Multi-time sprays and h-traceless maps on $J^1(T,M)$

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Abstract

It is known that the jet fibre bundle of order one $J^1(T,M)$ is a basic object in the study of classical and quantum field theories. In order to develop the subsequent multi-time Lagrangian theory of physical fields on $J^1(T,M)$, we need to generalize the main geometrical objects used in the classical rheonomic Lagrangian theory. In this direction, Section 1 presents the main properties of the differentiable structure of the jet fibre bundle of order one $J^1(T,M)$. Section 2 studies an important collection of geometrical objects on $J^1(T,M)$ as d-tensors, temporal and spatial sprays and h-traceless maps induced by these sprays, which naturally generalize analogous objects on the natural house $\mathbb{R}\times TM$ of the multi-time Lagrangian field theory. Section 3 studies the nonlinear connections Γ on $J^1(T,M)$, and discusses their relation with the temporal and spatial sprays. Section 4 opens the problem of prolongation of vector fields from $T\times M$ to 1-jet space $J^1(T,M)$, using adapted bases.

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1 The jet fibre bundle $J^1(T,M)$

Let us consider the smooth manifolds T and M of dimension p, respectively n, coordinated by $(t^\alpha)_{\alpha=\overline{1,p}}$, respectively $(x^i)_{i=\overline{1,n}}$. We remark that, throughout this paper, the set $\{1,2,..,p\}$ is indexed by $\alpha,\beta,\gamma,...$, and the set $\{1,2,...,n\}$ is indexed by $i,j,k,....$

Now, let (t_0,x_0) be an arbitrary point of the product manifold $T\times M$. We denote $C^\infty(T,M)$ the set of all smooth maps between T and M and define the equivalence relation

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$$f \sim_{(t_0,x_0)} g \Leftrightarrow f(t_0) = g(t_0) = x_0,\ df_{t_0} = dg_{t_0} \tag{1.1.1}$$

on $C^\infty(T,M)$. For every $f,g \in C^\infty(T,M)$, the relation $f \backsim_{(t_0,x_0)} g$ can be expressed locally by

$$\begin{aligned} & x^i\left(t_0^\beta\right) = y^i\left(t_0^\beta\right) = x_0^i, \\ & \frac{\partial x^i}{\partial t^\alpha}\left(t_0^\beta\right) = \frac{\partial y^i}{\partial t^\alpha}\left(t_0^\beta\right), \end{aligned} \tag{1.1.2}$$

where $t^\beta(t_0) = t_0^\beta$, $x^i(x_0) = x_0^i$, $x^i = x^i \circ f$ and $y^i = x^i \circ g$. The equivalence class of a smooth map $f \in C^\infty(T,M)$ is denoted by

$$[f]_{(t_0,x_0)} = \left\{g \in C^\infty(T,M) \mid g \backsim_{(t_0,x_0)} f\right\}.$$

If the quotient $J^1_{(t_0,x_0)}(T,M) = C^\infty(T,M) \diagup \sim_{(t_0,x_0)}$ is the factorization by the equivalence relation " $\sim_{(t_0,x_0)}$ ", we build the total spaces of 1-jet set, taking

$$J^1(T,M) = \bigcup_{(t_0,x_0)\in T\times M} J^1_{(t_0,x_0)}(T,M). \tag{1.1.3}$$

in order to organize the total space of 1-jets $J^1(T,M)$ as a vector bundle over the base space $T \times M$, we start with a smooth map $f \in C^\infty(T,M), x = f(t)$, $\left(t^1,...,t^p\right) \longrightarrow \left(x^1\left(t^1,...,t^p\right),...,x^n\left(t^1,...,t^p\right)\right)$, and expand the maps x^i using Taylor formula around the point $\left(t_0^1,...,t_0^p\right) \in \mathbb{R}^p$. We obtain

$$x^i\left(t^1,...,t^p\right) = x_0^i + \left(t^\alpha - t_0^\alpha\right)\frac{\partial x^i}{\partial t^\alpha}\left(t_0^1,...,t_0^p\right) + \mathcal{O}(2),\ \ \|t - t_0\| < \varepsilon.$$

Automatically the linear affine approximation $\widetilde{f} \in C^\infty(T,M), \widetilde{x} = \widetilde{f}(t)$,

$$\tilde{x}^i\left(t^1,...,t^p\right) = x_0^i + \left(t^\alpha - t_0^\alpha\right)\frac{\partial x^i}{\partial t^\alpha}\left(t_0^1,...,t_0^p\right),\ \ \|t - t_0\| < \varepsilon,$$

satisfies $\tilde{f} \sim_{(t_0,x_0)} f$, that is, it is a convenient representative of equivalence class $[f]_{(t_0,x_0)}$.

Let $\pi : J^1(T,M) \to T \times M$ be the projection $\pi\left([f]_{(t_0,x_0)}\right) = (t_0, f(t_0))$. It is obvious that the map π is well defined and surjective. Using this projection, for every local chart $U\times V \subset T\times M$ on the product manifold $T\times M$, we can define the bijection

$$\Phi_{U\times V} : \pi^{-1}(U\times V) \to U \times V \times \mathbb{R}^{np},$$

seting $\Phi_{U\times V}\left([f]_{(t_0,x_0)}\right)=\left(t_0,x_0,\dfrac{\partial x^i}{\partial t^\alpha}\left(t_0^\beta\right)\right),\ x_0=f\left(t_0\right).$

In conclusion, the 1-jet set $J^1\left(T,M\right)$ can be endowed with a differentiable structure of dimension $p+n+pn$, such that the maps $\Phi_{U\times V}$ to be diffeomorphisms. We emphasize that the local coordinates on $J^1\left(T,M\right)$ are $\left(t^\alpha,x^i,x^i_\alpha\right)$, where

$$
\begin{aligned}
&t^\alpha\left([f]_{(t_0,x_0)}\right)=t^\alpha\left(t_0\right),\\
&x^i\left([f]_{(t_0,x_0)}\right)=x^i\left(x_0\right),\\
&x^i_\alpha\left([f]_{(t_0,x_0)}\right)=\frac{\partial x^i}{\partial t^\alpha}\left(t_0^\beta\right).
\end{aligned}
\tag{1.1.4}
$$

In the above coordinates on $J^1\left(T,M\right)$, the projection $\pi:J^1\left(T,M\right)\to T\times M$ has the local expression $\pi\left(t^\alpha,x^i,x^i_\alpha\right)=\left(t^\alpha,x^i\right)$. Moreover, the differential π_* of the map π is locally determined by the Jacobi matrix

$$
\begin{pmatrix}\delta_{\alpha\beta} & 0 & 0\\ 0 & \delta_{ij} & 0\end{pmatrix}\in M_{p+n,p+n+pn}.
$$

It follows that π_* is a surjection ($rank\ \pi_*=p+n$) and therefore the projection π is a submersion. Consequently, the 1-jet total space $J^1\left(T,M\right)$ becomes a vector bundle over the base space $T\times M$, having the fibre type $\mathbb{R}^{pn}$.

Using (1.1.4), by a simple direct calculation, we obtain

Proposition 1.1. 1) The coordinate transformations $\left(t^\alpha,x^i,x^i_0\right)\longleftrightarrow\left(\tilde{t}^\alpha,\tilde{x}^i,\tilde{x}^i_\alpha\right)$ of the 1-jet vector bundle $E=J^1\left(T,M\right)$ are given by

$$
\begin{aligned}
&\tilde{t}^\alpha=\tilde{t}^\alpha\left(t^\beta\right),\\
&\tilde{x}^i=\tilde{x}^i\left(x^j\right),\\
&\tilde{x}^i_\alpha=\frac{\partial\tilde{x}^i}{\partial x^j}\frac{\partial t^\beta}{\partial\tilde{t}^\alpha}x^j_\beta,
\end{aligned}
\tag{1.1.5}
$$

where $\det\left(\partial\tilde{t}^\alpha/\partial t^\beta\right)\neq 0$ and $\det\left(\partial\tilde{x}^i/\partial x^j\right)\neq 0$. Consequently E is always an orientable manifold.

2) The canonical bases $\left\{\dfrac{\partial}{\partial t^\alpha},\dfrac{\partial}{\partial x^i},\dfrac{\partial}{\partial x^i_\alpha}\right\},\left\{\dfrac{\partial}{\partial\tilde{t}^\alpha},\dfrac{\partial}{\partial\tilde{x}^i},\dfrac{\partial}{\partial\tilde{x}^i_\alpha}\right\}$ *of the vector fields on E are related by*

$$\text{(1.1.6)}\qquad \begin{cases} \dfrac{\partial}{\partial t^\alpha} = \dfrac{\partial \tilde{t}^\beta}{\partial t^\alpha}\dfrac{\partial}{\partial \tilde{t}^\beta} + \dfrac{\partial \tilde{x}^j_\beta}{\partial t^\alpha}\dfrac{\partial}{\partial \tilde{x}^j_\beta} \\[2ex] \dfrac{\partial}{\partial x^i} = \dfrac{\partial \tilde{x}^j}{\partial x^i}\dfrac{\partial}{\partial \tilde{x}^j} + \dfrac{\partial \tilde{x}^j_\beta}{\partial x^i}\dfrac{\partial}{\partial \tilde{x}^j_\beta} \\[2ex] \dfrac{\partial}{\partial x^i_\alpha} = \dfrac{\partial \tilde{x}^j}{\partial x^i}\dfrac{\partial t^\alpha}{\partial \tilde{t}^\beta}\dfrac{\partial}{\partial \tilde{x}^j_\beta}, \end{cases}$$

3) The canonical bases $\left\{dt^\alpha, dx^i, dx^i_\alpha\right\}$, $\left\{d\tilde{t}^\alpha, d\tilde{x}^i, d\tilde{x}^i_\alpha\right\}$ of the 1-forms on E are related by

$$\text{(1.1.7)}\qquad \begin{cases} dt^\alpha = \dfrac{\partial t^\alpha}{\partial \tilde{t}^\beta} d\tilde{t}^\beta \\[2ex] dx^i = \dfrac{\partial x^i}{\partial \tilde{x}^j} d\tilde{x}^j \\[2ex] dx^i_\alpha = \dfrac{\partial x^i_\alpha}{\partial \tilde{t}^\beta} d\tilde{t}^\beta + \dfrac{\partial x^i_\alpha}{\partial \tilde{x}^j} d\tilde{x}^j + \dfrac{\partial x^i}{\partial \tilde{x}^j}\dfrac{\partial \tilde{t}^\beta}{\partial t^\alpha} d\tilde{x}^j_\beta. \end{cases}$$

Some physical aspects. At the end of this Section, we discuss certain physical aspects of the jet vector bundle of order one that we consider very eloquent for the subsequent theory.

Thus, from physical point of view, we regard the space T as a *"temporal"* manifold or a *"multi-time"* while the manifold M is regarded as a *"spatial"* one. In mechanics terms, the vector bundle $J^1(T,M)$ is regarded as a *bundle of configurations*, and its elements $[f]$ are regarded as classes of "*parametrized sheets*".

In order to motivate the terminology used, we study more deeply the jet vector bundle of order one, in the particular case $T = \mathbb{R}$ (i.e., the usual time axis represented by the set of real numbers). Let us suppose that $J^1(\mathbb{R},M) \equiv \mathbb{R}\times TM$ is coordinated by $\left(t, x^i, y^i\right)$. The gauge group of the bundle

$$\text{(1.1.8)}\qquad \pi : J^1(\mathbb{R},M) \to \mathbb{R}\times M, \left(t,x^i,y^i\right) \to \left(t,x^i\right),$$

is given by

$$\text{(1.1.9)}\qquad \begin{aligned} &\tilde{t} = \tilde{t}(t), \\ &\tilde{x}^i = \tilde{x}^i\left(x^j\right), \\ &\tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j}\frac{dt}{d\tilde{t}} y^j. \end{aligned}$$

We remark that the form of this gauge group stands out by the *relativistic* character of the time t. For that reason, we consider that the jet fibre bundle of order one $J^1(R,M)$ is the natural bundle of configurations of the *relativistic rheonomic Lagrangian mechanics* [10].

Comparatively, in the *classical rheonomic Lagrangian mechanics* [5], the bundle of configuration is the fibre bundle

$$\pi : \mathbb{R} \times TM \to M, \left(t, x^i, y^i\right) \to \left(x^i\right), \tag{1.1.10}$$

whose geometrical invariance group is

$$\begin{aligned} \tilde{t} &= t, \\ \tilde{x}^i &= \tilde{x}^i\left(x^j\right), \\ \tilde{y}^i &= \frac{\partial \tilde{x}^i}{\partial x^j} y^j. \end{aligned} \tag{1.1.11}$$

Obviously, the structure of the gauge group (1.1.11) emphasizes the *absolute* character of the time t from the classical rheonomic Lagrangian mechanics. At the same time, we point out that the gauge group (1.1.11) is a subgroup of (1.1.9). In other words, the gauge group of the jet bundle of order one, from the relativistic rheonomic Lagrangian mechanics is more general than that used in the classical rheonomic Lagrangian mechanics which ignores the temporal reparametrizations.

Finally, we invite the reader to compare both the classical and relativistic rheonomic Lagrangian mechanics developed in [5] and [10].

2 d-Tensors. Multi-time sprays. h-traceless maps

It is well known the importance of the tensors in development of a geometry on a fibre bundle. In the study of the 1-jet fibre bundle, a central role is played by the *distinguished tensors* or, briefly, *d-tensors.*

Definition 2.1. *A geometrical object $D = \left(D^{\alpha i(j)(\nu)\ldots}_{\gamma k(\beta)(t)\ldots}\right)$, on the 1-jet vector bundle E, whose local components verify the following rules of transformation*

$$D^{\alpha i(j)(\nu)\ldots}_{\gamma k(\beta)(l)\ldots} = \tilde{D}^{\delta p(m)(\eta)\ldots}_{\varepsilon r(\mu)(s)\ldots} \frac{\partial t^\alpha}{\partial \tilde{t}^\delta} \frac{\partial x^i}{\partial \tilde{x}^p} \frac{\partial x^j}{\partial \tilde{x}^m} \frac{\partial \tilde{t}^\mu}{\partial t^\beta} \frac{\partial \tilde{t}^\epsilon}{\partial t^\gamma} \frac{\partial \tilde{x}^r}{\partial x^k} \frac{\partial \tilde{x}^s}{\partial x^l} \frac{\partial t^\nu}{\partial \tilde{t}^\eta} \ldots, \tag{2.2.1}$$

is called a **d-tensor** field.

The utilization of parentheses for certain indices of the local components $D^{\alpha i(j)(\nu)\ldots}_{\gamma k(\beta)(l)\ldots}$ will be motivated at the end of Section 3 of this paper, before the introduction of a nonlinear connection Γ on E together with its adapted bases of vector and covector fields.

A d-tensor field D on $E = J^1(T, M)$ can be viewed like an object defined on $T \times M$ which depends on **partial derivatives** (**partial velocities**) x^i_α.

Example 2.2. *i) If $L : E \to R$ is a multi-time Lagrangian function with partial derivates of order one, then the local components*

$$G^{(\alpha)(\beta)}_{(i)(j)} = \frac{1}{2} \frac{\partial^2 L}{\partial x^i_\alpha \partial x^j_\beta} \tag{2.2.2}$$

represent a d-tensor field on E. We point out that taking $T = \mathbb{R}$ and L a regular time-dependent Lagrangian, then the d-tensor field $G^{(\alpha)(\beta)}_{(i)(j)}\left(t, x^i, y^j\right)$ is a natural generalization of that so-called metrical d-tensor field $g_{ij}\left(t, x^i, y^j\right)$ of a classical rheonomic Lagrange space $RL^n = \left(M, L\left(t, x^i, y^j\right)\right)$ [5].

ii) The geometrical object $\mathbf{C} = \left(C^{(i)}_{(\alpha)}\right)$, where $C^{(i)}_{(\alpha)} = x^i_\alpha$, represent a d-tensor field on E. This is called the **canonical Liouville d-tensor** *on the 1-jet vector bundle E. We emphasize that this d-tensor field naturally generalizes the Liouville d-vector field $C = y^i \dfrac{\partial}{\partial y^i}$ used in [5].*

iii) Let $h_{\alpha\beta}$ be a semi-Riemannian metric on the temporal manifold T, i.e., it has the signature $(p_1, p_2), p_1 + p_2 = p$. The geometrical object $L = \left(L^{(i)}_{(\alpha)\beta\gamma}\right)$, where $L^{(i)}_{(\alpha)\beta\gamma} = h_{\beta\gamma} x^i_\alpha$, is a d-tensor field which is called the **Liouville d-tensor associated to the metric** *h.*

iv) Using the preceding metric h, we construct the d-tensor $J = \left(J^{(i)}_{(\alpha)\beta j}\right)$, where $J^{(i)}_{(\alpha)\beta j} = h_{\alpha\beta}\delta^i_j$. This d-tensor is called the **h-normalization d-tensor** *of the jet bundle E. Note that the h-normalization d-tensor of $J^1(T, M)$ is a natural generalization of the tangent structure J from the Lagrange geometry [5].*

It is obvious that any d-tensor on E is a tensor on E. The converse is not true. As examples, we will build two tensors wich are not d-tensors. We refer to notions of temporal and spatial sprays wich allow the generalization of the notion of time-dependent spray used in [5], [14].

Definition 2.3. *A global tensor H, expressed locally by*

$$H = \delta^\beta_\alpha dt^\alpha \otimes \frac{\partial}{\partial t^\beta} - 2H^{(j)}_{(\beta)\alpha} dt^\alpha \otimes \frac{\partial}{\partial x^j_\beta}, \tag{2.2.3}$$

is called a **temporal spray** *on E.*

Taking into account that a temporal spray is a global tensor on E, by a direct calculation, we deduce

Proposition 2.4. *i)* The components $H^{(j)}_{(\beta)\alpha}$ of the temporal spray H transform by the rules

$$2\widetilde{H}^{(k)}_{(\mu)\gamma} = 2H^{(j)}_{(\beta)\alpha}\frac{\partial t^\alpha}{\partial \tilde{t}^\gamma}\frac{\partial \tilde{x}^k}{\partial x^j}\frac{\partial t^\beta}{\partial \tilde{t}^\mu} - \frac{\partial t^\alpha}{\partial \tilde{t}^\gamma}\frac{\partial \tilde{x}^k_\mu}{\partial t^\alpha}. \tag{2.2.4}$$

ii) Conversely, to give a temporal spray on E is equivalent to give a set of local functions $H = \left(H^{(j)}_{(\beta)\alpha}\right)$ which transform by the formulas *(2.2.4).*

iii) The global tensor

$$H = H^\beta_\alpha dt^\alpha \otimes \frac{\partial}{\partial t^\beta} - 2H^{(j)}_{(\beta)\alpha}dt^\alpha \otimes \frac{\partial}{\partial x^j_\beta}$$

is a temporal spray iff $J^{(j)}_{(\beta)\alpha i}H^\alpha_\gamma = J^{(j)}_{(\beta)\gamma i}$, where J is the normalization d-tensor of the fibre bundle E associated to an arbitrary semi-Riemannian metric h on T.

The previous proposition allows us to offer the following important example of temporal spray. The importance of this kind of temporal spray is determined by its using in the description of the classical harmonic maps between two Riemannian manifolds [3].

Example 2.5. *Using the transformation rules of the Christoffel symbols $H^\alpha_{\beta\gamma}$ attached to a semi-Riemannian metric $h_{\alpha\beta}$ on T, we deduce that the components $2H^{(j)}_{(\beta)\alpha} = -H^\gamma_{\alpha\beta}x^j_\gamma$ represent a temporal spray on E. This is called the* **canonical temporal spray associated to the metric** *h.*

Definition 2.6. *A global tensor G, locally defined by*

$$G = x^i_\alpha dt^\alpha \otimes \frac{\partial}{\partial x^i} - 2G^{(j)}_{(\beta)\alpha}dt^\alpha \otimes \frac{\partial}{\partial x^j_\beta}, \tag{2.2.5}$$

is called a **spatial spray** *on E.*

As in the case of the temporal spray, we can prove without difficulties the following statement.

Proposition 2.7. The components $G^{(j)}_{(\beta)\alpha}$ of the spatial spray G transform by the rules

$$2\tilde{G}^{(k)}_{(\mu)\gamma} = 2G^{(j)}_{(\beta)\alpha}\frac{\partial t^\alpha}{\partial \tilde{t}^\gamma}\frac{\partial \tilde{x}^k}{\partial x^j}\frac{\partial t^\beta}{\partial \tilde{t}^\mu} - \frac{\partial x^i}{\partial \tilde{x}^j}\frac{\partial \tilde{x}^k_\mu}{\partial x^i}\tilde{x}^j_\gamma. \tag{2.2.6}$$

ii) To give a spatial spray is equivalent to give a set of local functions $G = \left(G^{(j)}_{(\beta)\alpha}\right)$ which change by the law *(2.2.6).*

iii) A global tensor on E, defined locally by

$$G = G^i_\alpha dt^\alpha \otimes \frac{\partial}{\partial x^i} - 2G^{(j)}_{(\beta)\alpha} dt^\alpha \otimes \frac{\partial}{\partial x^j_\beta},$$

is a spatial spray iff $J^{(j)}_{(\beta)\alpha i} G^i_\gamma = L^{(j)}_{(\beta)\alpha\gamma}$, where J (resp. L) is the normalization (resp. Liouville) d-tensor associated to an arbitrary semi-Riemannian metric h.

Example 2.8. *If γ^i_{jk} are the Christoffel symbols of the semi-Riemannian metric φ_{ij} on the spatial manifold M, the local coefficients $2G^{(j)}_{(\beta)\alpha} = \gamma^j_{kl} x^k_\alpha x^l_\beta$ define a spatial spray which is called the* **canonical spatial spray associated to the metric** *φ. We point out that this kind of spatial spray is also used in the description of the classical harmonic maps between two Riemannian manifolds [3].*

Definition 2.9. *A pair (H,G) which consists of a temporal spray and a spatial one, is called a* **multi-time spray** *on E.*

To characterize the multi-time sprays on E and to underline again the importance of the canonical temporal and spatial sprays attached to the metrics h and φ, we prove the following theorem.

Theorem 2.10. Let (T,h), (M,φ) be semi-Riemannian manifolds and let $H = \left(H^{(i)}_{(\alpha)\beta}\right)$ *(resp.* $G = \left(G^{(i)}_{(\alpha)\beta}\right)$*)* be an arbitrary temporal (resp. spatial) spray on E. In these conditions, we have

$$\begin{aligned} H^{(i)}_{(\alpha)\beta} &= -\frac{1}{2} H^\gamma_{\alpha\beta} x^i_\gamma + D^{(i)}_{(\alpha)\beta}, \\ G^{(i)}_{(\alpha)\beta} &= \frac{1}{2} \gamma^i_{jk} x^j_\alpha x^k_\beta + F^{(i)}_{(\alpha)\beta}, \end{aligned} \tag{2.2.7}$$

where $D^{(i)}_{(\alpha)\beta}$, $F^{(i)}_{(\alpha)\beta}$ are certain d-tensors on E.

Proof. The theorem comes from the following true statements:

i) An affine combination of temporal (spatial) sprays is a temporal (spatial) spray;

ii) The product between a scalar and a temporal (spatial) spray is a temporal (spatial) spray;

iii) The difference between two temporal (spatial) sprays is a d-tensor. ■

In order to generalize the notion of path of a spray from Lagrangian geometry, we fix $h_{\alpha\beta}$ a semi-Riemannian metric on the temporal manifold T. In this context, we give the following definition.

Definition 2.11. *A geometrical object $H = \left(H^k\right)$ $\left(\text{resp. } G = \left(G^k\right)\right)$ is called a* temporal *(resp.* spatial*)* $h-$spray *if the local components modify by the rules*

$$2\tilde{H}^k = 2H^j \frac{\partial x^k}{\partial \tilde{x}^j} - \tilde{h}^{\gamma\mu} \frac{\partial t^\alpha}{\partial \tilde{t}^\gamma} \frac{\partial \tilde{x}^k_\mu}{\partial t^\alpha}, \tag{2.2.8}$$

respectively

$$2\tilde{G}^k = 2G^j \frac{\partial \tilde{x}^k}{\partial x^j} - \tilde{h}^{\gamma\mu} \frac{\partial x^i}{\partial \tilde{x}^j} \frac{\partial \tilde{x}^k_\mu}{\partial x^i} \tilde{x}^j_\gamma. \tag{2.2.9}$$

Example 2.12. *Starting with* $H = \left(H^{(i)}_{(\alpha)\beta}\right)$ *(resp.* $G = \left(G^{(i)}_{(\alpha)\beta}\right)$*) like a temporal (resp. spatial) spray, the entity* $H = \left(H^i\right)$ (resp. $G = \left(G^i\right)$*), where* $H^i = h^{\alpha\beta}H^{(i)}_{(\alpha)\beta}$ $\left(resp. G^i = h^{\alpha\beta}G^{(i)}_{(\alpha)\beta}\right)$, *represents a temporal (resp. spatial)* $h-$*spray which will be called the* $h-$**trace of the temporal** *(resp.* **spatial***)* **spray** H *(resp.* G*). Particularly, the components* $H^k = -h^{\alpha\beta}H^\gamma_{\alpha\beta}x^k_\gamma$ *(resp.* $G^k = h^{\alpha\beta}\gamma^k_{ij}x^i_\alpha x^j_\beta$*) represent the* **canonical temporal** *(resp.* spatial*)* $h-$**spray attached to the metric** h *(resp.* φ*).*

The previous example show that the $h-$trace of a temporal or a spatial spray represents a temporal or a spatial $h-$spray. Conversely, we prove the following result.

Theorem 2.13. *If* $\dim T = 1$, *any temporal (spatial)* $h-$*spray is the* $h-$*trace of a unique temporal (spatial) spray.*

Proof. Let $G = \left(G^k\right)$ be a spatial $h-$spray. We denote $G^{(k)}_{(1)1} = h_{11}G^k$. Obviously, the relation $G^k = h^{11}G^{(k)}_{(1)1}$ is true. In these conditions, using the transformation rules (2.2.9), we deduce

$$2\tilde{G}^{(k)}_{(1)1} = 2G^{(j)}_{(1)1}\left(\frac{d\tilde{t}}{dt}\right)^2 - \frac{dt}{d\tilde{t}}\frac{dy^k}{dt.}$$

This means that $G = \left(G^{(k)}_{(1)1}\right)$ is a spatial spray. The uniqueness is clear.

By analogy, we treat the case of the temporal $h-$sprays, taking $H^k = h^{11}H^{(k)}_{(1)1}$, where $H^{(k)}_{(1)1} = h_{11}H^k$. ∎

Remark 2.14. *The previous theorem shows that, in the case* $\dim T = 1$, *there is a 1-1 corespondence between sprays and* $h-$*sprays while, for* $\dim T > 2$, *this statement is not true.*

In the sequel, let us fixe a temporal spray $H = \left(H^{(i)}_{(\alpha)\beta}\right)$ and a spatial spray $G = \left(G^{(i)}_{(\alpha)\beta}\right)$ on E. The following notions show that the 1-jet fibre bundle is the natural house for important objects with geometrical and physical meaning.

Definition 2.15. *A solution* $f \in C^\infty(T, M)$ *of the PDEs system of order two*

$$x^i_{\alpha\beta} + G^{(i)}_{(\alpha)\beta} + G^{(i)}_{(\beta)\alpha} + H^{(i)}_{(\alpha)\beta} + H^{(i)}_{(\beta)\alpha} = 0 \tag{2.2.10}$$

where the map f *is locally expressed by* $(t^\alpha) \to \left(x^i(t^\alpha)\right)$ *and* $x^i_{\alpha\beta} = \dfrac{\partial^2 x^i}{\partial t^\alpha \partial t^\beta}$, *is called* **an affine map of the multi-time spray** (H, G).

A reason which offers the naturalness of the notion of an affine map of a multi-time spray on E, is that, in the particular case $T = \mathbb{R}$, the equations of the affine maps generalize the equations of the paths of a single-time spray from the rheonomic Lagrangian geometry [5].

Example 2.16. *Considering the canonical multi-time spray*

$$
\left\{
\begin{array}{l}
H^{(i)}_{(\alpha)\beta} = -\dfrac{1}{2} H^{\gamma}_{\alpha\beta} x^i_{\gamma} \\
G^{(i)}_{(\alpha)\beta} = \dfrac{1}{2} \gamma^i_{jk} x^j_{\alpha} x^k_{\beta},
\end{array}
\right.
\tag{2.2.11}
$$

the equations of the affine maps of this spray reduce to

$$
x^i_{\alpha\beta} - H^{\gamma}_{\alpha\beta} x^i_{\gamma} + \gamma^i_{jk} x^j_{\alpha} x^k_{\beta} = 0,
\tag{2.2.12}
$$

that is, the equations whose solutions are exactly the maps $f \in C^{\infty}(T,M)$ which carry the geodesics of $(T, h_{\alpha\beta})$ into the geodesics of the space (M, φ_{ij}).

Taking $h = (h_{\alpha\beta}(t))$ a temporal semi-Riemannian metric and doing a contraction by $h^{\alpha\beta}$ in (2.2.10) we can introduce the next concept.

Definition 2.17. *A map $f \in C^{\infty}(T,M)$ is called a* **h-traceless map of the multi-time spray** (H,G), **with respect to semi-Riemannian metric** h, *if f is a solution of the PDEs system of order two*

$$
h^{\alpha\beta} \left\{ x^i_{\alpha\beta} + 2G^{(i)}_{(\alpha)\beta} + 2H^{(i)}_{(\alpha)\beta} \right\} = 0.
\tag{2.2.13}
$$

Example 2.18. *In case of Riemannian manifolds $(T,h), (M,\varphi)$ and the canonical multi-time spray of preceding example, we recover the classical notion of harmonic map [3].*

It is obvious that the affine map of a multi-time spray (H,G) is a h-traceless map of the same spray with respect to any semi-Riemannian metric $h_{\alpha\beta}$ on the temporal space T.

In the particular case $(T,h) = (\mathbb{R}, \delta)$ the notions of h-traceless map and affine map identify. Consequently, both notions naturally generalize that so-called a path of a time-dependent spray, used in [5].

Let us denote $S^{(i)}_{(\alpha)\beta} = G^{(i)}_{(\alpha)\beta} + H^{(i)}_{(\alpha)\beta} + \dfrac{1}{2} H^i_{\alpha\beta} x^i_{\gamma}$ and $S^i = h^{\alpha\beta} S^{(i)}_{(\alpha)\beta}$. Obviously, we deduce that $S = (S^i)$ is a spatial $h-$spray. In this context, we obtain without difficulties the following result.

Theorem 2.19. *The equations of the h-traceless maps of the multi-time spray (G,H), with respect to the semi-Riemannian metric h, can be rewritten in the form*

$$
Œ_h x^i + 2S^i = 0,
\tag{2.2.14}
$$

where $Œ_h x^i = h^{\alpha\beta}\left(x^i_{\alpha\beta} - H^{\gamma}_{\alpha\beta} x^i_{\gamma}\right)$ is an ultrahyperbolic differential operator.

This theorem plays a central role in the development of the generalized metrical multi-time Lagrange theory of physical fields since the Euler-Lagrange equations of a multi-time dependent Lagrangian $\mathcal{L} = L\sqrt{|h|}$, where $L : J^1(T,M) \to \mathbb{R}$ is a

Kronecker h-regular Lagrange function, can be written in the form (2.14). In this sense, the PDEs equations (2.14) are Euler-Lagrange prolongations of the PDEs equations (2.10) . Hence, the extremals of $\mathcal{L}$ can be regarded as **ultra-harmonic maps**, offering them a profound geometrical and physical character. For more details, see also [7], [9].

In a furthercoming paper, we shall use the same theorem to offer a proper geometrical interpretaton of solutions of PDEs, in metrical multi-time Lagrangian geometry terms (see [11]). In this fashion, we will offer a final answer to the Udrişte-Neagu open problem [8], [17] , whose essential physical aspects are presented in [15], [16].

3 Nonlinear connections

The form of the coordinate transformations on $E = J^1(T, M)$ determines complicated rules of transformation of the local components of diverse geometrical objects of this space. To avoid such complications we introduce a *suitable nonlinear connection* which is coherent to *adapted bases*. These bases have the quality to simplify the transformation rules of the components of the geometrical objects taken in study.

With a view to doing this, we take $u \in E$ and consider the differential map $\pi_{*,u} : T_uE \to T_{(t,x)}(T \times M)$ of the canonical projection $\pi : E \to T \times M$, $\pi(u) = (t, x)$. At the same time, let us consider the vector subspace $V_u = Ker\pi_{*,u} \subset T_uE$. Because the map $\pi_{*,u}$ is a surjection, we have $\dim_{\mathbb{R}} V_u = pn$, $\forall u \in E$. Moreover, a basis in V_u is determined by $\left\{\dfrac{\partial}{\partial x^i_\alpha}\right\}$. In conclusion, the map

$$\mathcal{V} : u \in E \to V_u \in T_uE \tag{3.3.1}$$

is a differential distribution wich is called *the vertical distribution* of the 1-jet fibre bundle E.

Definition 3.1. *A* nonlinear connection *on E is a differential distribution*

$$\mathcal{H} : u \in E \to H_u \subset T_uE \tag{3.3.2}$$

which verifies the relation

$$T_uE = H_u \oplus V_u,\ \forall u \in E. \tag{3.3.3}$$

The distribution $\mathcal{H}$ is called the **horizontal distribution** *on E.*

Remark 3.2. *i) The above definition implies that $\dim_R H_u = p + n$, $\forall u \in E$.*

ii) The vector fields set $\mathcal{X}(E)$ can be decompose in the following direct sum $\mathcal{X}(E) = \Gamma(\mathcal{H}) \oplus \Gamma(\mathcal{V})$, where $\Gamma(\mathcal{H})$ (resp. $\Gamma(\mathcal{V})$) is the set of the sections on $\mathcal{H}$ (resp.$\mathcal{V}$).

Now, supposing that there is a nonlinear connection $\mathcal{H}$ on E, we have the isomorphism

$$\pi_{*,u}\mid_{H_u}: H_u \to T_{\pi(u)}\left(T\times M\right), \tag{3.3.4}$$

which allows us to prove the following

Theorem 3.3. *i) There exist the unique horizontal vector fields* $\dfrac{\delta}{\delta t^\alpha}, \dfrac{\delta}{\delta x^i} \in \Gamma\left(\mathcal{H}\right)$, *linearly independent, having the properties*

$$\pi_*\left(\frac{\delta}{\delta t^\alpha}\right) = \frac{\partial}{\partial t^\alpha},\ \pi_*\left(\frac{\delta}{\delta x^i}\right) = \frac{\partial}{\partial x^i}. \tag{3.3.5}$$

ii) The vector fields $\dfrac{\delta}{\delta t^\alpha}$ *and* $\dfrac{\delta}{\delta x^i}$ *can be uniquely written in the form*

$$\begin{cases} \dfrac{\delta}{\delta t^\alpha} = \dfrac{\partial}{\partial t^\alpha} - M^{(j)}_{(\beta)\alpha}\dfrac{\partial}{\partial x^j_\beta} \\ \dfrac{\delta}{\delta x^i} = \dfrac{\partial}{\partial x^i} - N^{(j)}_{(\beta)i}\dfrac{\partial}{\partial x^j_\beta}. \end{cases} \tag{3.3.6}$$

iii) The components $M^{(j)}_{(\beta)\alpha}$ *and* $N^{(j)}_{(\beta)i}$ *modify by the rules*

$$\begin{cases} \widetilde{M}^{(j)}_{(\beta)\mu}\dfrac{\partial \tilde{t}^\mu}{\partial t^\alpha} = M^{(k)}_{(\gamma)\alpha}\dfrac{\partial \tilde{x}^j}{\partial x^k}\dfrac{\partial t^\gamma}{\partial \tilde{t}^\beta} - \dfrac{\partial \tilde{x}^j_\beta}{\partial t^\alpha} \\ \widetilde{N}^{(j)}_{(\beta)k}\dfrac{\partial \tilde{x}^k}{\partial x^i} = N^{(k)}_{(\gamma)i}\dfrac{\partial \tilde{x}^j}{\partial x^k}\dfrac{\partial t^\gamma}{\partial \tilde{t}^\beta} - \dfrac{\partial \tilde{x}^j_\beta}{\partial x^i}. \end{cases} \tag{3.3.7}$$

iv) To give a nonlinear connection $\mathcal{H}$ *on* E *is equivalent to give a set of local functions* $\Gamma = \left(M^{(j)}_{(\beta)\alpha}, N^{(j)}_{(\beta)i}\right)$ *which transform by (3.3.7).*

Example 3.4. *Studying the transformation rules of the local components*

$$\begin{aligned} M^{(j)}_{(\beta)\alpha} &= -H^\gamma_{\alpha\beta}x^j_\gamma, \\ N^{(j)}_{(\beta)i} &= \gamma^j_{ik}x^k_\beta, \end{aligned} \tag{3.3.8}$$

we conclude that $\Gamma_0 = \left(M^{(j)}_{(\beta)\alpha}, N^{(j)}_{(\beta)i}\right)$ *represents a nonlinear connection on* E, *which is called the* **canonical nonlinear connection attached to the semi-Riemannian metrics** $h_{\alpha\beta}$ **and** φ_{ij}.

Let us consider the 1-forms $\delta x^i_\alpha = dx^i_\alpha + M^{(i)}_{(\alpha)\beta}dt^\beta + N^{(i)}_{(\alpha)j}dx^j$. One easily deduces that the set of 1-forms $\left\{dt^\alpha, dx^i, \delta x^i_\alpha\right\}$ is a basis in the set of 1-forms.

Definition 3.5. *The basis of vector fields* $\left\{\dfrac{\delta}{\delta t^\alpha}, \dfrac{\delta}{\delta x^i}, \dfrac{\partial}{\partial x^i_\alpha}\right\} \subset \mathcal{X}\left(E\right)$ *and its dual basis of 1-forms* $\left\{dt^\alpha, dx^i, \delta x^i_\alpha\right\} \subset \mathcal{X}^*\left(E\right)$ *are called the* **adapted bases** *on* E, *determined by the nonlinear connection* Γ.

The big advantage of the adapted bases is that the transformation laws of their elements are simple and natural ones.

Proposition 3.6. The transformation laws of the elements of the adapted bases attached to the nonlinear connection Γ are

$$
\begin{aligned}
\frac{\delta}{\delta t^\alpha} &= \frac{\partial \tilde{t}^\beta}{\partial t^\alpha}\frac{\delta}{\delta \tilde{t}^\beta},\\
\frac{\delta}{\delta x^i} &= \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\delta}{\delta \tilde{x}^j},\\
\frac{\partial}{\partial x^i_\alpha} &= \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial t^\alpha}{\partial \tilde{t}^\beta}\frac{\partial}{\partial \tilde{x}^j_\beta},
\end{aligned}
\tag{3.3.9}
$$

$$
\begin{aligned}
dt^\alpha &= \frac{\partial t^\alpha}{\partial \tilde{t}^\beta}d\tilde{t}^\beta,\\
dx^i &= \frac{\partial x^i}{\partial \tilde{x}^j}d\tilde{x}^j,\\
\delta x^i_\alpha &= \frac{\partial x^i}{\partial \tilde{x}^j}\frac{\partial \tilde{t}^\beta}{\partial t^\alpha}\delta \tilde{x}^j_\beta.
\end{aligned}
\tag{3.3.10}
$$

The simple transformation rules (3.9) and (3.10) determine us to describe the objects with geometrical and physical meaning from the subsequent generalized metrical multi-time Lagrange theory of physical fields [7], [9], in adapted components. In a such prospect, we emphasize that, using adpted bases of nonlinear connection Γ, a d-tensor $D = \left(D^{\alpha i(j)(\mu)\ldots}_{\gamma k(\beta)(l)\ldots}\right)$ on E can be regarded as a global geometrical object, locally defined by

$$
D = D^{\alpha i(j)(\mu)\ldots}_{\gamma k(\beta)(l)\ldots}\frac{\delta}{\delta t^\alpha}\otimes\frac{\delta}{\delta x^i}\otimes\frac{\partial}{\partial x^j_\beta}\otimes dt^\gamma\otimes dx^k\otimes \delta x^l_\mu\otimes \ldots. \tag{3.3.11}
$$

The utilization of certain indices between parenthesis in the description of the local components of d-tensor D is suitable for contractions. To illustrate this fact,we consider, for example, the local components of the metrical d-tensor (2.2). These define the geometrical object

$$
G = G^{(\alpha)(\beta)}_{(i)(j)}\delta x^i_\alpha\otimes\delta x^j_\beta \tag{3.3.12}
$$

On the other hand, considering the local components of the h-normalization d-tensor $J^{(i)}_{(\alpha)\beta j}$, we obtain the representative object

$$
J = J^{(i)}_{(\alpha)\beta j}\frac{\delta}{\delta x^i_\alpha}\otimes dt^\beta\otimes dx^j. \tag{3.3.13}
$$

Finally, let us study the relation between multi-time sprays and nonlinear connections. In this context, the components $M^{(j)}_{(\beta)\alpha}$ (resp. $N^{(j)}_{(\beta)i}$) of the nonlinear connection Γ are called the *temporal* (resp. *spatial*) *nonlinear connection.* In this terminology, using the transformation formulas (2.2.4), (2.2.6) and (3.3.7) we can easily prove the following statements.

Theorem 3.7. *i)* If $M^{(i)}_{(\alpha)\beta}$ are the components of a temporal nonlinear connection, then the components

$$H^{(i)}_{(\alpha)\beta} = \frac{1}{2} M^{(i)}_{(\alpha)\beta} \tag{3.3.14}$$

represent a temporal spray.

ii) Conversely, if $H^{(i)}_{(\alpha)\beta}$ are the components of a temporal spray, then

$$M^{(i)}_{(\alpha)\beta} = 2H^{(i)}_{(\alpha)\beta} \tag{3.3.15}$$

are the components of a temporal nonlinear connection.

Theorem 3.8. *i)* If $G^{(i)}_{(\alpha)\beta}$ are the components of a spatial spray and $G^i = h^{\alpha\beta} G^{(i)}_{(\alpha)\beta}$ represent the $h-$trace of this spray, then the components

$$N^{(i)}_{(\alpha)j} = \frac{\partial G^i}{\partial x^j_\gamma} h_{\gamma\alpha} \tag{3.3.16}$$

represent a spatial nonlinear connection.

ii) Conversely, the spatial nonlinear connection $N^{(i)}_{(\alpha)j}$ induces the spatial spray

$$2G^{(i)}_{(\alpha)\beta} = N^{(i)}_{(\alpha)j} x^j_\beta. \tag{3.3.17}$$

The previous theorems allow us to conclude that a multi-time spray (H, G) induces naturaly a nonlinear connection Γ on E, which is called the **canonical nonlinear connection associated to the multi-time dependent spray** (H, G). We point out that the canonical nonlinear connection Γ attached to the multi-time dependent spray (H, G) is a natural generalization of the canonical nonlinear connection N induced by a time-dependent spray G from the classical rheonomic Lagrangian geometry [5].

4 Jet prolongation of vector fields

A general vector field X^* on $J^1(T, M)$ can be written under the form

$$X^* = X^\alpha \frac{\partial}{\partial t^\alpha} + X^i \frac{\partial}{\partial x^i} + X^{(i)}_{(\alpha)} \frac{\partial}{\partial x^i_\alpha},$$

where the componets $X^{\alpha}, X^{i}, X^{(i)}_{(\alpha)}$ are functions of $\left(t^{\alpha}, x^{i}, x^{i}_{\alpha}\right)$.

The prolongation of a vector field X on $T\times M$ to a vector field on the 1-jet bundle $J^{1}(T,M)$ was solved by Olver [12] in the following sense.

Definition 4.1. *Let X be a vector field on $T\times M$ with corresponding (local) one-parameter group exp(εX). The* **1-th prolongation** *of X, denoted by $pr^{(1)}X$, will be a vector field on the 1-jet space $J^{1}(T,M)$, and is defined to be the infinitesimal generator of the corresponding prolonged one-parameter group $pr^{(1)}\left[\exp(\varepsilon X)\right]$, i.e.*

$$\left[pr^{(1)}X\right]\left(t^{\alpha}, x^{i}, x^{i}_{\alpha}\right) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} pr^{(1)}\left[\exp(\varepsilon X)\right]\left(t^{\alpha}, x^{i}, x^{i}_{\alpha}\right).$$

In order to write the components of the prolongation, Olver used the *α-th total derivative* D_{α} of an arbitrary function $f\left(t^{\alpha}, x^{i}\right)$ on $T\times M$, which is defined by the relation

$$D_{\alpha}f = \frac{\partial f}{\partial t^{\alpha}} + \frac{\partial f}{\partial x^{i}}x^{i}_{\alpha}$$

.

Thus, starting with $X = X^{\alpha}(t,x)\dfrac{\partial}{\partial t^{\alpha}} + X^{i}(t,x)\dfrac{\partial}{\partial\, x^{i}}$ like a vector field on $T\times M$, Olver introduced the 1-th prolongation of X as the vector field

$$pr^{(1)}X = X + X^{(i}_{(\alpha)}\left(t^{\beta}, x^{i}, x^{i}_{\beta}\right)\frac{\partial}{\partial x^{i}_{\alpha}},$$

where

$$X^{(i)}_{(\alpha)} = D_{\alpha}X^{i} - \left(D_{\alpha}X^{\beta}\right)x^{i}_{\beta} = \frac{\partial X^{i}}{\partial t^{\alpha}} + \frac{\partial X^{i}}{\partial x^{j}}x^{j}_{\alpha} - \left(\frac{\partial X^{\beta}}{\partial t^{\alpha}} + \frac{\partial X^{\beta}}{\partial x^{j}}x^{j}_{\alpha}\right)x^{i}_{\beta}.$$

If we assume that is given a nonlinear connection $\Gamma = \left(M^{(i)}_{(\alpha)\beta}, N^{(i)}_{(\alpha)j}\right)$ on $J^{1}(T,M)$, then the α-th total derivative used by Olver can be written as

$$D_{\alpha}f = \frac{\delta f}{\delta t^{\alpha}} + \frac{\delta f}{\delta x^{i}}x^{i}_{\alpha},$$

and, consequently, $D_{\alpha}f$ represent the local components of a distinguished 1-form on $J^{1}(T\times M)$, wich is expressed by $Df = (D_{\alpha}f)\,dt^{\alpha}$. Now, let there be given a vector field X on $T\times M$. From a geometrical point of view, we can define a 1-jet prolongation of X as the *horizontal lift* X^{H} of X. This is defined by

$$X^{H} = X^{\alpha}\frac{\delta}{\delta t^{\alpha}} + X^{i}\frac{\delta}{\delta x^{i}} = X - \left(M^{(j)}_{(\beta)\alpha}X^{\alpha} + N^{(j)}_{(\beta)i}X^{i}\right)\frac{\partial}{\partial x^{j}_{\beta}}.$$

<u>**Open problem**</u>

Study the prolongations of vectors, 1-forms, tensors, G-structures from the basis manifold $T \times M$ to the manifold $J^1(T, M)$.

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