Warped product submanifolds of cosymplectic manifolds

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Abstract. Cosymplectic manifolds provide a natural setting for time dependent mechanical systems as they are locally product of a Kaehler manifold and a one dimensional manifold. Thus study of warped product submanifolds of cosymplectic manifolds is significant. In this paper we have proved results on the non-existence of warped product submanifolds of certain types in cosymplectic manifolds.

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1 Introduction

Bishop and O'Neill in 1969 introduced the notion of warped product manifolds. These manifolds are generalization of Riemannian product manifolds and occur naturally. (e.g. surfaces of revolution and Kenmotsu manifolds are warped product manifolds). In fact, the sphere and even $\mathbb{R}^n - \{0\}$ are locally isometric to warped product manifolds [5]. These manifolds also have applications in physics. It is shown that the space around a black hole or a massive star can be modeled on a warped product manifold. The existence or non-existence of these submanifolds thus assumes significance. Recently B. Sahin showed that there does not exist semi-slant warped product submanifolds of a Kaehler manifold [11]. In this paper we have extended this study to the warped product submanifolds of cosymplectic manifolds which are an important class of manifolds as they themselves are locally product of a Kaehler manifold and a one dimensional manifold and provide a natural framework for time dependent mechanical system.

2 Preliminaries

Let $\bar{M}$ be a$(2n+1)$-dimensional almost contact manifold with an almost contact structure (ϕ, ξ, η) i.e., a global vector field ξ , a $(1,1)$ tensor field ϕ and a 1-form η on $\bar{M}$ such that

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$$
(2.1) \qquad \begin{cases} \phi^2 X = -X + \eta(X)\xi, \\ \\ \eta(\xi) = 1 \end{cases}
$$

It is easy to see that on an almost contact manifold $\phi(\xi) = 0$ and $\eta \circ \phi = 0$.

On the product manifold $\bar{M} \times \mathbb{R}$, there induces an almost complex structure J defined as

$$
J(X, f\frac{d}{dt}) = (\phi X - f\xi,\ \eta(X)\frac{d}{dt})
$$

where X is a vector field on $\bar{M}$ and t, the co-ordinate function on $\mathbb{R}$.

The manifold $\bar{M}$ is said to be *normal* if the almost complex structure J on $\bar{M} \times \mathbb{R}$ has no torsion i.e., J is integrable, in other words the tensor $[\phi, \phi] + 2d\eta \otimes \xi$ vanishes identically on $\bar{M}$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ [13]. On an almost contact manifold there exists a Riemannian metric g satisfying

$$
(2.2) \qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)
$$

for any vector fields $X,\ Y$ tangent to $\bar{M}$. Clearly in this case η is dual of ξ i.e., $g(X, \xi) = \eta(X)$, for any vector fields X tangent to $\bar{M}$. The almost contact manifold with the above metric g is said to be an *almost contact metric manifold.*

The fundamental two form Φ on $\bar{M}$ is defined as $\Phi(X, Y) = g(X, \phi Y)$, for any vector fields $X,\ Y$ tangent to $\bar{M}$. The manifold $\bar{M}$ is called an *almost cosymplectic manifold* if η and Φ are closed i.e., $d\eta = 0$ and $d\Phi = 0$, where d is the exterior differential operator. If $\bar{M}$ is almost cosymplectic and normal it is called *cosymplectic* [2]. It is well known that an almost contact metric manifold is cosymplectic if and only if $\bar{\nabla}\phi$ vanishes identically, where $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$. From the formula $\bar{\nabla}_X \phi = 0$, it follows that $\bar{\nabla}_X \xi = 0$. These manifolds are known to have many applications and in fact found to provide a natural geometrical framework to describe time-dependent mechanical systems [4]. It is also shown that these manifolds are locally a product of Kaehler manifold and a real line or a circle.

Let M be a submanifold of an almost contact metric manifold $\bar{M}$ with almost contact metric structure (ϕ, ξ, η, g). Then Gauss and Weingarton formulae are given by

$$
(2.3) \qquad \bar{\nabla}_X Y = \nabla_X Y + h(X, Y)
$$

$$
(2.4) \qquad \bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X Y,
$$

for any $X,\ Y$ in TM and N in $T^{\perp}M$, where TM is the Lie algebra of vector field in M and $T^{\perp}M$ the set of all vector fields normal to M. $\nabla^{\perp}$ is the connection in the normal bundle, h the second fundamental form and A_N is the Weingarton endomorphism associated with N. It is easy to see that

$$
(2.5) \qquad g(A_N X, Y) = g(h(X, Y), N).
$$

For any $X \in TM$, we write

$$\phi X = PX + FX, \tag{2.6}$$

where PX is the tangential component and FX is the normal component of ϕX. Similarly for $N \in T^{\perp}M$, we write

$$\phi N = tN + fN, \tag{2.7}$$

where tN is the tangential component and fN is the normal component of ϕN. We shall always consider ξ to be tangent to M. It is easy to verify

$$fFX = -FPX$$

The submanifold M is said to be *invariant* if F is identically zero, that is, $\phi X \in TM$ for any $X \in TM$. On the other hand M is said to be *anti-invariant* if P is identically zero, that is, $\phi X \in T^{\perp}M$, for any $X \in TM$.

For each non zero vector X tangent to M at x, such that X is not proportional to ξ, we denotes by $\theta(X)$, the angle between ϕX and PX.

M is said to be *slant* [3] if the angle $\theta(X)$ is constant for all $X \in TM - \{\xi, 0\}$ and $x \in M$. The angle θ is called *slant angle* or *Wirtinger angle*. Obviously if $\theta = 0$, M is invariant and if $\theta = \pi/2$, M is an anti-invariant submanifold. If the slant angle of M is different from 0 and $\pi/2$ then it is called *proper slant*.

A characterization of slant submanifolds is given by following.

Theorem 2.1 [3]. *Let M be a submanifold of an almost contact metric manifold $\bar{M}$, such that $\xi \in TM$. Then M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$P^2 = \lambda(-I + \eta \otimes \xi). \tag{2.8}$$

Furthermore, in such case, if θ is slant angle, then $\lambda = \cos^2\theta$.

Following relations are straight forward consequence of equation (2.8)

$$\begin{cases} g(PX, PY) = \cos^2\theta[g(X, Y) - \eta(X)\eta(Y)] \\ g(FX, FY) = \sin^2\theta[g(X, Y) - \eta(X)\eta(Y)] \end{cases} \tag{2.9}$$

for any X, Y tangent to M.

N. Papaghuic [10] introduced the notion of semi-slant submanifolds of almost Hermitian manifolds, which were latter extended to almost contact metric manifold by J.L. Cabrerizo et.al [3]. We say M is a *semi-slant submanifold* of $\bar{M}$ if there exist an orthogonal direct decomposition of TM as

$$TM = D_1 \oplus D_2 \oplus \{\xi\}$$

where D_1 is an invariant distribution i.e., $\phi(D_1) = D_1$ and D_2 is slant with slant angle $\theta \neq 0$. Similarly we say M is *anti-slant submanifold* of $\bar{M}$ if D_1 is an anti-invariant distribution of M i.e., $\phi D_1 \subseteq T^{\perp}M$ and D_2 is slant with slant angle $\theta \neq 0$.

3 Warped and doubly warped product manifolds

The study of warped product submanifolds was initiated by R. L. Bishop and B. O'Neill [1]. They defined these as follows

Definition 3.1. Let (B, g_B) and (F, g_F) be two Riemannian manifolds with Riemannian metrics g_B and g_F respectively and f, a positive differentiable function on B. The warped product $B \times_f F$ of B and F is the Riemannian manifold $(B \times F, g)$, where

$$g = g_B + f^2 g_F. \tag{3.1}$$

More explicitly, if U is tangent to $M = B \times_f F$ at (p, q), then

$$\|U\|^2 = \|d\pi_1 U\|^2 + f^2(p)\|d\pi_2 U\|^2$$

where $\pi_i (i = 1,\ 2)$ are the canonical projections of $B \times F$ onto B and F respectively.

The following lemma provides some basic formulas on warped product manifolds.

Lemma 3.1 [1]. *Let $M = B \times_f F$ be a warped product manifold. If $X, Y \in TB$ and $V, W \in TF$ then*

(i) $\nabla_X Y \in TB$

(ii) $\nabla_X V = \nabla_V X = (X \ln f)V$

(iii) $nor(\nabla_V W) = -\frac{g(V,W)}{f}\nabla f$

where $nor(\nabla_V W)$ is the component of $\nabla_V W$ in TB and ∇f is the gradient of f.

Corollary 3.1 [1]. *On a warped product manifold $M = B \times_f F$*

(i) *B is totally geodesic in M,*

(ii) *F is totally umbilical in M.*

B.Y. Chen [6] studied CR-submanifolds of Kaehler manifolds as warped product submanifolds namely (a) $N_\perp \times_f N_T$ and (b) $N_T \times_f N_\perp$. B. Sahin [11], extending the non-existence theorem for warped product CR-submanifolds $N_\perp \times_f N_T$ showed that there do not exists proper semi-slant warped product submanifolds of Kaehler manifolds. Doubly warped product manifolds were introduced as a generalization to warped product manifolds by B. Unal [12]. A *doubly warped product manifold* of B and F, denoted as ${}_fB \times {}_bF$ is endowed with a metric g defined as

$$g = f^2 g_B + b^2 g_F \tag{3.2}$$

where b and f are positive differentiable functions on B and F respectively.

In this case formula (iii) of Lemma 3.1 is generalized as

$$\nabla_X Z = (Z \ln f)X + (X \ln b)Z \tag{3.3}$$

for each X in TB and Z in TF [12].

If neither b nor f is constant we have a non trivial doubly warped product $M = {}_fB \times {}_bF$. Obviously in this case both B and F are totally umbilical submanifolds of M.

We, now consider a doubly warped product of two Riemannian manifolds N_1 and N_2 embedded into a cosymplectic manifold $\bar{M}$ such that the structure vector field ξ is tangential to the submanifold $M = {}_{f_2}N_1 \times {}_{f_1}N_2$ and prove

Theorem 3.1. *If $M = {}_{f_2}N_1 \times {}_{f_1}N_2$ is a doubly warped product submanifold of a cosymplectic manifold $\bar{M}$ where N_1 and N_2 are any Riemannian submanifolds of $\bar{M}$. Then f_2 is constant if the structure vector field ξ is tangent to N_1 and f_1 is constant if ξ is tangent to N_2.*

Proof. We have using Gauss formula and the fact that $\bar{\nabla}_U \xi = 0$, for $U \in TM$

$$\nabla_U \xi = 0. \tag{3.4}$$

Thus in case $\xi \in T(N_1)$ and $U \in T(N_2)$ equations (3.3) and (3.4) imply that $(\xi \ln f_1)U + (U \ln f_2)\xi = 0$, which shows that f_2 is constant. Similarly for $\xi \in T(N_2)$ and $U \in T(N_1)$ we have $(\xi \ln f_2)U + (U \ln f_1)\xi = 0$, showing that f_1 is constant. This completes the proof. □

It follows from the above theorem that

Corollary 3.2. *There does not exist a warped product submanifold of the type $N_1 \times {}_fN_2$ of cosymplectic manifolds $\bar{M}$ where N_1 and N_2 are any Riemannian submanifolds of $\bar{M}$ with ξ tangential to N_2.*

To study the warped product submanifolds $N_1 \times {}_fN_2$ with structure vector field ξ tangential to N_1, we first obtain some useful formulae for later use.

Lemma 3.2. *Let $M = N_1 \times {}_fN_2$ be a proper warped product submanifold of a cosymplectic manifold $\bar{M}$, with $\xi \in T(N_1)$, where N_1 and N_2 are any Riemannian submanifolds of $\bar{M}$, then*

(i) $\xi \ln f = 0$,

(ii) $A_{FZ}X = -th(X, Z)$,

(iii) $g(h(X, Z), FY) = g(h(X, Y), FZ)$,

(iv) $g(h(X, Z), FW) = g(h(X, W), FZ)$

for any $X,\ Y \in T(N_1)$ and $Z,\ W \in T(N_2)$.

Proof. The first result is an immediate consequence of the fact that $\bar{\nabla}_U \xi = 0$, for $U \in TM$. For (ii) consider $\bar{\nabla}_X \phi Z = \phi \bar{\nabla}_X Z$, which implies

$$\nabla_X PZ + h(X, PZ) - A_{FZ}X + \nabla^{\perp}_X FZ = P\nabla_X Z + F\nabla_X Z + th(X, Z) + fh(X, Z).$$

Comparing tangential and normal parts and using the fact $\nabla_X PZ = P\nabla_X Z$, we get

$$A_{FZ}X = -th(X, Z)$$

(iii) and (iv) follow by taking the product in (ii) by Y and W respectively. □

Now in the following section we shall investigate semi-slant warped product submanifolds of cosymplectic manifolds.

4 Semi-slant warped product submanifolds

From Corollary 3.2, it follows that there does not exist semi-slant warped product submanifolds of the type $N_T \times_f N_\theta$ and $N_\theta \times_f N_T$ of a cosymlectic manifold $\bar{M}$ where N_T is an invariant and N_θ is a proper slant submanifold of $\bar{M}$ with ξ being tangential to N_θ and N_T respectively. Thus we are left with two cases:

(i) $N_T \times_f N_\theta$, and

(ii) $N_\theta \times_f N_T$

with ξ in $T(N_T)$ and $T(N_\theta)$ respectively.

For case (i), we have

Theorem 4.1. *There does not exist a proper warped product submanifold $N_T \times_f N_\theta$ where N_T is invariant and N_θ is a proper slant submanifold of a cosymplectic manifold $\bar{M}$ such that ξ is tangent to N_T.*

Proof. For $X \in T(N_T)$ and $Z \in T(N_\theta)$

$$\begin{aligned}(X \ln f)\|Z\|^2 &= -g(\nabla_Z Z, X) = \eta(\bar{\nabla}_Z Z)\eta(X) - g(\phi\bar{\nabla}_Z Z, \phi X)\\ &= -g(\bar{\nabla}_Z \phi Z, \phi X) = -g(\nabla_Z PZ - A_{FZ}Z, \phi X)\\ &= g(h(Z, \phi X), FZ).\end{aligned}$$

That is,

$$g(h(Z, \phi X), FZ) = (X \ln f)\|Z\|^2. \tag{4.1}$$

Now,

$$\begin{aligned}g(h(Z, \phi X), FZ) &= g(\bar{\nabla}_Z \phi X, FZ) = g(\phi\bar{\nabla}_Z X, FZ)\\ &= g(\phi(\nabla_Z X + h(X, Z)), FZ)\\ &= g(F\nabla_Z X, FZ) + g(\phi h(X, Z), FZ)\\ &= \sin^2\theta[g(\nabla_Z X, Z) - \eta(\nabla_Z X)\eta(Z)] - g(h(X, Z), fFZ)\end{aligned}$$

i.e.,

$$g(h(Z,\phi X),FZ) = (X\ln f)\sin^2\theta\|Z\|^2 + g(h(X,Z),FPZ) \tag{4.2}$$

On using (4.1) and (4.2), we get

$$g(h(X,Z),FPZ) = (X\ln f)\cos^2\theta\|Z\|^2 \tag{4.3}$$

Which on using formula (iv) of Lemma 3.2 yields

$$g(h(X,PZ),FZ) = (X\ln f)\cos^2\theta\|Z\|^2. \tag{4.4}$$

Replacing Z by PZ in above gives

$$g(h(X,P^2Z),FPZ) = (X\ln f)\cos^2\theta\|PZ\|^2.$$

Which on using (2.8) and (2.9), implies

$$g(h(X,Z),FPZ) = -(X\ln f)\cos^2\theta\|Z\|^2.$$

As the warped product submanifold is assumed to be a proper submanifold, it follows from (3.4) and the last relation that either $\theta = \pi/2$ or f is constant on N_T. Hence theorem is proved. □

Now case(ii) is dealt in the following theorem.

Theorem 4.2. *There does not exist a proper warped product submanifold $N_\theta \times_f N_T$ where N_T is an invariant and N_θ is a proper slant submanifold of a cosymplectic manifold $\bar{M}$ such that ξ is tangent to N_θ.*

Proof. For $X \in T(N_T)$ and $Z \in T(N_\theta)$, we have

$$\nabla_Z X = \nabla_X Z = (Z\ln f)X.$$

Now since $\bar{\nabla}_X\xi = 0$, the above relation implies on using Gauss formula that

$$\xi\ln f = 0. \tag{4.5}$$

On using Lemma 3.1, we have

$$g(\bar{\nabla}_X X, Z) = g(\nabla_X X, Z) = -(Z\ln f)g(X,X), \tag{4.6}$$

but,

$$g(\bar{\nabla}_X X, Z) = g(\phi\bar{\nabla}_X X, \phi Z) + \eta(\bar{\nabla}_X X)\eta(Z)$$

which on using (4.6) simplifies as

$$\begin{aligned} g(\nabla_X X, Z) &= g(\bar{\nabla}_X \phi X, \phi Z) \\ &= g(\nabla_X \phi X, PZ) + g(h(X,\phi X), FZ) \\ &= g(h(X,\phi X), FZ). \end{aligned}$$

Therefore,

$$g(h(X,\phi X),FZ) = -(Z\ln f)\|X\|^2. \tag{4.7}$$

Changing X by ϕX in above we get

$$g(h(\phi X,X),FZ) = (Z\ln f)\|X\|^2. \tag{4.8}$$

Equations (4.7) and (4.8) imply that $Z\ln f = 0$, i.e., f is constant on N_θ, proving the result. □

From Corollary 3.2, Theorems 4.1 and 4.2 it follows that:

Theorem 4.3. *There does not exists a semi-slant warped product submanifold of a cosymplectic manifold other than a CR-product.*

5 Generic warped product submanifolds

Above results prompt us to consider more general warped product submanifolds of a cosymplectic manifold, namely $N_T \times {}_f N$ and $N \times {}_f N_T$ called generic warped product submanifolds, with ξ is tangent to N_T and N respectively where N is an arbitrary Riemannian submanifold of $\bar{M}$.

In fact, it can be realized that Theorem 4.2 is true even for variable θ, to be more precise, we can state

Theorem 5.1. *There does not exist a proper warped product submanifold $N \times {}_f N_T$ where N_T is an invariant and N is any Riemannian submanifold of a cosymplectic manifold $\bar{M}$ such that ξ is tangent to N.*

Proof. Let $M = N \times {}_f N_T$ be a warped product submanifolds of $\bar{M}$. Then by Lemma 3.1,

$$\nabla_X Z = \nabla_Z X = (Z\ln f)X \tag{5.1}$$

for each $X \in TN_T$ and $Z \in TN$. Thus,

$$g(X,\nabla_{\phi X}Z) = 0. \tag{5.2}$$

Making use of equations (2.2), (2.3), (2.4), (2.6) and the fact that $\bar{M}$ is cosymplectic, we deduce from equation (5.2) that

$$g(\phi X,\nabla_{\phi X}PZ) - g(h(PX,PX),FZ) = 0$$

which on applying formula (5.1) yields

$$g(h(PX,PX),FZ) = (PZ\ln f)g(X,X). \tag{5.3}$$

As $\bar{M}$ be cosymplectic we have

$$(\nabla_X P)Z = A_{FZ}X + th(X, Z) \tag{5.4}$$

and by equation (5.1),

$$(\nabla_X P)Z = (PZ \ln f)X - (Z \ln f)PX. \tag{5.5}$$

Equating the right hand side of equations (5.4) and (5.5) and taking product with $Y \in TN_T$, we get

$$(Z \ln f)g(X, Y) + (PZ \ln f)g(PX, Y) = g(h(PX, Y), FZ). \tag{5.6}$$

Interchanging X and Y in the above equation and adding the resulting equation in (5.6) while taking account of the fact that

$$h(X, PY) - h(PX, Y) = F[X, Y] = 0,$$

we obtain that

$$(Z \ln f)g(X, Y) = g(h(X, PY), FZ). \tag{5.7}$$

In particular, we have

$$g(h(PX, PY), FZ) = 0. \tag{5.8}$$

By equation (5.3) and (5.8), it follows that

$$(PZ \ln f) = 0$$

for each $Z \in TN$. This completes the proof. □

Now, the other case i.e., $N_T \times {}_f N$ with ξ is tangent to N_T is dealt in the following theorem:

Theorem 5.2. *There does not exist a proper warped product submanifold $N_T \times {}_f N$ where N_T is an invariant and N is any proper non anti-invariant Riemannian submanifold of a cosymplectic manifold $\bar{M}$ such that ξ is tangent to N_T.*

Proof. Let $M = N_T \times {}_f N$. For $U,\ V$ in TM, if we define

$$(\nabla_U P)V = \nabla_U PV - P\nabla_U V$$

Then by Lemma 3.1 (ii), it can be deduced that

$$(\nabla_X P)Z = 0, \tag{5.9}$$

and

$$(\nabla_Z P)X = (PX \ln f)Z - (X \ln f)PZ \tag{5.10}$$

for any $X \in TN_T$ and $Z \in TN$. Now, as $\bar{M}$ is cosymplectic, it is straight forward to see that

(5.11) $$(\nabla_U P)V = A_{FV}U + th(U, V).$$

Using equation (5.11) in equations (5.9) and (5.10), we get

(5.12) $$A_{FZ}X = -th(X, Z)$$

and

(5.13) $$(PX \ln f)Z - (X \ln f)PZ = th(X, Z).$$

Combining equations (5.12), (5.13) and taking product with PZ, gives

(5.14) $$g(h(X, PZ), FZ) = (X \ln f)\|PZ\|^2.$$

Now, as $\bar{\nabla}\phi = 0$ and

(5.15) $$\nabla_X Z = \nabla_Z X = (X \ln f)Z$$

we have on using equation (2.2)

$$g(\bar{\nabla}_{PZ}\phi Z, \phi X) = 0$$

i.e.,

$$g(\nabla_{PZ}PZ, \phi X) = g(A_{FZ}PZ, \phi X).$$

Replacing X by ϕX and making use of equation (5.15), the above equation yields,

(5.16) $$g(h(X, PZ), FZ) = -(X \ln f\|PZ\|^2.$$

Further from Lemma 3.2, we have

(5.17) $$\xi \ln f = 0.$$

From the equations (5.14), (5.16) and (5.17), it follows that the warped product $N_T \times {}_f N$ is trivial. This proves the theorem. □

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