Infinite periodic discrete minimal surfaces without self-intersections

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Abstract

A triangulated piecewise-linear minimal surface in Euclidean 3-space $\mathbb{R}^3$ defined using a variational characterization is critical for area amongst all continuous piecewise-linear variations with compact support that preserve the simplicial structure. We explicitly construct examples of such surfaces that are embedded and are periodic in three independent directions of $\mathbb{R}^3$.

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Key words: minimal surfaces, discrete surfaces, periodic surfaces.

1 Introduction

The goal of this article is to show existence of examples of discrete triply-periodic minimal surfaces, modelled on smooth triply-periodic minimal surfaces. For each smooth minimal surface considered, we find a variety of corresponding discrete minimal surfaces. We restrict ourselves to discrete surfaces with a high degree of symmetry with respect to their vertex density, thus appearing highly discretized. This has the advantages that we can give explicit mathematical proofs of minimality, and we can make symmetry changes that are forbidden in the smooth case.

1.1 Smooth minimal surfaces

Soap films not containing bounded pockets of air minimize area with respect to their boundaries, and are modelled by minimal surfaces. Minimal surfaces are those that are critical for area amongst compactly-supported boundary-fixing smooth variations. Here, equivalently, we define a *smooth minimal surface* in $\mathbb{R}^3$ as a C^∞ immersion $f : \mathcal{M} \to \mathbb{R}^3$ of a 2-dimensional manifold $\mathcal{M}$ whose mean curvature is identically zero. Some general introductions to smooth minimal surfaces are [6], [7], [8], [13], [15], [19] and [21]. Amongst these, one can find the definition of *conjugate* minimal surfaces.

The simplest example of a minimal surface is the flat plane. Another example is the catenoid, which is a surface of revolution that can be parametrized by

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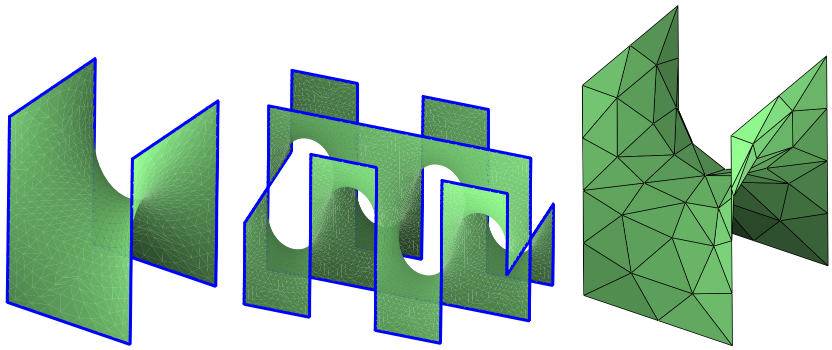

Figure 1: Smooth and discrete superman surfaces. (All the computer graphics here were made using K. Polthier's Javaview software [23].)

(1.1) $$\{(\cosh x \cos y, \cosh x \sin y, x) \in \mathbb{R}^3 \mid x \in \mathbb{R}, y \in (0, 2\pi] \subset \mathbb{R}\}\ ,$$

where $\mathbb{R}$ denotes the real numbers and $\mathbb{R}^3$ denotes Euclidean 3-space. The catenoid is conjugate to another well-known minimal surface called the helicoid.

A *triply-periodic* smooth surface is one that is periodic in three independent directions of $\mathbb{R}^3$, i.e. f and $f + \vec{v}_j$ have equal images for three independent constant vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^3$. There are a wide variety of smooth triply-periodic minimal surfaces, see the papers of Karcher, Polthier, Schoen, Fischer and Koch [4], [11], [13], [14], [29], for example. We show a few examples here: Figure 1, center, was named the superman surface by Meeks [15] (one special case of this is the Schwarz D surface); Figure 2, second row, second surface, is a generalized Schwarz P surface (one special case of this is the original Schwarz P surface); the smooth Schwarz CLP surface is shown in Figure 7, lower-right; and the smooth triply-periodic Fischer-Koch surface is shown in Figure 11, lower row.

1.2 Defining discrete minimal surfaces

Recently, finding discrete analogs of smooth objects has become an important theme in mathematics, appearing in a variety of places in analysis and geometry. So it is natural to consider discrete analogs of smooth minimal surfaces. But there is no single definitive approach; the definition one chooses depends on which properties of smooth minimal surfaces one wishes to emulate in the discrete case. Bobenko and Pinkall [2] use discrete integrable systems to define them, in analogy to integrable systems properties of smooth minimal (and constant mean curvature) surfaces, which does not yield area-critical discrete surfaces with respect to vertex variations. Here, rather, we define a discrete minimal surface in $\mathbb{R}^3$ to be a piecewise linear triangulated surface that is critical for area with respect to any compactly-supported boundary-fixing continuous piecewise-linear variation that preserves its simplicial structure, see [22], [26] and Section 2 here. (Although we do not use it, there is a broader variation-based definition by Polthier [25] using "non-conforming triangulations".)

1.3 Constructing discrete minimal surfaces

Just as for smooth surfaces, a triply-periodic discrete surface is one that is periodic in three independent directions of $\mathbb{R}^3$. We will construct a variety of discrete examples modelled on each single smooth example, via two approaches:

- **Method 1** vary the choice of simplicial structure of some compact portion of the surface, and/or
- **Method 2** vary the choice of rigid motions of $\mathbb{R}^3$ that create the complete surface from some compact portion.

To explain these two methods, imagine a compact portion M of a smooth triply-periodic minimal surface in $\mathbb{R}^3$, with piece-wise smooth boundary ∂M consisting of smooth curves γ_1,...,γ_n. Suppose each γ_j is either a straight line segment or a planar curve in a principal curvature direction of M (the latter is called a *planar geodesic*, as it is also a geodesic of M). A larger minimal surface M' is constructed from M by including images of M under 180° rotations about the lines containing linear γ_j and under reflections through the planes containing planar geodesic γ_j. (The fact that the larger portion M' is still a smooth minimal surface follows from complex analysis, see [11], [13], [19], [21], for example.) The larger portion M' again has a piece-wise smooth boundary $\partial M'$ consisting of line segments and planar geodesics, so this procedure can be repeated on M'. Repeating this on ever-bigger pieces, one builds a complete surface. M is often called a *fundamental domain* of the complete surface.

For example, the minimal surface on the left-hand side of Figure 1 is a fundamental domain M of a complete triply-periodic smooth minimal surface. The boundary ∂M contains the eight vertices $p_1 = (0, 0, 0)$, $p_2 = (x, 0, 0)$, $p_3 = (x, 0, z)$, $p_4 = (x, y, z)$, $p_5 = (x, y, 0)$, $p_6 = (0, y, 0)$, $p_7 = (0, y, z)$, $p_8 = (0, 0, z)$, for some given positive reals $x, y, z > 0$. Then ∂M is a polygonal loop consisting of eight line segments, from p_j to p_{j+1} for $j = 1, 2, ..., 7$, and finally from p_8 to p_1. One can construct the entire complete surface using only 180° rotations about boundary line segments. A larger piece of this complete surface is shown in Figure 1, center. For general values of x, y, z, the resulting complete triply-periodic surface is a superman surface. When $x = y$ this surface represents Schwarz' solution of Gergonne's problem (see [11], [14]), and when $x = y = \sqrt{2} \cdot z$ this surface is the Schwarz D surface.

To construct triply-periodic discrete minimal surfaces modelled on smooth superman surfaces, we first find a discrete version of the smooth fundamental domain M. One such example, found numerically using JavaView software [23], is in Figure 1, right side. However, there are many ways to choose the simplicial structure of the discrete version, and a number of them are shown in Figure 4. (The examples in Figure 4 all have coarser simplicial structures.) The corresponding complete triply-periodic discrete minimal surfaces are then constructed like in the smooth case, by 180° rotations about boundary line segments. This is an example of **Method 1**.

To demonstrate **Method 2**, consider the left-hand minimal surface M in Figure 2, second row. Its boundary ∂M is two squares in parallel planes, where one square projects to the other by projection orthogonal to the planes. As in the previous example, 180° rotations about boundary line segments produce a complete triply-periodic smooth minimal surface, and a larger piece of which is shown just to the

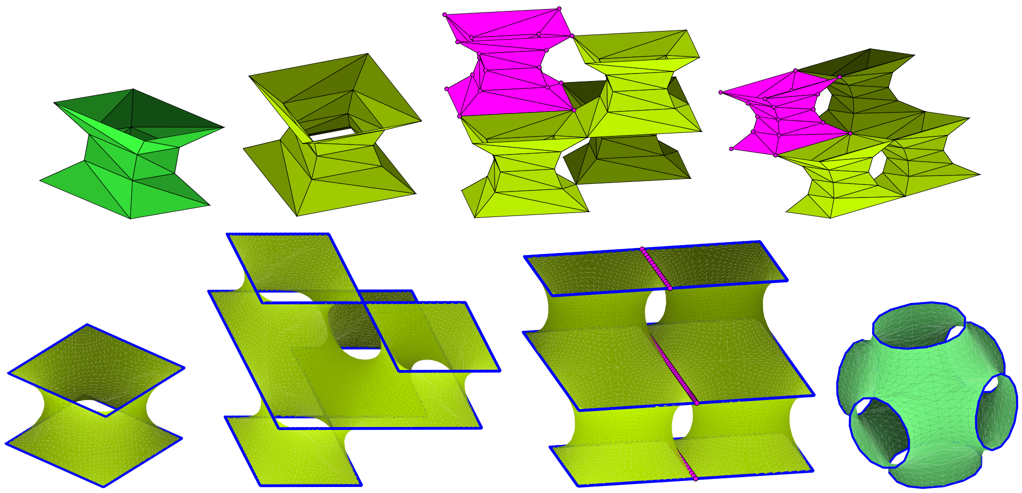

Figure 2: Discrete and smooth Schwarz P surfaces.

right of M in Figure 2. This example is a Schwarz P surface. The original Schwarz P surface occurs when the distance between the two parallel planes containing ∂M is $1/\sqrt{2}$ times the length of each edge in ∂M, and it is conjugate to the Schwarz D surface. (For later use, we mention that the lower-right surface $\hat{M}$ in Figure 2 is also part of a Schwarz P surface, now bounded by six planar geodesics, each lying within one face of a rectanguloid with square base. The entire complete surface can be built from $\hat{M}$ solely by applying reflections in equally-spaced planes parallel to the faces of the rectanguloid. The top one-fourth of $\hat{M}$ equals the bottom half of M. So one could choose either M or $\hat{M}$ as the fundamental piece for constructing a complete Schwarz P surface.)

As with the superman surface, to create discrete analogs of the Schwarz P surface, one can apply **Method 1** and choose amongst many different simplicial structures for this surface. Two such possibilities are the first two discrete minimal surfaces in Figure 2, first row. The first has 4 squares in parallel planes in its edge set, and the second has 5 squares in parallel planes in its edge set. In fact, one can make examples with any number$\geq$ 3 of such squares in its edge set, as we will see in Section 5. In Figure 2, first row, third figure, a larger portion of a resulting complete discrete triply-periodic minimal surface is shown.

But we return to demonstrating **Method 2** on the Schwarz P surface. Let P be a plane perpendicular to the two planes containing ∂M so that P also contains two disjoint boundary edges in ∂M. Consider the surface that results if we first include a reflected image of M across P, and then create a complete surface by 180° rotations about all resulting boundary line segments. A part of this surface is shown in Figure 2, second row, third figure (bounded by three boundary components, one square and two rectangles). In this part in Figure 2, there are three parallel dotted lines, along which the surface is not even a C^1 immersion, hence the mean curvature is not defined there. So this construction is forbidden in the smooth case. However, for the discrete analogs shown in Figure 2, such a construction actually does produce a discrete complete triply-periodic minimal surface. Part of this surface is shown in Figure 2, upper-right (having the same three boundary components). One can easily imagine including more reflections (not just across one single plane P), to make infinitely many different kinds of discrete minimal surfaces modelled on M. This is **Method 2**.

In Section 3, we state our main result. In Sections 4 and 5, as applications of **Method 1**, we will see a variety of different simplicial structures making discrete analogs of the smooth superman and Schwarz P surfaces. Section 6 contains examples modelled on other smooth minimal surfaces. In the last example of Section 5 and the first example of Section 6 there are further applications of **Method 2**.

Sullivan and Goodman-Strauss also considered triply periodic discrete minimal surfaces with the same definition, including discrete analogs of the Schwarz P and CLP surfaces [5]. Also, Schoen studied discrete Voronoi gyroids [30].

2 Discrete Minimal Surfaces

We will define discrete minimal surfaces so that they are area-critical for boundary-fixing variations. We first define discrete surfaces and their variations, beginning with an informal definition: A *discrete surface* in $\mathbb{R}^3$ is a C^0 mapping $f : \mathcal{M} \to \mathbb{R}^3$ of a 2-dimensional manifold $\mathcal{M}$ so that each face of some triangulation of $\mathcal{M}$ is mapped to a triangle in $\mathbb{R}^3$. The surface $f(\mathcal{M})$ is *embedded* if f is injective.

We define *embedded* in the discrete case without any conditions about nondegeneracy of f (nondegeneracy is meaningless here, as f is only C^0). However, we still use this word, to maintain the analogy to embeddedness of smooth surfaces.

Definition 2.1. *A* discrete surface *in $\mathbb{R}^3$ is a triangular mesh which has the topology of an abstract 2-dimensional locally-finite simplicial surface K combined with a geometric C^0 realization $\mathcal{T}$ in $\mathbb{R}^3$ that is piecewise-linear on each simplex. Because K is a simplicial "surface", each 1-dimensional simplex in K lies in the boundary of exactly one or two 2-dimensional simplices of K. The geometric realization $\mathcal{T}$ is determined by a set of vertices $\mathcal{V} = \{p_1, p_2, ...\} \subset \mathbb{R}^3$ corresponding to the 0-dimensional simplices of K, with $\mathcal{V}$ either finite or countably infinite. The simplicial surface K represents the connectivity of $\mathcal{T}$. The 0, 1, and 2 dimensional simplices of K represent the vertices, edges, and triangles of $\mathcal{T}$.*

Let $T = (p, q, r)$ denote an oriented triangle of $\mathcal{T}$ with $p, q, r \in \mathcal{V}$. Let $\overline{pq}$ denote an edge of T with endpoints p, q. Let $\mathrm{star}(p)$ be the triangles of $\mathcal{T}$ containing p as a vertex.

The boundary *$\partial\mathcal{T}$ of $\mathcal{T}$ is the union of those edges bounding only a single triangle of $\mathcal{T}$. The* interior vertices *(respectively,* boundary vertices*) of $\mathcal{T}$ are those that are not contained (respectively, are contained) in $\partial\mathcal{T}$.*

We say that $\mathcal{T}$ is complete *if $\partial\mathcal{T}$ is empty and if $\mathcal{T}$ is complete with respect to the distance function induced by its realization in $\mathbb{R}^3$.*

Definition 2.2. *Let $\mathcal{V} = \{p_1, p_2, ...\}$ be the set of vertices of a discrete surface $\mathcal{T}$. A* variation *$\mathcal{T}(t)$ of $\mathcal{T}$ is defined as a C^∞ variation of the vertices p_i*

$$p_i(t) : [0, \epsilon) \to \mathbb{R}^3 \text{ so that } p_i(0) = p_i \;\forall i = 1, ..., m.$$

The straightness of the edges and the flatness of the triangles are preserved as the vertices $p_i(t)$ move with respect to t.

When $\mathcal{T}$ is compact, we say that $\mathcal{T}(t)$ fixes the boundary $\partial\mathcal{T}$ *if $p_i(t)$ is constant in t for all $p_i \in \partial\mathcal{T}$. When $\mathcal{T}$ is complete, we say that $\mathcal{T}(t)$ is* compactly supported *if $p_i(t)$ is constant in t for all but a finite number of vertices p_i.*

The area of a discrete surface is

$$\operatorname{area}\mathcal{T} := \sum_{T\in\mathcal{T}} \operatorname{area} T\ ,$$

where area T denotes the Euclidean area of the triangle T as a subset of $\mathbb{R}^3$.

Lemma 2.1. *Let $\mathcal{T}(t)$ be a variation of a discrete surface $\mathcal{T}$. At each vertex p of $\mathcal{T}$, the gradient of area is*

$$\nabla_p \operatorname{area}\mathcal{T} = \frac{1}{2} \sum_{T=(p,q,r)\in\operatorname{star}(p)} J(r-q)\ , \tag{2.2}$$

where J is $90°$ rotation in the plane of each oriented triangle T. The first derivative of the area is then given by the chain rule

$$\frac{d}{dt} \operatorname{area}\mathcal{T}(t)\bigg|_{t=0} = \sum_{p\in\mathcal{V}} \left\langle \frac{d(p(t))}{dt}\bigg|_{t=0}, \nabla_p \operatorname{area}\mathcal{T} \right\rangle\ . \tag{2.3}$$

Proof. Let $p_i(t)$ be the corresponding variation of each vertex in the vertex set $\mathcal{V}(t)$ of the variation $\mathcal{T}(t)$. Then

$$\operatorname{area}(\mathcal{T}(t)) = \frac{1}{6} \sum_{p(t)\in\mathcal{V}(t)} \left(\sum_{(p(t),q(t),r(t))\in\operatorname{star}(p(t))} ||(r(t)-p(t)) \times (q(t)-p(t))|| \right)\ ,$$

and a computation implies

$$\frac{d}{dt}\operatorname{area}(\mathcal{T}(t)) = \frac{1}{2} \sum_{p(t)\in\mathcal{V}(t)} \left\langle \frac{d(p(t))}{dt}, \sum_{(p(t),q(t),r(t))\in\operatorname{star}(p(t))} ||r(t)-q(t)||\eta(t) \right\rangle\ ,$$

where $\eta(t)$ is the unit conormal in the plane of the triangle $(p(t), q(t), r(t))$ along the edge $r(t) - q(t)$, oriented in the same direction as $J(r(t) - q(t))$. Restricting to $t = 0$ proves the lemma. □

As defined in Section 1, a smooth immersion $f : \mathcal{M} \to \mathbb{R}^3$ of a 2-dimensional complete manifold $\mathcal{M}$ without boundary is minimal if f is area-critical for all compactly-supported smooth variations. In the case that $\mathcal{M}$ is compact with boundary, then f is minimal if it is area-critical for all smooth variations preserving $f(\partial\mathcal{M})$.

We wish to define discrete minimal surfaces $\mathcal{T}$ so that they have the analogous properties, for variations as in Definition 2.2. So when $\mathcal{T}$ is compact, we consider variations $\mathcal{T}(t)$ of $\mathcal{T}$ that fix $\partial\mathcal{T}$; and when $\mathcal{T}$ is complete, we consider variations $\mathcal{T}(t)$ of $\mathcal{T}$ that are compactly supported. By Lemma 2.1, the condition that makes $\mathcal{T}$ area-critical for any variation of these types is expressed in the following definition.

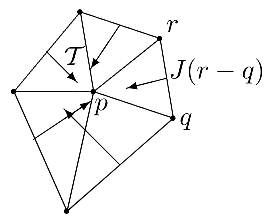

Figure 3: At each vertex p the gradient of discrete area is the sum of the 90°-rotated edge vectors $J(r-q)$, as in Equation (2.2).

Definition 2.3. *A discrete surface is* minimal *if*

$$\nabla_p \operatorname{area} \mathcal{T} = 0 \tag{2.4}$$

for all interior vertices p.

Remark 2.1. *If $\mathcal{T}$ is a discrete minimal surface that contains a discrete subsurface $\mathcal{T}'$ lying in a plane P, it follows from Equations (2.2) and (2.4) that the discrete minimality of $\mathcal{T}$ is independent of the choice of triangulation of the trace of $\mathcal{T}'$ within P. Thus whenever such a planar part $\mathcal{T}'$ occurs in the following examples, we will be free to triangulate $\mathcal{T}'$ any way we please, within its trace in P.*

3 Results

In order to state our main theorem, we give the following two definitions:

Definition 3.1. *A discrete triply-periodic minimal surface $\mathcal{T}$ has* common topology and symmetry *as a smooth triply-periodic minimal immersion $f : \mathcal{M} \to \mathbb{R}^3$ if there exists a homeomorphism*

$$\phi : f(\mathcal{M}) \to \mathcal{T}$$

such that the following statement holds: $R_s : \mathbb{R}^3 \to \mathbb{R}^3$ is a rigid motion preserving $f(\mathcal{M})$ if and only if there exists a rigid motion $R_d : \mathbb{R}^3 \to \mathbb{R}^3$ preserving $\mathcal{T}$ so that

$$R_d \circ \phi = \phi \circ R_s|_{f(\mathcal{M})} \ ,$$

and furthermore, R_s is a reflection (resp. translation, rotation, screw motion) if and only if R_d is a reflection (resp. translation, rotation, screw motion).

Definition 3.2. *We say that a subsurface $\mathcal{T}'$ of a complete discrete triply-periodic minimal surface $\mathcal{T}$ is a* fundamental domain *if $\mathcal{T}'$ can be extended to all of $\mathcal{T}$ by a discrete group of rigid motions $\{R_{d,\alpha}\}_{\alpha\in\Lambda}$ generated by*

1. *reflections across planes containing boundary edges and*
2. 180° *degree rotations about boundary edges*

so that each $R_{d,\alpha}$ is a symmetry of the full surface $\mathcal{T}$.

Remark 3.1. *In the above definition of a fundamental domain, we do not allow rigid motions that do not fix any edges of $\mathcal{T}$ (thus any fundamental domain of the example in Subsection 4.1 must contain at least 6 triangles, even in the most symmetric case $x = 1$). Also, we do not allow rigid motions that are not symmetries of the full surface $\mathcal{T}$ (thus any fundamental domain of the example in Subsection 5.2 must contain at least 32 triangles).*

We now state our results about embedded triply-periodic discrete minimal surfaces, which involve comparisons to the following smooth minimal surfaces: the superman surfaces (Figure 1), the Schwarz P surfaces (Figure 2), the Schwarz H surfaces, the Schwarz CLP surfaces (Figure 7), A. Schoen's I-Wp and F-Rd and H-T surfaces, and the triply-periodic Fischer-Koch surfaces (Figure 11). More complete information about these smooth surfaces can be found in [4], [6], [11], [13], [14], [15], [16], [18], [19], [28] and [29]. To prove this theorem, we need only collect the examples proven to exist in the remainder of this paper.

Theorem 3.1. *The following discrete embedded triply-periodic minimal surfaces exist:*

1. *those with common topology and symmetry as smooth superman surfaces whose fundamental domains contain 4, 5, 6 or 8 triangles (see Subsections 4.3, 4.4, 4.2, and 4.1, respectively, see also a second type for 6 triangles in the penultimate paragraph of 6.1);*
2. *those with common topology and symmetry as smooth Schwarz P surfaces whose fundamental domains contain 1, 2, 6 or 32 triangles (first example of 5.1, second example of 5.1, 5.2, and last paragraph of 6.1, respectively), and also a different class of discrete surfaces with common topology and symmetry as smooth Schwarz P surfaces whose fundamental domains contain $2n$ triangles for any positive integer n ($k = 4$, $z_0 = 0$, $j_0 = n$ in 5.3, with fundamental domains the $2n$ triangles between two adjacent meridans and below the plane $\{(x, y, 0) \,|\, x, y \in \mathbb{R}\}$);*
3. *those with common topology and symmetry as smooth Schwarz H surfaces whose fundamental domains contain $2n$ triangles for any positive integer n ($k = 3$, $z_0 = 0$, $j_0 = n$ in 5.3, with fundamental domains the $2n$ triangles between two adjacent meridans and below the plane $\{(x, y, 0) \,|\, x, y \in \mathbb{R}\}$);*
4. *those with common topology and symmetry as smooth Schwarz CLP surfaces whose fundamental domains contain 6 triangles (see 6.1);*
5. *one with common topology and symmetry as Schoen's smooth I-Wp surface whose fundamental domain contains 5 triangles (see 6.2);*
6. *one with common topology and symmetry as Schoen's smooth F-Rd surface whose fundamental domain contains 3 triangles (see 6.2);*
7. *those with common topology and symmetry as Schoen's smooth H-T surfaces whose fundamental domains contain 6 triangles (see 6.3);*

8. *those with common topology and symmetry as the smooth triply-periodic surfaces of Fischer-Koch whose fundamental domains contain* 8 *triangles (see 6.4).*

4 Discrete versions of the superman surface

In Sections 4, 5 and 6, we construct discrete triply-periodic minimal surfaces. All of the surfaces we construct are embedded.

To construct examples, we always start with a compact discrete fundamental piece $\mathcal{T}$, with given simplicial structure and boundary constraints. The complete triply-periodic discrete surface is then formed by including images of $\mathcal{T}$ under a discrete group of rigid motions of $\mathbb{R}^3$. This group of rigid motions is generated by a finite number of 180° rotations about lines and/or reflections across planes, and for each edge $\overline{pq}$ in $\partial\mathcal{T}$ this group contains either

- the 180° rotation about the line containing $\overline{pq}$, or
- a reflection across a plane containing $\overline{pq}$.

To ensure that the resulting complete discrete triply-periodic surface is minimal, Section 2 gave us the following two approaches:

1. Use symmetries of $\mathcal{T}$ and of the resulting complete discrete surface to show that Equation (2.4) holds at the vertices.
2. Locate the vertices of $\mathcal{T}$ so that $\mathcal{T}$ is area-critical with respect to its boundary constraints.

In the following examples, either approach produces the same conditions for minimality.

We wish to give explicit mathematical proofs of minimality here, so we are limited to examples with a high degree of symmetry with respect to their vertex density, and thus appear highly discretized. Discrete minimal surfaces that appear more like approximations of smooth minimal surfaces usually can only be found numerically. Numerical examples, with finer simplicial structures, of discrete versions of the superman, Schwarz P, F-Rd, I-Wp and H-T surfaces are shown in [25].

4.1 First example

The fundamental piece $\mathcal{T}$ here has eight boundary vertices

$$p_1 = (1, 0, 0), \;\; p_2 = (1, 1, 0), \;\; p_3 = (1, 1, x), \;\; p_4 = (0, 1, x),$$

$$p_5 = (0, 1, 0), \;\; p_6 = (0, 0, 0), \;\; p_7 = (0, 0, x), \;\; p_8 = (1, 0, x),$$

for any given fixed $x > 0$, and has one interior vertex

$$p_9 = (\frac{1}{2}, \frac{1}{2}, \frac{x}{2}) \; .$$

There are eight triangles in $\mathcal{T}$, which are

$$(p_j, p_{j+1}, p_9),\ j = 1, ..., 7\ , \quad (p_8, p_1, p_9).$$

The complete triply-periodic surface is generated by including the image of $\mathcal{T}$ under 180° rotations about each edge of $\partial\mathcal{T}$, and then continuing to include the images under 180° rotations about each resulting boundary edge until the surface is complete. It is evident from the symmetries of this surface that Equation (2.4) holds at every vertex. The fundamental piece $\mathcal{T}$ and a larger part of the resulting complete surface are shown on the left-hand side and center of the first row of Figure 4 for $x = 1$. The case for some given $x \in (0, 1)$ is shown on the right-hand side of the first row of Figure 4.

4.2 Second example

The fundamental piece $\mathcal{T}$ here has six boundary vertices

$$p_1 = (0, 0, 0),\ \ p_2 = (x, 0, 0),\ \ p_3 = (x, y, 0),$$

$$p_4 = (x, y, 1),\ \ p_5 = (0, y, 1),\ \ p_6 = (0, 0, 1),$$

for any given fixed $x, y > 0$, and has one interior vertex

$$p_7 = (\frac{x}{2}, \frac{y}{2}, \frac{1}{2})\ .$$

There are six triangles in $\mathcal{T}$, which are

$$(p_j, p_{j+1}, p_7),\ j = 1, ..., 5\ , \quad (p_6, p_1, p_7).$$

The complete triply-periodic surface is generated by 180° rotations about boundary edges, just as in the previous example. In this example as well, it is evident from the symmetries of this surface that Equation (2.4) holds at every vertex. Two fundamental pieces $\mathcal{T}$ of different sizes and larger parts of the resulting complete surfaces are shown in the second row of Figure 4 ($x = y = 1$ in the first case, and $x < 1 < y$ in the second case).

In the case that $x = y = 1$, this fundamental piece $\mathcal{T}$ has the same boundary as a fundamental piece of the smooth Schwarz D surface. Furthermore, for general x and y, this surface can be viewed as a discrete analog of the superman surface as follows: Consider the eight-straight-edged polygonal curve from the point $(0, 0, -1/2)$ to the point $(x, -y, -1/2)$ and then to $(x, -y, 1/2)$ and then to $(2x, 0, 1/2)$ and then to $(2x, 0, -1/2)$ and then to $(x, y, -1/2)$ and then to $(x, y, 1/2)$ and then to $(0, 0, 1/2)$ and then back to $(0, 0, -1/2)$. This polygonal curve is contained in this discrete surface (although not in its edge set) and is also the boundary of a smooth superman surface.

4.3 Third example

The fundamental piece $\mathcal{T}$ here has four boundary vertices

$$p_1 = (0, 0, 0),\ \ p_2 = (1, 1, 0),\ \ p_3 = (1, 1, z),\ \ p_4 = (1, 0, z),$$

for any given fixed $z > 0$, and has one interior vertex

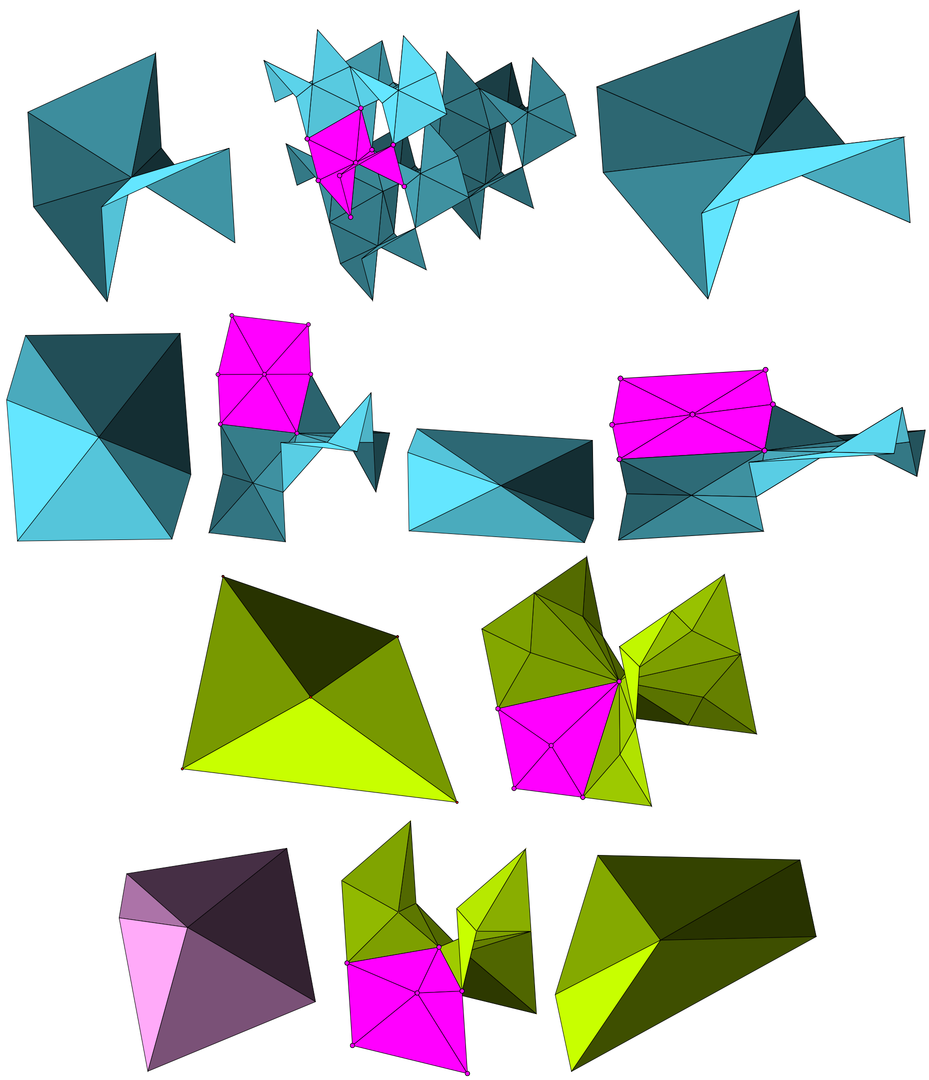

Figure 4: Four different discrete versions of the superman surface.

$$p_5 = (a, b, c) \ .$$

There are four triangles in $\mathcal{T}$, which are

$$(p_j, p_{j+1}, p_5), \ j = 1, ..., 3 \ , \quad (p_4, p_1, p_5).$$

Reflecting $\mathcal{T}$ across the plane $\{(x_1, 0, x_3) \in \mathbb{R}^3 \,|\, x_1, x_3 \in \mathbb{R}\}$ and attaching its image to $\mathcal{T}$, one has a larger discrete surface containing eight triangles and six boundary edges. One can extend this larger discrete surface to a complete triply-periodic surface by 180° rotations about boundary edges, just as in the previous examples. In this example, Equation (2.4) holds at each vertex p_1,...,p_4 in the resulting complete surface. However, getting this to hold at p_5 requires proper choices of a and b.

For simplicity, we restrict to the case $z = 1$. Then, by symmetry, we may assume $b = c$. A computation shows that Equation (2.4) holding at p_5 is equivalent to

$$(1-a)\sqrt{a^2 - 2ab + 3b^2} = (a-b)\sqrt{(1-a)^2 + (1-b)^2} \ , \tag{4.5}$$

$$(1-b)\sqrt{a^2 - 2ab + 3b^2} = (3b-a)\sqrt{(1-a)^2 + (1-b)^2} \ . \tag{4.6}$$

The solution to this is

$$b = \frac{1}{2} \ , \quad a = \frac{3-\sqrt{2}}{2} \ . \tag{4.7}$$

So when $z = 1$ and $b = c$ and Equation (4.7) holds, the area gradient is zero at each vertex p_j for $j = 1, 2, ..., 9$ in the extended complete triply-periodic discrete surface, and then symmetries of the surface imply the entire complete surface is minimal.

Since the above minimality condition (4.5)-(4.6) is a system of two equations in two variables a and b, we say the minimality condition here (when $z = 1$) is two-dimensional.

The fundamental piece $\mathcal{T}$ with $z = 1$ is shown on the left-hand side of the third row of Figure 4, and a larger part of the resulting complete surface is shown just to the right of this.

4.4 Fourth example

The fundamental piece $\mathcal{T}$ here has five boundary vertices

$$p_1 = (0,0,0), \ \ p_2 = (1,1,0), \ \ p_3 = (1,1,z), \ \ p_4 = (0,1,z), \ \ p_5 = (0,0,z)$$

for any given fixed $z > 0$, and has one interior vertex

$$p_6 = (a, 1-a, b) \ .$$

There are five triangles in $\mathcal{T}$, which are

$$(p_j, p_{j+1}, p_6), \ j = 1, ..., 4 \ , \quad (p_5, p_1, p_6).$$

The complete triply-periodic surface is generated by 180° rotations about boundary edges. In this example, Equation (2.4) holds at each vertex p_1,...,p_5 in the resulting complete surface, and making it hold also at p_6 requires proper choices of a and b.

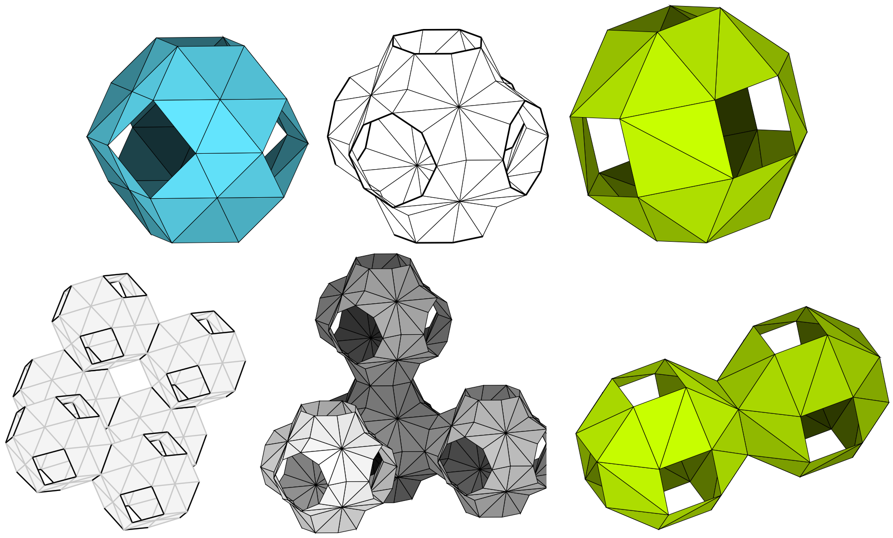

Figure 5: Three different discrete versions of the Schwarz P surface.

Like in the previous example, we can find a pair of explicit equations, in the variables a and b, that represent the minimality condition. These equations are similar to those of the previous example, and are slightly more complicated. One can then show the existence of a and b solving this minimality condition.

Two fundamental pieces $\mathcal{T}$ of different sizes ($z = 1$ in the first case, and $z < 1$ in the second case) are shown on the left and right-hand sides of the bottom row of Figure 4. A larger part of the resulting complete surface in the case $z = 1$ is shown in the bottom-middle of Figure 4.

5 Discrete versions of the Schwarz P surface

5.1 First two examples

Consider the vertices

$$p_1 = (3, 0, 6), \;\; p_2 = (6, 0, 3), \;\; p_3 = (6, 3, 0),$$

$$p_4 = (3, 6, 0), \;\; p_5 = (0, 6, 3), \;\; p_6 = (0, 3, 6), \;\; p_7 = (3, 3, 3),$$

and let $\mathcal{T}_1$ be the planar fundamental domain with the six triangles

$$(p_j, p_{j+1}, p_7), \; j = 1, ..., 5 \;, \quad (p_6, p_1, p_7).$$

Also, consider the vertices

$$p_1 = (3, 0, 6), \;\; p_2 = (4, 0, 4), \;\; p_3 = (6, 0, 3), \;\; p_4 = (6, 2, 2),$$

$$p_5 = (6,3,0), \;\; p_6 = (4,4,0), \;\; p_7 = (3,6,0), \;\; p_8 = (2,6,2),$$

$$p_9 = (0,6,3), \;\; p_{10} = (0,4,4), \;\; p_{11} = (0,3,6), \;\; p_{12} = (2,2,6), \;\; p_{13} = (3,3,3),$$

and let $\mathcal{T}_2$ be the fundamental domain with the twelve triangles

$$(p_j, p_{j+1}, p_{13}), \; j = 1, ..., 11 \;, \quad (p_{12}, p_1, p_{13}).$$

We can extend $\mathcal{T}_j$ (for either $j = 1, 2$) to a complete discrete surface by including the images of $\mathcal{T}_j$ under the reflections across the planes $\{(x, y, 6k) \,|\, x, y \in \mathbb{R}\}$, $\{(x, 6k, z) \,|\, x, z \in \mathbb{R}\}$ and $\{(6k, y, z) \,|\, y, z \in \mathbb{R}\}$ for all integers k. Furthermore, Equation (2.4) holds at all vertices of the extended surface, so it is minimal. (See the first two columns of Figure 5.)

The surface produced by $\mathcal{T}_1$ (resp. $\mathcal{T}_2$) is a simpler (resp. more complicated) version of a discrete Schwarz surface. Note that they are analogous to the bottom-right picture in Figure 2. (The second example $\mathcal{T}_2$ was also shown in [26].)

5.2 Third example

Consider the ten vertices

$$p_j = (a, (-1)^j a, 1), \;\; p_{j+2} = (a, 1, (-1)^{j+1} a), \;\; p_{j+4} = (a, (-1)^{j+1} a, -1),$$

$$p_{j+6} = (1, (-1)^j a, a), \;\; p_{j+8} = (1, (-1)^{j+1} a, -a), \qquad\qquad j = 1, 2,$$

and let $\hat{\mathcal{T}}$ be the discrete surface with the eight triangles

$$(p_1, p_2, p_7), \;\; (p_2, p_8, p_7), \;\; (p_2, p_3, p_8), \;\; (p_3, p_4, p_8),$$

$$(p_4, p_9, p_8), \;\; (p_4, p_5, p_9), \;\; (p_5, p_6, p_9), \;\; (p_6, p_{10}, p_9).$$

Then let $\mathcal{T}$ be the discrete surface, with 24 vertices and 32 faces, that is made by including the four images of $\hat{\mathcal{T}}$ under the rotations about the axis $\{(0, 0, r) \,|\, r \in \mathbb{R}\}$ of angles 0°, 90°, 180° and 270°. This $\mathcal{T}$ is shown in the upper-right of Figure 5.

One can then generate a complete triply-periodic surface by including the images of $\mathcal{T}$ under reflections across the planes $\{(x, y, k) \,|\, x, y \in \mathbb{R}\}$, $\{(x, k, z) \,|\, x, z \in \mathbb{R}\}$ and $\{(k, y, z) \,|\, y, z \in \mathbb{R}\}$ for all integers k. The result of applying one such reflection is shown in the lower picture of the right-most column of Figure 5.

The condition for this discrete triply-periodic surface to be minimal is that

$$a = \frac{3\sqrt{2} - \sqrt{3}}{6\sqrt{2} - \sqrt{3}} \,,$$

i.e. for this value of a, Equation (2.4) holds at every vertex of the surface.

5.3 Examples based on discrete minimal catenoids

Here we give two closely-related types of examples based on discrete minimal catenoids. One type is a discrete analog of the Schwarz P surface. The other type is actually an analog of the smooth Schwarz H surface, not the Schwarz P surface. To construct these examples, we will use discrete catenoids [26], which are described in terms of the hyperbolic cosine function, just as the smooth catenoid was in Equation (1.1).

The vertices of a discrete minimal catenoid lie on congruent planar polygonal meridians, and the meridians are contained in planes that meet along a single line (the axis) at equal angles. Every meridian is the image of every other meridian by some rotation about the axis. By drawing edges between corresponding vertices of adjacent meridians (i.e. so that these edges are perpendicular to the axis), we have a piecewise linear continuous surface tessellated by planar isosceles trapezoids. We can triangulate each trapezoid any way we please without affecting minimality, as noted in Remark 2.1, so we shall triangulate each trapezoid by drawing a single diagonal edge across it.

Two examples of discrete catenoids are shown in the first two pictures in the upper row of Figure 2. Both of these pictures have adjacent meridians in planes meeting at $90°$ angles. The first (resp. second) one has four (resp. five) vertices in each meridian. Another example is shown in the left-most picture of Figure 6, where the adjacent meridians lie in planes meeting at $120°$ angles, and there are four vertices in each meridian.

To explicitly describe discrete catenoids, we need only specify:

1. The axis ℓ: let us fix $\ell = \{(0,0,z) \,|\, z \in \mathbb{R}\}$.
2. The angle θ between planes of adjacent meridians: let us fix $\theta = \frac{2\pi}{k}$ for some integer $k \geq 3$.
3. The locations of the vertices along one meridian.

We can place one meridian in the plane $\{(x,0,z) \,|\, x, z \in \mathbb{R}\}$, and locating its vertices at the following points will ensure minimality of the surface (see [26]):

$$p_j = (r \cosh\left(\frac{1}{r} a(z_0 + j\delta)\right), 0, z_0 + j\delta)$$

with $j = j_0, j_0+1, ..., j_1$ for some integers j_0 and j_1 ($j_0 < j_1$), and with

$$a = \frac{r}{\delta} \operatorname{arccosh}\left(1 + \frac{1}{r^2} \frac{\delta^2}{1+\cos\theta}\right),$$

where $r > 0$ and $\delta > 0$ and $z_0 \in \mathbb{R}$ are constant. The edges along this meridian are $\overline{p_j p_{j+1}}$ for j between j_0 and $j_1 - 1$.

For our application, we shall restrict to either $k = 4$, as in Figure 2, or to $k = 3$, as in Figure 6. We shall further assume that either

- $z_0 = 0$ and $j_0 = -j_1 < 0$, or
- $z_0 = \frac{\delta}{2}$ and $j_0 = -j_1 - 1 < -1$.

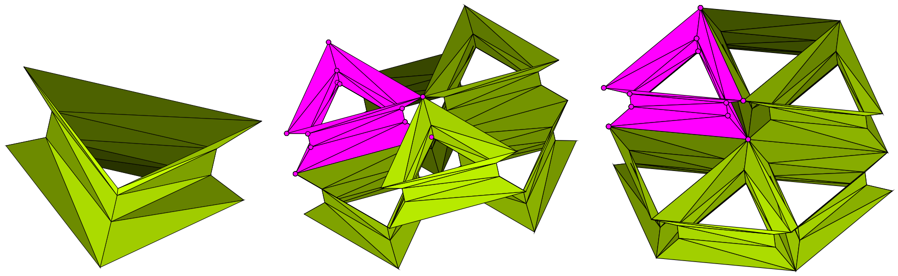

Figure 6: Discrete version of the Schwarz H surface.

Either of these conditions will produce a discrete minimal surface $\mathcal{T}$ whose trace has dihedral symmetry. One can then extend $\mathcal{T}$ by 180° rotation about boundary lines to a complete embedded discrete surface in $\mathbb{R}^3$. To conclude minimality of this complete surface, it remains only to check that Equation (2.4) holds at any vertex contained in any edge about which a 180° rotation was made, and this is clear from the symmetry of the surface.

The case when $k = 4$ and $z_0 = 0$ and $j_0 = -j_1 = -2$ is shown in the second picture of the first row of Figure 2, and a larger portion of the resulting complete minimal surface is shown in the picture just to the right of it. The case when $k = 3$ and $z_0 = \delta/2$ and $j_0 = -j_1 - 1 = -2$ is shown in the left-most picture of Figure 6, and a larger portion of the resulting complete minimal surface is shown in the middle of Figure 6. When $k = 4$, the analogy to the smooth Schwarz P surface is clear. When $k = 3$, one can imagine a smooth embedded minimal annulus with the same boundary as $\mathcal{T}$, and this surface is called the Schwarz H surface.

As explained in Section 1, there are infinitely many different ways (by using combinations of reflections and 180° rotations that are not allowed in the smooth case) to extend $\mathcal{T}$ to a complete discrete minimal surface. Two such ways are shown in the upper right of Figure 2, and another two ways are shown in the center and right-hand side of Figure 6. The two examples in Figure 2 and the central one in Figure 6 can be extended to complete triply-periodic discrete minimal surfaces by 180° rotations about boundary edges. The right-most example in Figure 6 can be extended to a complete triply-periodic discrete minimal surface by using horizontal translations perpendicular to the axis ℓ that generate a 2-dimensional hexagonal grid, and then by applying vertical translations parallel to ℓ of length $2\delta(j_1 - j_0)$. The upper-right examples in both Figures 2 and 6 are applications of **Method 2**.

Remark 5.1. *When $k = 4$ and $j_0 = -j_1 = 1$, and when r and δ are chosen properly, this $\mathcal{T}$ can produce the same surface as $\mathcal{T}_1$ produced in Subsection 5.1. The way of triangulating the planar isosceles trapezoids was different in Subsection 5.1, but by Remark 2.1 this is irrelevant to the minimality of the surfaces, and the two examples are the same in the sense that they have the same traces in $\mathbb{R}^3$.*

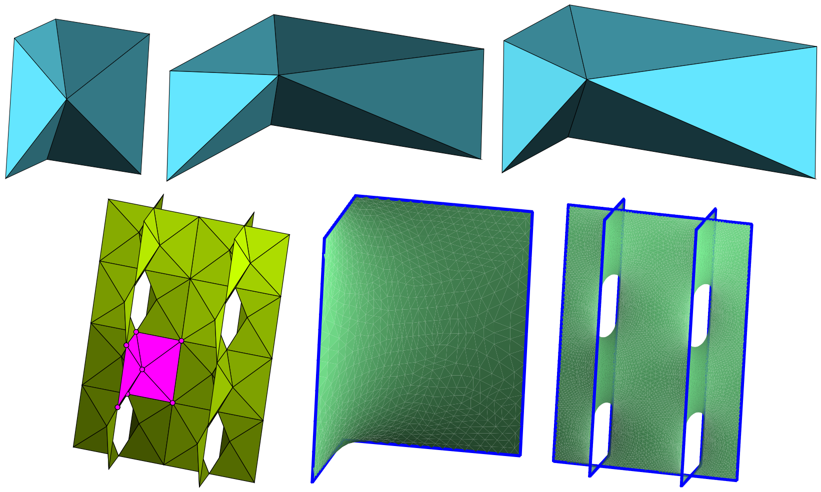

Figure 7: Discrete and smooth Schwarz CLP surfaces.

6 Other examples

6.1 Discrete Schwarz CLP surface

The fundamental piece $\mathcal{T}$ here has six boundary vertices

$$p_1 = (x, 0, 0),\;\; p_2 = (0, 0, 0),\;\; p_3 = (0, y, 0),$$

$$p_4 = (0, y, 1),\;\; p_5 = (0, 0, 1),\;\; p_6 = (x, 0, 1)$$

for any given fixed $x, y > 0$, and has one interior vertex

$$p_7 = (a, b, \frac{1}{2})\;.$$

There are six triangles in $\mathcal{T}$, which are

$$(p_j, p_{j+1}, p_7),\; j = 1, ..., 5\;, \quad (p_6, p_1, p_7)\;.$$

The complete triply-periodic surface is generated by 180° rotations about boundary edges, continuing to make such rotations until the surface is complete. Every vertex in $\partial\mathcal{T}$, and every vertex that is an image of a vertex in $\partial\mathcal{T}$ under these rotations, satisfies Equation (2.4), because of the symmetry of the surface. The condition for Equation (2.4) to hold at the interior vertex p_7 and all images of p_7 under these rotations is that

$$\frac{2ya}{\sqrt{a^2 + \frac{1}{4}}} + \frac{a}{\sqrt{a^2 + (y-b)^2}} + \frac{a-x}{\sqrt{b^2 + (x-a)^2}} = 0\;,$$

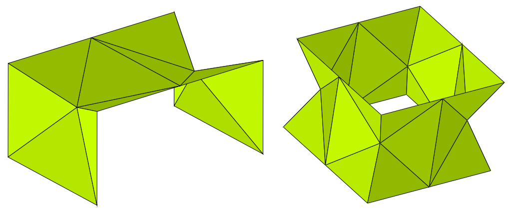

Figure 8: Variants of the discrete Schwarz CLP surface.

$$\frac{2xb}{\sqrt{b^2+\frac{1}{4}}}+\frac{b}{\sqrt{b^2+(x-a)^2}}+\frac{b-y}{\sqrt{a^2+(y-b)^2}}=0\ .$$

When $x=y=\sqrt{2}/2$, one explicit solution is $a=b=\frac{\sqrt{2}-1}{2}$. The fundamental piece $\mathcal{T}$ and a larger part of the resulting complete surface are shown in the left-most column of Figure 7 for these values of x, y, a, and b.

For general choices of x and y, there is always a solution to the above system of two equations with respect to the two variables a and b, thus giving $\nabla_{p_7}\,\text{area}\,\mathcal{T}=0$. Thus, for general x and y, the minimality condition for this example is two-dimensional. Fundamental pieces $\mathcal{T}$ for other choices of x and y are shown in the upper-center and upper-right of Figure 7 ($x=y$ in the center and $x\neq y$ on the right).

We can also apply **Method 2** here. For example, suppose we include the reflection of $\mathcal{T}$ across the plane P containing the three points $(x,0,0)$, $(x,1,0)$, $(x,0,1)$ along with $\mathcal{T}$ to get a discrete minimal surface $\mathcal{T}_1$ with twelve triangles, see the left-hand side of Figure 8. (Such a reflection across P would not be allowed for the smooth Schwarz CLP surface.) We can then extend $\mathcal{T}_1$ to a complete triply-periodic discrete minimal surface by 180° rotations about boundary edges, and this surface is yet another discrete superman surface.

For a second example of applying **Method 2**, suppose we include the reflection of $\mathcal{T}_1$ across the plane Q containing the three points $(0,y,0)$, $(1,y,0)$, $(0,y,1)$ along with $\mathcal{T}_1$ to get a discrete minimal surface $\mathcal{T}_2$ with twenty-four triangles, see the right-hand side of Figure 8. (Such a reflection again would not be allowed in the smooth case.) We can then extend $\mathcal{T}_2$ to a complete triply-periodic discrete minimal surface by 180° rotations about boundary edges, and this gives yet another discrete Schwarz P surface.

6.2 Discrete I-Wp and F-Rd surfaces

The fundamental piece $\mathcal{T}$ of this I-Wp example has six vertices

$$p_1=(b,0,b),\ \ p_2=(b,0,0),\ \ p_3=(b,b,0),$$

$$p_4=(1,1,a),\ \ p_5=(1,a,1),\ \ p_6=\overline{p_1p_4}\cap\overline{p_3p_5}\ .$$

There are five triangles in $\mathcal{T}$, which are

$$(p_1,p_2,p_3),\ \ (p_1,p_3,p_6),\ \ (p_3,p_4,p_6),\ \ (p_4,p_5,p_6),\ \ (p_5,p_1,p_6).$$

By including the two images of $\mathcal{T}$ under the two reflections across the planes $\{(x,x,z)\,|\,x,z\in R\}$ and $\{(x,y,x)\,|\,x,y\in R\}$, we have a larger discrete surface with fifteen triangles. Reflecting this larger piece across all planes of the form $\{(x,y,k)\,|\,x,y\in\mathbb{R}\}$, $\{(x,k,z)\,|\,x,z\in\mathbb{R}\}$, $\{(k,y,z)\,|\,y,z\in\mathbb{R}\}$ for all integers k, we arrive at a complete embedded triply-periodic surface in $\mathbb{R}^3$. See the right-hand side of Figure 9.

The minimality condition that Equation (2.4) holds at each vertex of the complete triply-periodic surface is

$$1+a+a^2-3b-2ab+2b^2=0\ , \tag{6.8}$$

$$a^2+a(2-3b)+b\left(3b-3+\sqrt{(1+a-b)^2+2(1-b)^2}\right)=0\ . \tag{6.9}$$

Thus, to make the surface minimal, we must find a and b satisfying Equations (6.8)-(6.9), so the minimality condition is two-dimensional. Equation (6.8) holds if

$$a=\frac{1}{2}\left(2b-1+\sqrt{-3+8b-4b^2}\right)\ ,$$

and then Equation (6.9) will hold if b satisfies

$$\left(3-\sqrt{-3+8b-4b^2}\right)(1-b)=\sqrt{2}\sqrt{3-4b+2b^2+\sqrt{-3+8b-4b^2}}\ .$$

One can find such a real number b in a completely explicit form (although not in such simple forms like in Subsections 4.3, 5.2 and 6.1).

One can similarly find a discrete analog, shown on the left-hand side of Figure 9, of the smooth triply-periodic minimal F-Rd surface. With the simplicial structure chosen in Figure 9, one can again explicitly solve the minimality condition, in the same way as we did for the I-Wp example.

6.3 Trigonal example

The fundamental piece $\mathcal{T}$ of this H-T example has six boundary vertices

$$p_1=(\frac{a}{2},\frac{\sqrt{3}a}{2},b),\ \ p_2=(\frac{1}{2},\frac{\sqrt{3}}{2},c),\ \ p_3=(2-\frac{3s}{2},\frac{\sqrt{3}s}{2},0),$$

$$p_4=(2-\frac{3s}{2},-\frac{\sqrt{3}s}{2},0),\ \ p_5=(\frac{1}{2},-\frac{\sqrt{3}}{2},c),\ \ p_6=(\frac{a}{2},-\frac{\sqrt{3}a}{2},b)$$

for any given fixed $b>0$, and has one interior vertex

$$p_7=\frac{1}{2}(p_2+p_5)\ .$$

There are six triangles in $\mathcal{T}$, which are

$$(p_j,p_{j+1},p_7),\ j=1,...,5\ ,\quad (p_6,p_1,p_7).$$

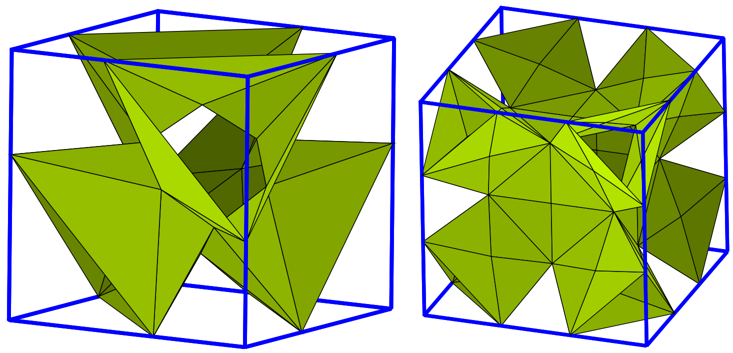

Figure 9: Discrete versions of A. Schoen's F-Rd surface and I-Wp surfaces.

Including the images of $\mathcal{T}$ under the $120°$ and $240°$ rotations about the axis $\{(0,0,z)\,|\,z \in \mathbb{R}\}$, and also including the images of $\mathcal{T}$ and these two rotated copies of $\mathcal{T}$ under reflection across the plane $\{(x,y,0)\,|\,x,y \in \mathbb{R}\}$, one has the larger piece shown on the left-hand side of Figure 10. This larger piece has five boundary components, each contained in a plane, and these five planes bound a trigonal prism (a prism of height $2b$ over an equilateral triangle with edge-lengths $2\sqrt{3}$). Including the images of this larger piece by reflecting across these five planes, and also by including all subsequent images of reflections across planes containing subsequent boundary components, one arrives at a triply-periodic discrete surface, which is embedded when $a, s \in (0,1)$ and $c \in (0,b)$. A larger portion of this complete discrete surface is shown in the central figure of Figure 10.

The minimality condition involves three equations in the three variables a, c, s, and so is three-dimensional. We will not show the equations here, but they can be solved explicitly. For example, when $b = 1$, the following choices ensure minimality:

$$a = s = \frac{2+\sqrt{2}}{4}\ , \quad c = \frac{3}{4} \qquad (\text{and } \ b = 1)\ .$$

In fact, these choices also ensure that all of the vertices of $\mathcal{T}$ lie in the same plane, and hence the fundamental piece $\mathcal{T}$ is planar and could be freely triangulated within its trace (see Remark 2.1).

Furthermore, rather than using a portion of the complete surface within a trigonal prism as a building block for the complete surface, one could have instead used a portion within a hexagonal prism as the building block. The figure on the right-hand side of Figure 10 will produce exactly the same complete surface (again by reflecting across planes containing boundary components). In the case of smooth H-T surfaces, this same duality exists between building blocks in trigonal and hexagonal prisms, as noted in [11].

6.4 Discrete Fischer-Koch example

An interesting triply-periodic smooth embedded minimal surface was found by W. Fischer and E. Koch [4], and is shown in the bottom row of Figure 11. Here we give a discrete minimal analog of this surface, shown in the top row of Figure 11.

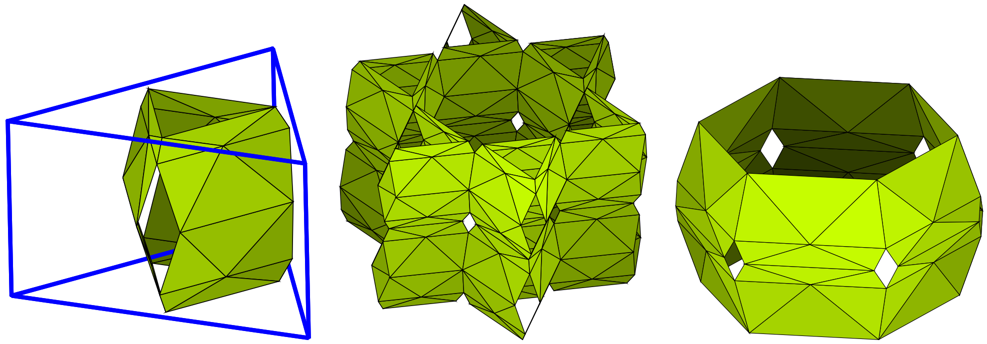

Figure 10: Discrete version of A. Schoen's H-T surface.

The fundamental piece $\mathcal{T}$ of this example has eight boundary vertices

$$p_1 = (0,0,-1),\ \ p_2 = (0,0,-2),\ \ p_3 = (a,0,-2),\ \ p_4 = (a,0,1),$$

$$p_5 = (0,0,1),\ \ p_6 = (0,0,2),\ \ p_7 = (\frac{a}{2}, \frac{\sqrt{3}a}{2}, 2),\ \ p_8 = (\frac{a}{2}, \frac{\sqrt{3}a}{2}, -1)$$

for any given fixed $a > 0$, and has one interior vertex

$$p_9 = (\frac{\sqrt{3}b}{2}, \frac{b}{2}, 0)$$

with $0 < b < a$. There are eight triangles in $\mathcal{T}$, which are

$$(p_j, p_{j+1}, p_9),\ j = 1, ..., 7\ ,\quad (p_8, p_1, p_9).$$

The complete triply-periodic surface is generated by 180° rotations about boundary edges. In this example, the symmetry Equation (2.4) holds at each vertex p_1,...,p_8 in the resulting complete triply-periodic surface. However, getting this to hold at p_9 requires the proper choice of b. This minimality condition at p_9 is one-dimensional, and one can prove existence of a value $b \in (0, a)$ solving it.

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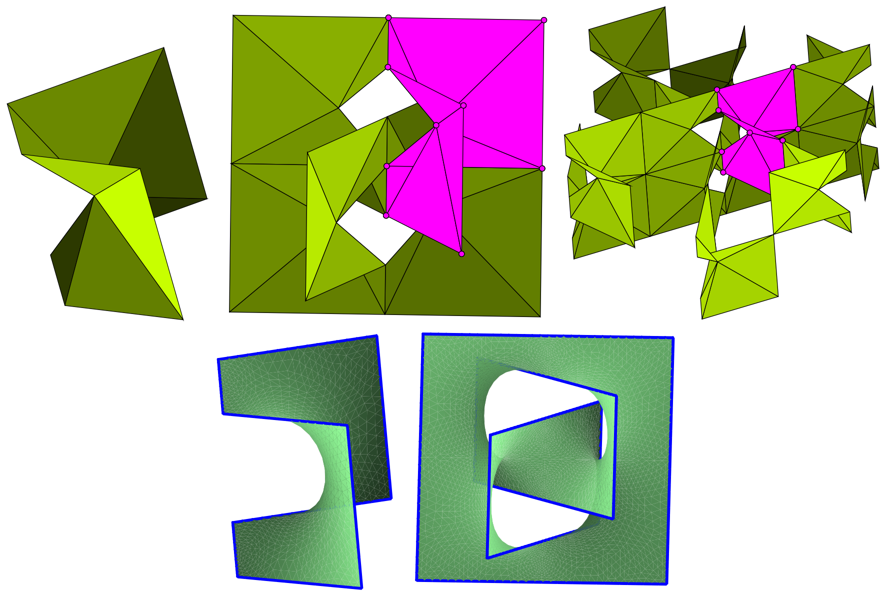

Figure 11: Discrete and smooth triply-periodic Fischer-Koch surfaces.

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