functions f_1 and f_2 the condition

$$\frac{1}{r^k} f_i(\rho(A \cap S_{l_{1,0}}(p,r), A \cap S_{l_{2,0}}(p,r))) \xrightarrow[r \to 0^+]{} 0 \tag{2.9}$$

for $i = 1, 2$, $A \in \widetilde{M}_{p,k} \cap D_p(E,l_0)$, where $l_{1,0}$ and $l_{2,0}$ are the metrics of the set E defined by the formulas:

$$l_{1,0}(x,y) = f_1(\rho(x,y)) \quad \text{and} \quad l_{2,0}(x,y) = f_2(\rho(x,y)) \quad \text{for} \quad x,y \in E. \tag{2.10}$$

In connection with the above the next question arises: what is the connection between the conditions (2.8) and (2.9)? The following may be true:

In the classes $\widetilde{M}_{p,k}$ of sets having the Darboux property in the metric spaces $(E, l_{1,0})$ and $(E, l_{2,0})$ the conditions (2.8) and (2.9) are equivalent.

The above problems may have essential meaning for the solution of the problem of compatibilty (equivalence) of the tangeny relations $T_{l_1}(a_1, b_1, k, p)$ and $T_{l_2}(a_2, b_2, k, p)$ of sets for the functions l generating different metrics on the set E and fulfilling the condition

$$f_1(\rho(A,B)) \le l(A,B) \le f_2(d_\rho(A \cup B)) \quad \text{for} \quad A, B \in E_0, \tag{2.11}$$

where f_1, f_2 are subadditive increasing and continuous real functions defined in a certain right-hand side neighbourhood of 0 such that $f_1(0) = f_2(0) = 0$.

B. Let *id* be the identity function defined in a certain right-hand side neighbourhood of the point 0. It is easy to notice that this function fulfils all assumptions concerning the function f. Let us suppose that the function l in particular belongs to the class $\mathbf{F}_{id,\rho}$.

Let $A, B \in E_0$ be arbitrary regular arcs tangent at the point p of the generalized metric space (E, l) in sense of the W. Waliszewski's definition. Then for $k = 1$

$$\frac{1}{r} l(A \cap S_\rho(p,r)_{a(r)}, B \cap S_\rho(p,r)_{b(r)}) \xrightarrow[r \to 0^+]{} 0.$$

Hence and from the fact that $l \in \mathbf{F}_{id,\rho}$ we have

$$\frac{1}{r} \rho(A \cap S_\rho(p,r)_{a(r)}, B \cap S_\rho(p,r)_{b(r)}) \xrightarrow[r \to 0^+]{} 0. \tag{2.12}$$

From the Theorem 2.2 on the compatibility of the tangency relations of sets it follows that for

$$\frac{a(r)}{r} \xrightarrow[r \to 0^+]{} 0 \quad \text{and} \quad \frac{b(r)}{r} \xrightarrow[r \to 0^+]{} 0,$$

the condition (2.12) can be written in the equivalent form

$$\frac{1}{r} \rho(A \cap S_\rho(p,r), B \cap S_\rho(p,r)) \xrightarrow[r \to 0^+]{} 0. \tag{2.13}$$

Let $x \in A \cap S_\rho(p,r)$, $y \in B \cap S_\rho(p,r)$. From this and from (2.13) we get

$$\frac{1}{r} \rho(x,y) \xrightarrow[r \to 0^+]{} 0.$$

From the above condition it follows that

$$\frac{2r^2 - \rho^2(x,y)}{2r^2} \xrightarrow[r \to 0^+]{} 1,$$

that is

$$\frac{\rho^2(p,x) + \rho^2(p,y) - \rho^2(x,y)}{2\rho(p,x)\rho(p,y)} \xrightarrow[A \times B \ni (x,y) \to (p,p)]{} 1. \tag{2.14}$$

The condition (2.14) presents the next well-known definition of the tangency of simple arcs in the metric space (E, ρ). It means that (see [1, 8]), if the condition (2.14) is fulfilled, then the simple arc $A \in E_0$ is tangent to the simple arc $B \in E_0$ at the point p of the metric space (E, ρ).

From the above considerations it follows that the W. Waliszewski's definition essentially generalizes the definition (2.14) of the tangency of simple arcs in the metric space (E, ρ).

The left side of the formula (2.14), by $\rho(p, x) \to 0$ and $\rho(p, y) \to 0$, is equal to $\cos\alpha$, where $\alpha \in [0, \pi]$ is the so-called the Alexandrov's angle between the arcs $A, B \in E_0$ in the metric space (E, ρ) (see [1]).

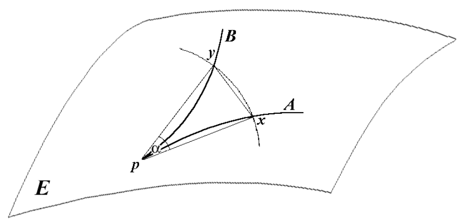

Figure 1

From this it follows that the W. Waliszewski's definition of the tangency of sets for the regular arcs in the metric space (E, ρ) is strictly related to cosine of the angle of Alexandrov between these arcs (see Figure 1).

Let us assume now that E is a differential Riemannian manifold with given the symmetric tensor field g of the valence $(0, 2)$. Using the metric tensor we may define in manifold E, among others, the following notions : length of a tangent vector, length of an arc and distance of points of this manifold.

By ρ we denote the metric of the manifold E generated by its metric tensor. Let A, B be regular arcs defined respectively by the vector equations: $r = r_1(t)$, $r = r_2(t)$ for $t \in [0, 1]$ and let $p = r_1(0) = r_2(0)$.

The angle between these arcs, the so-called the Riemannian angle, is defined as an angle $\gamma \in [0, \pi]$ between vectors tangent to these arcs at the point p, as follows:

$$\cos\gamma = \frac{(\acute{r}_1(0) \mid \acute{r}_2(0))}{|\acute{r}_1(0)|\,|\acute{r}_2(0)|}\ , \tag{2.15}$$

where $(\acute{r}_1(0) \mid \acute{r}_2(0))$ denotes the scalar product of the vectors $\acute{r}_1(t)$, $\acute{r}_2(t)$ at the point $t = 0$.

Two regular arcs $A, B \in E_0$ are tangent at the point $p \in E$ corresponding to the parameter $t = 0$, if they have at this point equal tangent vectors or ones differing only in the positive factor, i.e. $\acute{r}_1(0) = \lambda\,\acute{r}_2(0)$ for $\lambda > 0$.

Hence and from (2.15) it follows that the regular arcs $A, B \in E_0$ are tangent at the point $p \in E$, if $\cos\gamma = 1$, where α denotes the Riemannian angle between these arcs.

M.R. Bridson and A. Haefliger in the book "*Metric spaces of non-positive curvature*" (see [2]) proved that the Riemannian angle between the regular arcs (geodesics) in a smooth Riemann's manifold is equal to the Alexandrov's angle between them (see Figure 2).

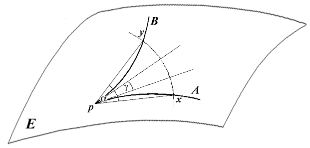

Figure 2

From this it follows (by the connection between the W. Waliszewski's definition of the tangency of sets and the Alexandrov's and Riemann's angles) that for investigation of the tangency of regular arcs in Riemannian manifolds we can use the W. Waliszewski's definition.

Furthermore, this definition can be use to examine the tangency of an arbitrary order $k \geq 1$ of regular arcs in Riemannian manifolds.

Moreover, it is worth emphasizing that the W. Waliszewski's definition of the tangency of sets generalizes the known earlier definitions of the tangency of regular arcs to the sets, which do not have a parametric structure.

Because the class $\widetilde{M}_{p,1}$ contains the class of regular arcs, then from the above considerations it follows that the results obtained by me in the paper [4] concerning the tangency of sets can be used in investigations of the tangency (of any order) of regular arcs in Riemannian manifolds.

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