B.Y. Chen's inequality for semi-slant submanifolds in $T-$space forms

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Abstract. In this paper, B. Y. Chen inequality for semi-slant submanifolds in $T-$space forms are established by using subspaces orthogonal to the structure vector fields.

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1 Introduction

Given a Riemannian manifold M, for each point $p \in M$, put

$$(\inf K)(p) = \{\inf K(\pi): \text{ plane section } \pi \subset T_pM\},$$

where $K(\pi)$ denotes the sectional curvature of M associated with π. Let

$$\delta_M(p) = \tau(p) - \inf K(p),$$

being τ the scalar curvature of M. Then, δ_M is a well-defined Riemannian invariant, which was recently introduced by Chen ([14], [13]).

For submanifolds M in a real space form $\widetilde{R}^m(c)$ of constant sectional curvature c, Chen gave the following basic inequality involving the instrinsic invariant δ_M and the squared mean curvature of immersion,

$$\delta_M(p) \leq \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + \frac{(n+1)(n-2)}{2} c,$$

where n denotes the dimension of M and H is the mean curvature vector.

Several authors have studied the above inequalities on different mathematical structures (see, [8], [11], [14], [12], [15], [18]- [20], [24], [27]).

A slant submanifold is defined as a submanifold N of an almost Hermitian manifold (M, J) with constant Wirtinger angle (= Kahler angle). The Wirtinger angle $\theta(X)$ of a tangent vector X to N at a point $p \in N$ is the angle between JX and the

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tangent space of N at p. Special cases are complex submanifolds ($\theta = 0$) and totally real submanifolds ($\theta = \frac{\pi}{2}$) by Chen in [16]. Several authors have studied on slant submanifold (see, for instances, [4], [5], [9], [8], [11], [16], [23]).

In 1994, Papaghiuc defines a class of submanifolds, called the semi-slant submanifolds, of a Kahlerian manifold to be a submanifold whose tangent bundle is the direct sum of a complex distribution and a slant distribution with the slant angle $\theta = 0$. The author obtains the necessary and sufficient conditions for the complex and slant distributions to be integrable. He also obtains a necessary and sufficient condition for a semi-slant submanifold to be the Riemannian product of a complex submanifold and a slant submanifold [25].

The notion of a semi-slant submanifold of a Sasakian manifold was introduced by Cabrerizo et. al in [16]. The authors define and study both bi-slant and semi-slant submanifolds of an almost contact metric manifold and, in particular, of a Sasakian manifold. They prove a characterization theorem for semi-slant submanifolds and we obtain integrability conditions for the distributions which are involved in the definition of such submanifolds. We also study an interesting particular class of semi-slant submanifolds.

In [11], Cioroboiu established Chen inequalities for semi-slant submanifolds in Sasakian space forms by using subspace orthogonal to the Reeb vector field ξ.

In [8], the authors established a version of Chen inequality for submanifold of an $S-$space form tangent to the structure vector field of the ambient space and applications to the case of slant immersions are obtained.

In this paper B. Y. Chen's sharp estimation for the sectional curvature of a submanifold in Riemannian space forms in terms of the scalar curvature is extended to semi-slant submanifolds in $T-$space forms. The paper is organized as follows. In section 2, we give a brief account of $T-$manifolds and their submanifolds, for later use. In section 3, B. Y. Chen inequality for semi-slant submanifolds in $T-$space forms are established by using subspaces orthogonal to the structure vector fields.

2 Preliminaries

Let $(\widetilde{M}, g)$ be a Riemannian manifold with $dim(\widetilde{M}) = 2m + s$ and denote by $T\widetilde{M}$ the Lie algebra of vector field in $\widetilde{M}$. Then $\widetilde{M}$ is said to be an $S-$Manifold if there exist on $\widetilde{M}$ an $f-$structure ϕ [26] of rank $2m$ and s global vector fields $\xi_1, ..., \xi_s$(structure vector fields) such that [2]

(i) If $\eta^1, ..., \eta^s$ are dual $1-$forms of $\xi_1, ..., \xi_s$, then:

$$\phi\xi_i = 0, \quad \eta^i \circ \phi = 0, \quad \phi^2 = -I + \sum_{i=1}^{s} \eta^i \otimes \xi_i, \tag{2.1}$$

$$\widetilde{g}(\phi X, \phi Y) = \widetilde{g}(X, Y) - \sum_{i=1}^{s} \eta^i(X)\eta^i(Y), \tag{2.2}$$

$$\widetilde{g}(X, \xi_i) = \eta^i(X), \tag{2.3}$$

for any $X, Y \in T\widetilde{M}$, $i = 1, ..., s$.

(ii) The $f-$structure ϕ is normal, that is

$$[\phi, \phi] + 2\sum_{i=1}^{s} \xi_i \otimes d\eta^i = 0$$

where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ.

(iii) $\eta^1 \wedge ... \wedge \eta^s \wedge (d\eta^i)^s \neq 0$ and for each i, $d\eta^i = 0$.

In a $T-$manifold $\widetilde{M}$, besides the relations (2.1) and (2.2) the following also hold:

$$(\nabla_X \phi) Y = 0 \tag{2.4}$$

$$\nabla_X \xi_i = 0 \tag{2.5}$$

for any vector fields $X, Y \in T\widetilde{M}$.

Let $\widetilde{D}$ denote the distribution determined by $-\phi^2$ and $\widetilde{D}^\perp$ the complementary distribution. $\widetilde{D}^\perp$ is determined by $\phi^2 + I$ and spanned by $\xi_1, ..., \xi_n$. If $X \in \widetilde{D}$, then $\eta^i(X) = 0$ for any i and if $X \in \widetilde{D}^\perp$, then $\phi X = 0$.

A plane section Π in $T_p\widetilde{M}$ of an $T-$manifold $\widetilde{M}$ is called a $\phi-$section if $\Pi \perp \widetilde{D}^\perp$ and $\phi(\Pi) = \Pi$. $\widetilde{M}$ is of constant $\phi-$sectional curvature [2] if at each point $p \in \widetilde{M}$, the sectional curvature $\widetilde{K}(\Pi)$ does depend on the choice of the $\phi-$section Π of $T_p\widetilde{M}$. If $\widetilde{K}(\Pi)$ is constant for all non-null vectors in Π, we call $\widetilde{M}$ to be of constant $\phi-$sectional curvature at point p. The function of c defined by $c(p) = \widetilde{K}(\Pi)$ is called the $\phi-$sectional curvature of $\widetilde{M}$. A $T-$manifold $\widetilde{M}$ with constant $\phi-$sectional curvature c is said to be a $T-$space form and is denoted by $\widetilde{M}(c)$.

The curvature tensor $\widetilde{R}$ of a $T-$ space form $\widetilde{M}(c)$ is given in [22],

$$\begin{aligned}\widetilde{g}(\widetilde{R}(X,Y)Z,W) &= \frac{c}{4}\{\widetilde{g}(X,Z)\widetilde{g}(Y,W) - \widetilde{g}(Y,Z)\widetilde{g}(X,W)\\ &-\widetilde{g}(X,Z)\sum u^i(Y)u^i(W) - \widetilde{g}(Y,W)\sum u^i(Z)u^i(X)\\ &+\widetilde{g}(X,W)\sum u^i(Y)u^i(Z) + \widetilde{g}(Y,Z)\sum u^i(X)u^i(W)\\ &+\Big(\sum u^i(Z)u^i(X)\Big)\Big(\sum u^i(Y)u^i(W)\Big) - \Big(\sum u^i(W)u^i(X)\Big)\\ &+\Big(\sum u^i(Y)u^i(Z)\Big) + \widetilde{g}(W,\phi X)\widetilde{g}(Y,\phi Z) + \widetilde{g}(Y,\phi W)\widetilde{g}(X,\phi Z)\\ &-2\widetilde{g}(X,\phi Y)\widetilde{g}(W,\phi Z)\}\end{aligned} \tag{2.6}$$

When $s = 0$, an $T-$manifold $\widetilde{M}$ becomes a keahler manifold. When $s = 1$, an $T-$manifold $\widetilde{M}$ becomes a cosymplectic manifold [22].

The Equation of Gauss for submanifold M of $\widetilde{M}$ is given by

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) &= R(X,Y,Z,W) + g(h(X,W),h(Y,Z))\\ &-g(h(X,Z),h(Y,W)),\end{aligned} \tag{2.7}$$

for any vectors X, Y, Z and W tangent to M, where we denote as usual $R(X,Y,Z,W) = g(R(X,Y)Z,W)$.

From now on, we suppose that the structure vector fields are tangent to M and we denote by $n+s$ the dimension of M. We consider $n \geq 2$. Hence, if we denote by $L = D_1 \oplus D_2$ the orthogonal distribution to $\widetilde{D}^{\perp}$ in TM. We can write orthogonal direct decomposition $TM = L \oplus \widetilde{D}^{\perp}$.

For an orthonormal basis $\{e_1, ..., e_n, e_{n+1}, ..., e_{n+s}\}$ of T_pM, the scalar curvature τ at p is defined by

$$\tau = \sum_{i<j} K(e_i \wedge e_j), \tag{2.8}$$

where $K(e_i \wedge e_j)$ denotes the sectional curvature of M associated with the plane section spanned by $e_i,\ e_j$. In particular, if we put $e_{n+\alpha} = \xi_\alpha$, for $\alpha = 1, 2, ..., s$, then (2.8) implies:

$$2\tau = \sum_{i \neq j} K(e_i \wedge e_j) + 2 \sum_{i=}^{n} \sum_{\alpha=1}^{s} K(e_i \wedge \xi_\alpha). \tag{2.9}$$

We denote by H the mean curvature vector, that is

$$H(p) = \frac{1}{n+s} \sum_{i=1}^{n+s} h(e_i, e_i).$$

Also, we set

$$h_{ij}^r = g(h(e_i, e_j), e_r)$$

and

$$\|h\|^2 = \sum_{i,j=1}^{n+s} g(h(e_i, e_j), h(e_i, e_j)).$$

For any $X \in TM$, we put $\phi X = PX + FX$, where PX and FX are the tangential and normal component of ϕX, respectively. We denote

$$\|P\|^2 = \sum_{i,j=1}^{n+s} g^2(Pe_i, e_j).$$

It is well-known that

$$g(PX, Y) = -g(X, PY), \tag{2.10}$$

for any $X \in TM$.

Now, we give the relevant definitions from [11].

Definition 2.1. *A differentiable distribution D on M is called a slant distribution if for each $x \in M$ and each nonzero vector $x \in D_x$, the angle $\theta_D(X)$ between ϕX and the vector subspace D_x is constant, which is independent of the choice of $x \in M$ and $x \in D_x$. In this case, the constant angle θ_D is called the slant angle of the distribution D.*

Definition 2.2. *A submanifold M tangent to structure vector fields is said to be a bislant submanifold of $\widetilde{M}$ if there exist two orthogonal distributions D_1 and D_2 on M such that*
(i) TM admits the orthogonal direct decomposition $TM = D_1 \oplus D_2 \oplus \widetilde{D}^{\perp}$
(ii) for any $i = 1, 2$ D_i is slant distribution with slant angle θ_i.

Definition 2.3. *A submanifold M tangent to structure vector fields is said to be a semi-slant submanifold of $\widetilde{M}$ if there exist two orthogonal distributions D_1 and D_2 on M such that*
(i) TM admits the orthogonal direct decomposition $TM = D_1 \oplus D_2 \oplus \widetilde{D}^{\perp}$
(ii) the distribution D_1 is an invariant distribution, that is, $\phi(D_1) = D_1$
(iii) the distribution D_2 is slant with angle $\theta \neq 0$.

Definition 2.4. *A submanifold M is said to be a slant if for any $p \in M$ and any $X \in T_pM$, linearly independent on structure vector fields, the angle between ϕX and T_pM is a constant $\theta \in [0, \pi/2]$, called the slant angle of M in $\widetilde{M}$.*

Invariant and anti-invariant immersions are slant immersions with slant angles $\theta = 0$ and $\theta = \pi/2$, respectively. A slant immersion which is neither invariant nor anti-invariant is called a proper slant immersion [4].

In [3], the invariant distribution of a semi-slant submanifold is a slant distribution with zero angle. Thus, it is obvious that, in fact, semi-slant submanifolds are particular cases of bislant submanifolds. Moreover, it is clear that if $\theta = \pi/2$, then the semi-slant submanifold is a semi-invariant submanifold.

(a) If $D_2 = 0$, then M is an invariant submanifold.
(b) If $D_1 = 0$ and $\theta = \pi/2$, then M is an anti-invariant submanifold.
(c) If $D_1 = 0$ and $\theta \neq \pi/2$, then M is a proper slant submanifold with slant angle θ. A semi-slant submanifold is said to be proper if both D_1 and D_2 are nontrivial and $\theta \neq \pi/2$.

3 B.Y. Chen's inequality

First, we repeat an algebraic lemma from [14] without proofs.

Lemma 3.1. *[14] Let $a_1, ..., a_k, c$ be $k+1 (k \geq 2)$ real numbers such that*

$$\left(\sum_{i=1}^{k} a_i\right)^2 = (k-1)\left(\sum_{i=1}^{k} a_i^2 + c\right).$$

Then, $2a_1a_2 \geq c$, with the equality holding if and only if $a_1 + a_2 = a_3 = ... = a_k$.

Now, we can prove the following version for the semi-slant submanifolds of $T-$manifolds of Theorem 3 of [12].

Theorem 3.1. *Let $\varphi : M^{n+s} \to \widetilde{M}^{2m+s}(c)$ be an isometric immersion from a Riemannian $(n+s)-$dimensional manifold in to an $T-$space form $\widetilde{M}^{2m+s}(c)$, such that the structure vector fields are tangent to M. Then,*
(i) For any plane section π invariant by P and tangent to D_1

$$\tau - K(\pi) \leq \frac{(n+s)^2(n+s-2)}{2(n+s-1)}\|H\|^2 + \frac{c}{4}\left(\frac{n(n-1)}{2} + \frac{3}{2}\left((d_1+1) + d_2\cos^2\theta + \frac{s(1-s)}{2}\right)\right) \tag{3.1}$$

(ii) For any plane section π invariant by P and tangent to D_2

$$\tau - K(\pi) \leq \frac{(n+s)^2(n+s-2)}{2(n+s-1)}\|H\|^2 + \frac{c}{4}\left(\frac{n(n-1)}{2} + \frac{3}{2}\left(d_1 + (d_2+1)\cos^2\theta + \frac{s(1-s)}{2}\right)\right) \tag{3.2}$$

The equality case of inequalities (3.1) and (3.2) holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, ..., e_n, e_{n+1} = \xi_1, ..., e_{n+s} = \xi_s\}$ of T_pM and an orthonormal basis $\{e_{n+s+1}, ..., e_{2m+s}\}$ of $T_p^{\perp}M$ such that the shape operators of M in $\widetilde{M}(c)$ at p have the following forms:

$$A_{n+s+1} = \begin{pmatrix} a & 0 & 0 & . & . & . & 0 \\ 0 & -a & 0 & . & . & . & 0 \\ 0 & 0 & & & O_{n+s-2} & & \end{pmatrix},$$

$$A_r = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & . & . & 0 \\ h_{21}^r & -h_{21}^r & 0 & . & . & 0 \\ 0 & 0 & & O_{n+s-2} & & \end{pmatrix}, \quad r = n+s+2, ..., 2m+r.$$

Proof. Let $p \in M$, $\{e_1, ..., e_n, e_{n+1} = \xi_1, ..., e_{n+s} = \xi_s\}$ an orthonormal basis of T_pM, and $\{e_{n+s+1}, ..., e_{2m+s}\}$ an orthonormal basis of $T_p^{\perp}M$. Let M be a proper semi-slant submanifold of $\widetilde{M}(c)$ and $\dim M = n + s = 2d_1 + 2d_2 + s$. We consider an adapted semi-slant orthonormal frames

$$\begin{aligned} e_1, e_2 &= \phi e_1, ..., e_{2d_1-1}, e_{2d_1} = \phi e_{2d_1-1}, e_{2d_1+1}, \\ e_{2d_1+2} &= \frac{1}{\cos\theta} P e_{2d_1+1}, ..., e_{2d_1+2d_2-1}, e_{2d_1+2d_2} \\ &= \frac{1}{\cos\theta} P e_{2d_1+2d_2-1} \\ e_{2d_1+2d_2+1} &= \xi_1, ..., e_{2d_1+2d_2+s} = \xi_s \end{aligned} \tag{3.3}$$

one can obtain easily,

$$g^2(\phi e_i, e_{i+1}) = \begin{cases} 1, & \text{for } i \in \{1, 2, ..., 2d_1 - 1\} \\ \cos^2\theta & \text{for } i \in \{2d_1+1, ..., 2d_1+2d_2-1\}. \end{cases} \tag{3.4}$$

Then

$$\sum_{i,j=1}^{n+s} g^2(e_j, \phi e_i) = 2\left(d_1 + d_2\cos^2\theta\right). \tag{3.5}$$

From (2.6), (2.9) and (3.5) we obtain

$$(3.6)\qquad 2\tau = (n+s)^2\|H\|^2 - \|h\|^2 + \frac{c}{4}\left(n(n-1) + 3\left(d_1 + d_2\cos^2\theta + s(1-s)\right)\right).$$

Put

$$(3.7)\qquad \varepsilon = 2\tau - \frac{(n+s)^2(n+s-2)}{2(n+s-1)}\|H\|^2 - \frac{c}{4}\left(n(n-1) + 3\left(d_1 + d_2\cos^2\theta + s(1-s)\right)\right).$$

Hence, (3.6) and (3.7) imply:

$$(3.8)\qquad (n+s)^2\|H\|^2 = (n+s-1)\left(\varepsilon + \|h\|^2\right)$$

Let $p \in M$, $\pi \subset T_pM$, $\dim \pi = 2$, and π orthogonal to $\widetilde{D}^{\perp}$ and invariant by P.

We consider two cases.

Case I. The plane section π is tangent to D_1. We may assume that $\pi = \mathrm{sp}\{e_1, e_2\}$ and $e_{n+s+1} = \dfrac{H}{\|H\|}$. Then, relation (3.8) becomes

$$(3.9)\qquad \left(\sum_{i=1}^{n+s} h_{ii}^{n+s+1}\right)^2 = (n+s-1)\left[\sum_{i,j=1}^{n+s}\sum_{r=n+s+1}^{2m+s}(h_{ij}^r)^2 + \varepsilon\right].$$

Using Lemma 3.1, we derive from (3.9)

$$(3.10)\qquad 2h_{11}^{n+s+1}h_{22}^{n+s+1} \geq \sum_{i\neq j}(h_{ij}^{n+s+1})^2 + \sum_{i,j=1}^{n+s}\sum_{r=n+s+2}^{2m+s}(h_{ij}^r)^2 + \varepsilon$$

From (2.6) and (3.4), we obtain

$$(3.11)\qquad K(\pi) = h_{11}^{n+s+1}h_{22}^{n+s+1} - \left(h_{12}^{n+s+1}\right)^2 + \sum_{r=n+s+2}^{2m+s}\left(h_{11}^{n+s+1}h_{22}^{n+s+1} - \left(h_{12}^{n+s+1}\right)^2\right) + c\}$$

Now, from (3.10) and (3.11) it follows that

$$K(\pi) \geq \sum_{r=n+s+1}^{2m+s}\sum_{j>2}\left[\left(h_{1j}^r\right)^2 + \left(h_{2j}^r\right)^2\right] + \frac{1}{2}\sum_{i\neq j>2}\left(h_{ij}^{n+s+1}\right)^2 + \frac{1}{2}\sum_{r=n+s+1}^{2m+s}\sum_{i,j>2}(h_{ij}^r)^2 + \frac{1}{2}\sum_{r=n+s+1}^{2m+s}\left(h_{11}^2 + h_{22}^r\right)^2 + \frac{\varepsilon}{2} + \frac{4c}{4} \geq \frac{\varepsilon}{2} + c$$

or, equivalently,

$$\varepsilon = \tau - \frac{(n+s)^2(n+s-2)}{2(n+s-1)}\|H\|^2 - \frac{c}{8}\left(n(n-1) + 3\left(d_1 + d_2\cos^2\theta + s(1-s)\right)\right).$$

$$
\begin{aligned}
(3.12) \qquad K(\pi) &\geq \frac{\varepsilon}{2}+\frac{4c}{4} \\
&= \tau-\frac{(n+s)^2(n+s-2)}{2(n+s-1)}\|H\|^2 \\
&\quad -\frac{c}{4}\left(\frac{n(n-1)}{2}+\frac{3}{2}\left(d_1+d_2\cos^2\theta+\frac{s(1-s)}{2}\right)+4\right).
\end{aligned}
$$

Case II. If the plane section π is tangent to D_2, similar to the proof of Case I, one can obtain (3.2)

The case of equality at appoint $p \in M$ holds if and only if it achieves the equalities in inequalities (3.10), (3.11) and (3.12). So, we have

$$h_{ij}^{n+s+1}=0,\ i\neq j>2;$$

$$h_{ij}^{r}=0,\ i\neq j>2,\ r=n+s+2,...,2m+r;\ i,j=3,...,n+s;$$

$$h_{11}^{r}+h_{22}^{r}=0,\ r=n+s+2,...,2m+r$$

$$h_{1j}^{n+s+1}=h_{2j}^{n+s+1}=0,\ j>2;$$

$$h_{11}^{n+s+1}+h_{22}^{n+s+1}=h_{33}^{n+s+1}=...=h_{n+s.n+s}^{n+s+1}.$$

Moreover we may choose e_1,e_2 such that $h_{12}^{n+s+1}=0$ and we denote $a=h_{11}^{r},\ b=h_{22}^{r}$ and $\mu=h_{33}^{n+s+1}=...=h_{n+s.n+s}^{n+s+1}$. Also, from Lemma 3.1 and (2.10), one can see that $h_{11}^{n+s+1}+h_{22}^{n+s+1}=h_{33}^{n+s+1}=...=h_{n+s.n+s}^{n+s+1}=0$. Hence, the shape operator take the desired forms. The converse follows from direct calculation. □

We can define

$$(\inf{}_L K)(p)=\{\inf K(\pi):\ \text{plane section } \pi\subset L_p\}.$$

Then $\inf{}_L K$ is a well-defined function on M. Let δ_M^L denote the difference between the scalar curvature and $\inf{}_L K$ that is

$$\delta_M^L(p)=\tau(p)-\inf{}_L K(p).$$

It is clear that $\delta_M^L\leq\delta_M$. Then, if $c=0$, from (3.1) and (3.2) we get directly following result:

Corollary 3.1. *Let $\varphi: M^{n+s}\to\widetilde{M}^{2m+s}(c)$ be an isometric immersion from a Riemannian $(n+s)-$dimensional manifold in to an $T-$space form $\widetilde{M}^{2m+s}(c)$, such that the structure vector fields are tangent to M. Then, for any plane section π invariant by P and tangent to D_1 or D_2*

$$\delta_M^L(p)\leq\frac{(n+s)^2(n+s-2)}{2(n+s-1)}\|H\|^2. \tag{3.13}$$

References

[1] D. E. Blair, *Contact manifold in Riemannian Geometry*, Lecture Notes in Mathematics, 509, Springer Verlag, 1976.

[2] D. E. Blair, *Geometry of manifolds with structural group $U(n)xO(s)$*, J. Diff. Geom, 4 (1970), 155-167.

[3] J. L. Cabrerizo, A. Carriazo, L. M. Fernández and M. Fernández, *Semi-slant submanifolds of a Sasakian manifold*, Geometriae Dedicata, 78 (1999), 183-199.

[4] J. L. Cabrerizo, A. Carriazo, L. M. Fernández and M. Fernández, *Slant submanifolds in Sasakian manifolds*, Glasgow Mathematical Journal 42, 1 (2000), 125-138.

[5] J. L. Cabrerizo, A. Carriazo, L. M. Fernández and M. Fernández, *Structure on a slant submanifold of a contact manifold*, Indian J. Pure Appl. Math. 31, 7 (2000), 857-864.

[6] J. L. Cabrerizo, L. M. Fernández and M. Fernández, *The curvature of submanifolds of an $S-$space form*, Acta Math. Hung. 62 (1993), 373-383.

[7] J. L. Cabrerizo, L. M. Fernández and M. Fernández, *The curvature tensor field on $f-$manifolds with complemented frames*, An. St. Univ. "Al. I. Cuza" Iasi Matematica 36 (1990), 151-161.

[8] J. L. Cabrerizo, L. M. Fernández, M. B. Hans-Uber, *B. Y. Chen's inequality for S-space-forms: applications to slant immersions*, Indian J. Pure Appl. Math. 34, 9 (2003), 1287-1298.

[9] J. L. Cabrerizo, L.M Fernández, M. B, Hans-Uber, *Some slant submanifolds of $S-$manifolds*, Acta Math. Hung. 107, 4 (2005), 267-285.

[10] A. Carriazo, *New developments in slant submanifolds theory, in: Applicable Mathematics in the Golden Age* (Edited by J. C. Mishra), Narosa Publishing House (New Delhi, 2002), 339–356.

[11] D. Cioroboiu, *B. Y. Chen inequalities for semi-slant submanifolds in Sasakian space forms*, Int. J. Math. Math. Sci, 27 (2003), 1731-1738.

[12] B. Y. Chen, *A general inequality for submanifolds in complex space forms and its applications*, Arch. Math. 67 (1996), 519-528.

[13] B. Y. Chen, *A Riemannian invariant and its applications to submanifold theory*, Results in Mathematics 27 (1995), 17-26.

[14] B. Y. Chen, *Some pinching and classification theorems for minimal submanifolds*, Arch. Math. 60 (1993), 568-578.

[15] B. Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken, *Totally real submanifolds of CP^n satisfying a basic equality*, Arch. Math. 63 (1994), 553-564.

[16] B. Y. Chen, *Geometry of Slant Submanifolds*, Katholieke Universiteit Leuven, Leuven, 1990.

[17] M. Dajczer and A. Florit, *On Chen's basic equality*, Illinois J. Math. 42 (1) (1998), 97-106.

[18] F. Defever, I. Mihai, and L. Verstraelen, *B. Y. Chen's inequality for C-totally real submanifolds of Sasakian-space-forms*, Bollettino U.M.I. (7)11-B (1997), 365-374.

[19] F. Dillen, M. Petrovic, and L. Verstraelen, *Einstein conformally flat and semi-symmetric submanifolds satisfying Chen's equality*, Israel J. Math. 100 (1997), 163-169.

[20] F. Dillen and L. Vrancken, *Totally real submanifolds in*$S^6(1)$ *satisfying Chen's equality*, Trans. Amer. Math. Soc. 348 (4) (1996), 1633-1646.

[21] J-B. Jun, U. C. De and G. Pathak, *On* $S-$ *manifolds*, J. Korean Math. Soc. 42, 3 (2005), 435-445.

[22] M. Kobayashi, *Semi-invariant submanifolds in an* $f-$*manifold with complemented frames*, Tensor N. S, 49 (1990), 154-177.

[23] A. Lotta, *Slant submanifolds in contact geometry*, Bull. Math. Soc. Roumanie 39 (1996), 183-198.

[24] C. Özgür and K. Arslan, *On some class of hypersurfaces in* E^{n+1} *satisfying Chen's equality*, Turkish J.Math, 26 (2002), 283-293.

[25] N. Papaghiuc, *Semi-slant submanifolds of a Kaehlerian manifold*, An. Stiint. Al.I.Cuza. Univ. Iasi 40 (1994), 62-78.

[26] K. Yano, *On a structure defined by a tensor field* f *of type* $(1,1)$ *satisfing* $f^3+f=0$, Tensor N.S, 14 (1963), 99-109.

[27] D. W. Yon, *A basic inequality of submanifolds in quaternionic space forms*, Balkan Journal of Geometry and Its Applications 9, 2 (2004), 92-102.

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