2-Killing vector fields on Riemannian manifolds

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Abstract. The aim of this paper is to define a class of vector fields on a Riemannian manifold which enlarges the class of Killing vector fields. We study the relations between 2-Killing vector fields and monotone vector fields, introduced by S.Z. Németh ([1], [2]), we give the characterization of a 2-Killing vector field on $\mathbb{R}^n$ and an example of such vector field which is not Killing.

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1 Monotone vector fields

Let us consider a Riemannian manifold (M, g), a vector field X on M and a geodesic $\gamma : I \to M$, where I is an interval of real numbers. With this ingredients, we define the function

$$\varphi_{X,\gamma} : I \to \mathbb{R}, \quad \varphi_{X,\gamma}(t) = g(X(\gamma(t)), \dot{\gamma}\,(t)).$$

The vector field X is called *monotone* if, for any geodesic γ on M, the function $\varphi_{X,\gamma}$ is monotone. If, for any geodesic γ on M, $\varphi_{X,\gamma}$ is a *increasing (decreasing) function*, the vector field X is called *increasing (decreasing)* vector field. The next theorem give us a characterization of monotone vector fields.

Theorem 1.1 (S.Z. Németh [1]). *If (M, g) is a Riemannian manifold and X a vector field on M, then the next assertions are equivalent*
i) X is a increasing (decreasing) vector field on M.
ii) For any vector field $U \in \mathcal{X}(M)$, $g(\nabla_U X, U) \geq 0$ (≤ 0).

A characterization of vector fields which have the property that $\varphi_{X,\gamma}$ is a constant function is

Theorem 1.2. *If (M, g) is a Riemannian manifold and X a vector field on M, then the next assertions are equivalent*
i) For any geodesic γ on M, the function $\varphi_{X,\gamma}$ is a constant.
ii) For any vector field $U \in \mathcal{X}(M)$, $g(\nabla_U X, U) = 0$.

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iii) For any vector fields $U, V \in \mathcal{X}(M)$, $g(\nabla_U X, V) + g(\nabla_V X, U) = 0$.
iv) X *is a Killing vector field.*

We present two examples of monotone vector fields.

Example 1.1. Any Killing vector field is monotone.

Example 1.2. If (M, g) is a Riemannian manifold and $f : M \to \mathbb{R}$ is a convex function of C^2 class, then grad$f$ is a monotone vector field.

2 2-Killing vector fields

A *Killing vector field* X on Riemannian manifold (M, g) is characterized by the fact that the Lie derivative of the metric tensor g, with respect to X, is zero. We define *2-Killing vector fields by*

Definition 2.1. *Let* (M, g) *be a Riemannian manifold. A vector field* $X \in \mathcal{X}(M)$ *is called 2-Killing if* $L_X L_X g = 0$, *where* L *is the Lie derivative.*

We give the next characterization

Theorem 2.1. *A vector field* $X \in \mathcal{X}(M)$ *is 2-Killing if and only if*

$$R(X, U, X, U) = g(\nabla_U \nabla_X X, U) + g(\nabla_U X, \nabla_U X), \forall\, U \in \mathcal{X}(M),$$

where R *is the curvature tensor of Riemannian manifold* (M, g).

Proof. As $L_X L_X g$ is a symmetric tensor, we have

$$L_X L_X g = 0 \Longleftrightarrow \forall\, U \in \mathcal{X}(M), (L_X L_X g)(U, U) = 0. \tag{2.1}$$

Let ∇ be the Levi-Civita connection of Riemannian manifold (M, g). For any two vector fields V, W, the Lie derivative of the metric tensor g is given by

$$\begin{aligned}(L_X g)(V, W) = Xg(V, W) - g([X, V], W) - g(V, [X, W]) = \\ = g(\nabla_V X, W) + g(\nabla_W X, V).\end{aligned} \tag{2.2}$$

Using again the definition of the Lie derivative and the relation (2.2), we obtain

$$\begin{aligned}(L_X L_X g)(U, U) = 2\{g(\nabla_X \nabla_U X, U) + g(\nabla_U X, \nabla_U X) - \\ - g(\nabla_{[X,U]} X, U)\}.\end{aligned} \tag{2.3}$$

The curvature tensor R is given by

$$\begin{aligned}R(X, U, X, U) = R(U, X, U, X) = g(R(U, X)X, U) = \\ = g(\nabla_U \nabla_X X - \nabla_X \nabla_U X - \nabla_{[U,X]} X, U),\end{aligned} \tag{2.4}$$

therefore

$$g(\nabla_X \nabla_U X, U) + g(\nabla_{[U,X]} X, U) = g(\nabla_U \nabla_X X, U) - R(X, U, X, U), \tag{2.5}$$

which implies

(2.6) $$g(\nabla_X \nabla_U X, U) - g(\nabla_{[X,U]} X, U) = g(\nabla_U \nabla_X X, U) - R(X, U, X, U).$$

From (2.3) and (2.6) one gets

(2.7) $$\begin{aligned}(L_X L_X g)(U, U) = 2\{g(\nabla_U \nabla_X X, U) - R(X, U, X, U)+\\ +g(\nabla_U X, \nabla_U X)\}.\end{aligned}$$

From (2.1) and (2.7) we obtain that X is a 2-Killing vector field if and only if

(2.8) $$R(X, U, X, U) = g(\nabla_U \nabla_X X, U) + g(\nabla_U X, \nabla_U X), \forall\ U \in \mathcal{X}(M).$$

Remark. In every point of the Riemannian manifold M the relation (2.8) can be regarded as an equality between two quadratic forms. From the fact that the bilinear forms associated with this two quadratic forms are equals, we obtain another characterization of the 2-Killing vector fields

(2.9) $$\begin{aligned}2R(X, U, X, V) = g(\nabla_U \nabla_X X, V) + g(\nabla_V \nabla_X X, U)+\\ +2g(\nabla_U X, \nabla_V X), \forall\ U, V \in \mathcal{X}(M).\end{aligned}$$

Similar to the case of Killing vector fields, on a compact Riemannian manifold (M, g) there is a small class of 2-Killing vector fields X which satisfies the relation

$$\mathrm{Ric}(X, X) \leq 0,$$

where Ric is the *Ricci tensor* of Riemannian manifold (M, g).

We give a proof for the above assertion.

Theorem 2.2. *Let X be a 2-Killing vector field on the compact n-dimensional Riemannian manifold (M, g). If $\mathrm{Ric}(X, X) \leq 0$, then X is a parallel vector field.*

Proof. Let us consider a point $x \in M$ and $\{e_1, e_2, ..., e_n\}$ an orthonormal frame in $T_x M$. From theorem 2.1 one gets

(2.10) $$R(X, e_i, X, e_i) = g(\nabla_{e_i} \nabla_X X, e_i) + g(\nabla_{e_i} X, \nabla_{e_i} X), \forall\ i \in \overline{1, n}.$$

Summing by $i \in \overline{1, n}$, we obtain

(2.11) $$\mathrm{Ric}(X, X) = \mathrm{div}(\nabla_X X) + \mathrm{Tr}(g(\nabla X, \nabla X)).$$

Integrating the equality (2.11) on the compact Riemannian manifold M, one gets

(2.12) $$\int_M \mathrm{Ric}(X, X) \mathrm{d}v_g = \sum_{i=1}^{n} \int_M \mathrm{Tr}(g(\nabla X, \nabla X)) \mathrm{d}v_g.$$

As $\mathrm{Ric}(X, X) \leq 0$ and $\sum_{i=1}^{n} \int_M \mathrm{Tr}(g(\nabla X, \nabla X)) \mathrm{d}v_g \geq 0$, the relation (2.12) implies

(2.13) $$\mathrm{Ric}(X, X) = 0$$

and

$$\mathrm{Tr}(g(\nabla X, \nabla X)) = 0. \tag{2.14}$$

The relation (2.14) justifies the fact that X is a parallel vector field.

We study the relations between 2-Killing vector fields and monotone vector fields.

Theorem 2.3. *We consider a Riemannian manifold (M, g) and $X \in \mathcal{X}(M)$ a 2-Killing vector field, which have the property that $\nabla_X X$ is increasing vector field. If there is a point $x \in M$ and a tangent vector $v \in T_xM$, $v \neq 0$ so that $R(X_x, v, X_x, v) \leq 0$, then*

i) $\nabla_v X = 0$ and $R(X_x, v, X_x, v) = 0$,

ii) The vector field $\nabla_X X$ is not strictly increasing.

Proof. As X is a 2-Killing vector field, one gets

$$R(X, U, X, U) = g(\nabla_U \nabla_X X, U) + g(\nabla_U X, \nabla_U X), \forall\ U \in \mathcal{X}(M). \tag{2.15}$$

Let us consider a point $x \in M$ and a tangent vector $v \in T_xM$ such that

$$R(X_x, v, X_x, v) \leq 0. \tag{2.16}$$

From (2.15) we obtain

$$g(\nabla_v X, \nabla_v X) = R(X_x, v, X_x, v) - g(\nabla_v \nabla_X X, v). \tag{2.17}$$

As $\nabla_X X$ is a increasing vector field, using theorem 1.1, one gets

$$g(\nabla_v \nabla_X X, v) \geq 0. \tag{2.18}$$

The relations (2.16), (2.17) and (2.18) implies

$$g(\nabla_v X, \nabla_v X) = 0, \text{ therefore } \nabla_v X = 0, \tag{2.19}$$

$$R(X_x, v, X_x, v) = 0 \text{ and} \tag{2.20}$$

$$g(\nabla_v \nabla_X X, v) = 0, \tag{2.21}$$

which lead us to the conclusion that $\nabla_X X$ is not strictly increasing vector field.

Theorem 2.4. *Let X be a 2-Killing vector field on the n-dimensional Riemannian manifold (M, g) of negative sectional curvature. Then*

i) $\nabla_X X$ is a decreasing vector field,

ii) If the sectional curvature is strictly negative and $\nabla_X X$ is not vanishing on M, then $\nabla_X X$ is a strictly decreasing vector field.

Proof. i) From the fact that X is a 2-Killing vector field, one gets

$$g(\nabla_U \nabla_X X, U) = R(X, U, X, U) - g(\nabla_U X, \nabla_U X) \leq 0, \forall\, U \in \mathcal{X}(M). \tag{2.22}$$

Using theorem 1.1 we obtain that $\nabla_X X$ is a decreasing vector field.

ii) If $\nabla_X X$ is not a strictly decreasing vector field, then there is a point $x \in M$ and a tangent vector $v \in T_x M$, so that

$$g(\nabla_v \nabla_X X, v) = 0. \tag{2.23}$$

From (2.22) and (2.23) one gets

$$R(X_x, v, X_x, v) = 0 \text{ and} \tag{2.24}$$

$$\nabla_v X = 0. \tag{2.25}$$

Using the fact that the sectional curvature of the Riemannian manifold (M, g) is strictly negative, from the relation (2.24) we obtain that the tangent vector v is colinear with X_x, therefore

$$\exists\, \lambda \in \mathbb{R}^* \text{ so that } v = \lambda X_x. \tag{2.26}$$

From (2.25) and (2.26) one gets the relation $\nabla_{X_x} X = 0$, which is a contradiction with the fact that the vector field $\nabla_X X$ is not vanishing on M. In conclusion, $\nabla_X X$ is a strictly decreasing vector field.

We give an example of a 2-Killing vector field, which is not a Killing vector field, on a open set of Riemannian manifold $(\mathbb{R}^n, \langle\ ,\ \rangle)$, where $\langle\ ,\ \rangle$ is the standard metric tensor on $\mathbb{R}^n$.

Theorem 2.5. *A vector field $X = X^i \partial_i$ on the Riemannian manifold $(\mathbb{R}^n, \langle\ ,\ \rangle)$ is a 2-Killing vector field if and only if*

$$\partial_i X^l\ \partial_l X^j + \partial_j X^l\ \partial_l X^i + X^l \partial_l(\partial_i X^j + \partial_j X^i) + 2\sum_{k=1}^{n} \partial_i X^k\ \partial_j X^k = 0,$$
$$\forall\, i, j \in \overline{1, n}.$$

Proof. From theorem 2.1 we obtain that $X \in \mathcal{X}(\mathbb{R}^n)$ is a 2-Killing vector field if and only if $\langle \nabla_U \nabla_X X, V\rangle + \langle \nabla_V \nabla_X X, U\rangle + 2\langle \nabla_U X, \nabla_V X\rangle = 0$, $\forall\, U, V \in \mathcal{X}(\mathbb{R}^n)$, where ∇ is the Levi-Civita connection of the Riemannian manifold $\mathbb{R}^n$, which is equivalent with

$$\langle \nabla_{\partial_i} \nabla_X X, \partial_j\rangle + \langle \nabla_{\partial_j} \nabla_X X, \partial_i\rangle + 2\langle \nabla_{\partial_i} X, \nabla_{\partial_j} X\rangle = 0, \forall\, i, j \in \overline{1, n}. \tag{2.27}$$

Using the fact that $\nabla_U V = U(V^i)\partial_i$, for any two vector fields $U, V \in \mathcal{X}(\mathbb{R}^n)$, one gets the conclusion of the theorem.

Remark. i) If $n = 1$, then any vector field is given by $X = f\frac{d}{dt}$, where f is a differentiable function. A 2-Killing vector field on $\mathbb{R}$ satisfies

$$ff^{''} = -2(f^{'})^2. \tag{2.28}$$

The solutions of the equation (2.28) are

$$f = a = \text{constant and} \tag{2.29}$$

$$f : \mathbb{R}\backslash\{b/a\} \to \mathbb{R}, f(t) = (at - b)^{\frac{1}{3}}, \tag{2.30}$$

where a and b are real numbers.

We obtain the 2-Killing vector fields

$$X \in \mathcal{X}(\mathbb{R}), X = a\frac{\mathrm{d}}{\mathrm{d}t} \text{ and} \tag{2.31}$$

$$X \in \mathcal{X}(\mathbb{R}\backslash\{b/a\}), X = (at - b)^{\frac{1}{3}}\frac{\mathrm{d}}{\mathrm{d}t}. \tag{2.32}$$

ii) An example of a 2-Killing vector field, which is not a Killing vector field, on $(\mathbb{R}^*)^n$ is $X = (x^k)^{\frac{1}{3}}\partial_k$.

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