Conformal connections on Lyra manifolds

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Abstract. We give an algebraic characterization of the case when conformal Weyl and conformal Lyra connections have the same curvature tensor. It is determined a (1,3)-tensor field invariant to certain transformation of semi-symmetric connections, compatible with Weyl structures on conformal manifolds. It is studied the case when this tensor is vanishing.

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Introduction

The invariance of curvature type tensors under conformal transformation of metrics plays a central role in conformal geometry and has deep geometric significance.

The conformal Weyl curvature tensor

$$C(X,Y,Z,W) = R(X,Y,Z,W) - \frac{1}{2}[g(X,W)S(Y,Z) - g(Y,W)S(X,Z) + \\ + g(Y,Z)S(X,W) - g(X,Z)S(Y,W)] + \\ + \frac{k}{(n-1)(n-2)}[g(X,W)g(Y,Z) - g(Y,W)g(X,Z)]$$

is invariant under conformal transformation of metrics $g \to \overline{g} = e^{2\xi}g$.

The conharmonic curvature tensor

$$K(X,Y,Z,W) = R(X,Y,Z,W) - \frac{1}{n-2}[S(X,W)g(Y,Z) - S(Y,W)g(X,Z) + \\ + S(Y,Z)g(X,W) - S(X,Z)g(Y,W)]$$

is invariant under conharmonic transformation of metrics $g \to \overline{g} = e^{2\xi}g$, where $\xi_p^p = g^{ij}\xi_{ij} = 0, \xi_{hk} = \xi_{h,k} - \xi_h\xi_k + \frac{1}{2}\xi_i\xi^i g_{hk}, \xi_i = \frac{\partial \xi}{\partial x^i}$.

The concircular curvature tensor

$$L(X,Y,Z,W) = R(X,Y,Z,W) - \frac{k}{n(n-1)}[g(X,W)g(Y,Z) - g(Y,W)g(X,Z)]$$

is invariant under concircular transformation of metrics $g \to \overline{g} = e^{2\xi}g$, where $T_{rs} = \xi_{r,s} - \xi_r\xi_s, T = \frac{1}{2}Tr(T)g$, S is the Ricci tensor and k is the scalar curvature.

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1 Semi-symmetric connections on Lyra manifolds

Let $\pi \in \Lambda^1(M)$. A linear connection ∇ is called π-semi-symmetric if

$$T(X,Y) = \pi(X)Y - \pi(Y)X, \quad \forall X,Y \in \mathcal{X}(M).$$

If, moreover, ∇ is metric ($\nabla_X g = 0$), then the triple (M, g, ∇) is called Lyra manifold associated to π.

A. Friedman, J.A. Schouten introduced the notion of semi-symmetric connection. The research is continued by H.A. Hayden. The subject was developed from different perspectives. The main directions of study are:

a) The geometrical significance of semi-symmetric connection:

Theorem A [12] *The necessary and sufficient condition such that a Riemannian manifold admits a metric semi-symmetric connection with vanishing curvature tensor is that the space is conformally flat (i.e. $C = 0$).*

Theorem B [12] *The necessary and sufficient condition such that a Riemannian manifold admits a metric semi-symmetric connection ∇ such that M is a group manifold (i.e. $R(X,Y)Z = 0, (\nabla_X T)(Y,Z) = 0$) is that the space (M,g) has constant curvature.*

Along the same line T. Imai got the following results

Theorem C [3] *If a Riemannian manifold (M,g) admits a metric semi-symmetric connection ∇ such that $S^{\nabla} = 0$, then:*

a) $R^{\nabla} = C$ (the curvature tensor associated to this connection coincides with the conformal Weyl curvature tensor of the Riemann space).

b) There exists $\overline{g} \in \hat{g}$ such that $\overline{R} = C$ (the curvature tensor of the Levi-Civita connection associated to $\overline{g}$ coincides with the conformal Weyl curvature tensor of the Riemann space).

If the 1-form π is closed one can introduce the notion of sectional curvature.

Theorem D [3] *If a Riemannian manifold (M,g) admits π-semi-symmetric connection ∇ such that π is closed and the sectional curvature corresponding to ∇ is constant, then the Riemann space is conformally flat.*

In [13] P. Zhao, H. Song, X. Yang studied semi-symmetric recurrent connections. They considered ∇ and $\overline{\nabla}$ two semi-symmetric metric recurrent connections on a Riemannian space such that $\nabla \to \overline{\nabla}$ is a projective transformation and determined an invariant of this transformation.

b). Properties of semi-symmetric connections on manifolds endowed with special structures:

Let $M(\varphi, \xi, \eta, g)$ be a Sasaki manifold. A metric connection is called S-connection if $(\nabla_X \varphi)(Y) = \eta(Y)X - g(X,Y)\xi$.

If, moreover, $T(X,Y) = \eta(Y)\varphi(X) - \eta(X)\varphi(Y)$, then ∇ is called metric semi-symmetric S-connection and is given by

$\nabla_X Y = \overset{\circ}{\nabla}_X Y - \eta(X)\varphi(Y)$,

where $\overset{\circ}{\nabla}$ is the Levi-Civita connection.

Theorem E [7] *If a Sasaki manifold $M(\varphi, \xi, \eta, g)$ admits a metric semi-symmetric S- connection, whose curvature tensor is vanishing, then:*
a) the conformal Weyl curvature tensor coincides with the conharmonic curvature tensor;
b) the concircular curvature tensor coincides with the Riemann curvature tensor.

R.N. Singh and K.P. Pandey [9] gave the relativististic significance of a semi-symmetric metric S- connection whose curvature tensor is vanishing. S.D.Singh, A.K. Pandey [8] studied semi-symmetric metric connections in an almost Norden metric manifolds. P.N. Pandey and B.B. Chaturvedi [6] considered semi-symmetric connections on Kähler manifolds. F. Ünal and A. Uysal [10] studied semi-symmetric connections on Weyl manifolds.

2 Weyl manifolds

Let M be a connected paracompact differentiable manifold of dimension $n \geq 3$.

Let g be a pseudo-Riemannian metric on M and $\hat{g} = \{e^{2\xi} g \mid \xi \in \mathcal{F}(M)\}$ the conformal class defined by g.

A Weyl structure on the conformal manifold $(M, \hat{g})$ is a mapping

$$W : \hat{g} \mapsto \Lambda^1(M), W(e^{2\xi} g) = W(g) - 2d\xi, \forall \xi \in \mathcal{F}(M).$$

We call the triple $(M, \hat{g}, W)$ a Weyl manifold.

Remark 2.1. There exists an unique torsion free connection ∇ on M, compatible with the Weyl structure W :
$\nabla g + W(g) \otimes g = 0$,
called the conformal Weyl connection. This is required to be invariant under the transformation $g \mapsto e^{2\xi} g$.

H.Weyl introduced the 2-form $\psi(M) = dW(g), g \in \hat{g}$ (a gauge invariant). If $\psi(M) = 0$, then the cohomology class $ch(W) = [W(g)] \in H^1(M, d)$ does not depend on the choice of the metric $g \in \hat{g}$.

$\psi(M)$ and $ch(M)$ are obstructions for a Weyl structure to be a Riemannian structure.

Theorem F [2] *Let $(M, \hat{g}, W)$ be a Weyl manifold and ∇ the conformal Weyl connection. The following assertions are equivalent:*
1) $\psi(M) = 0, ch(M) = 0$;
2) There is a Riemannian metric $\overline{g} \in \hat{g}$ such that $\nabla \overline{g} = 0$.

Let $(M, \hat{g}, W)$ be Weyl manifold and ∇ be the conformal Weyl connection.

Let $\overline{\nabla}$ be the π semi-symmetric connection compatible with the Weyl structure W i.e.
$\overline{\nabla} g + W(g) \otimes g = 0$,
called conformal π semi-symmetric connection or the conformal Lyra connection.

Let $E \in \mathcal{T}^{1,2}(M)$. The $\mathcal{F}(M)$-module $\mathcal{X}(M)$ becomes an algebra, denoted $\mathcal{U}(M, E)$ if $X \circ Y = E(X, Y), \forall X, Y \in \mathcal{X}(M)$.

If ∇ and ∇' are linear connections on M and $E = \nabla - \nabla'$, then $\mathcal{U}(M, E)$ is called the deformation algebra associated to the pair (∇, ∇').

Our purpose is to study properties of semi-symmetric connections on Weyl manifolds.

Theorem 2.1. *Let $(M, \hat{g}, W)$ be a Weyl manifold, $n \geq 4$ and $\mathcal{U}(M, \overline{\nabla} - \nabla)$ be the Weyl-Lyra deformation algebra associated to the 1-form π. Let $\overline{R}, R$ be the curvature tensors associated to the connections $\overline{\nabla}, \nabla$.*

Then $\overline{R} = R$, if $\psi(M) = 0$ and $R_p : T_p \times T_p \times T_p M \longrightarrow T_p M$ is surjective, $\forall p \in M$, if and only if the Weyl-Lyra algebra is associative.

Proof. " $\Rightarrow$ "

Let $\overline{A} = \overline{\nabla} - \nabla$. One has

$$g\left(\overline{A}(X, Y), Z\right) = \pi(Y)\, g(X, Z) - \pi(Z)\, g(X, Y), \forall X, Y, Z \in \mathcal{X}(M).$$

Using the second Bianci identities and $\overline{\nabla}_X \overline{R} = \overline{\nabla}_X R$ we have

$$(\delta_i^s R_{ljk}^r + \delta_j^s R_{lki}^r + \delta_k^s R_{lij}^r)\pi_r + (g_{il} R_{rjk}^s + g_{jl} R_{rki}^s + g_{kr} R_{rij}^s)\pi^r = 0.$$

This relation leads to

$$(n-3)\, g_{rh} R_{ljk}^h \pi_r + (g_{kl} S_{rj} - g_{jl} S_{rk})\, \pi^r = 0$$

and

$$(n-2)\, S_{rk} \pi^r = 0.$$

Therefore

$$(n-3)\, g_{rh} R_{ljk}^h \pi^r = 0.$$

Since R_p is surjective, one has $\overline{A} = 0$.

" $\Leftarrow$ "

The condition $(X \circ Y) \circ Z = X \circ (Y \circ Z), \forall X, Y, Z \in \mathcal{X}(M)$, implies

$$g_{jk} \pi_s \pi^s \delta_i^r = (g_{ik} \pi_j + g_{jk} \pi_i - g_{ij} \pi_k)\, \pi^r.$$

This becomes

$$(g_{ik} \pi_j + g_{jk} \pi_i - g_{ij} \pi_k)\, \pi^r = 0.$$

Hence $\pi = 0$ and $\overline{A} = 0$. Therefore $R = \overline{R}$.

A linear connection ∇ is compatible with the Weyl structure W and is associated to the 1-form ω if

$$(\star)(\nabla_X g)(Y, Z) + W(g)(X) g(Y, Z) + \omega(Y) g(X, Z) + \omega(Z) g(X, Y) = 0.$$

There exists an unique connection ∇ σ-semi-symmetric satisfying $(\star)$:

$\nabla_X Y = \overset{\circ}{\nabla}_X Y + \frac{1}{2} W(g)(X) Y + (\frac{1}{2} W(g) + \sigma)(Y) X - g(X, Y)(\frac{1}{2} W(g) + \sigma - \omega)^{\#}$,

where $\overset{\circ}{\nabla}$ is the Levi-Civita connection associated to g.

Proposition 2.2. *Let $(M, \hat{g}, W)$ and $(M, \hat{g}, \overline{W})$ be Weyl manifolds. Let ∇ (resp. $\overline{\nabla}$) be the σ (resp. $\overline{\sigma}$)-semi-symmetric connection compatible with the Weyl structure W (resp. $\overline{W}$), associated to the 1-form ω (resp. $\overline{\omega}$). Then*

$$(\star\star) \qquad \overline{\nabla}_X Y = \nabla_X Y + p(X)Y + q(Y)X - g(X,Y)r^{\#},$$

holds, where $p = \frac{1}{2}(\overline{W}(g) - W(g)), q = p + \overline{\sigma} - \sigma, r = q - \overline{\omega} + \omega$.

Theorem 2.3 *Let* $(M, \hat{g})$ *be a conformal manifold ,* $n \geq 3$*.The tensor*

$$B^i_{jsl} = A^i_{jsl} + \frac{2}{n-2}\{\Omega^{mi}_{js}(A_{ml} - \frac{k}{2(n-1)}g_{ml}) - \Omega^{mi}_{jl}(A_{ms} - \frac{k}{2(n-1)}g_{ms})\}$$

is invariant under the transformation $(\star\star)$,
where $\Omega = \frac{1}{2}(I \otimes I - g \otimes \tilde{g})$ *is the Obata operator,* $(g.\tilde{g})(X, \sigma) = g(X, \sigma^{\#})$,
$A^i_{jsl} = R^i_{jsl} - \frac{1}{n}\delta^i_j R^p_{psl}$, $A_{ij} = A^s_{ijs}$ *and* k *is the scalar curvature.*

Proof. From $(\star\star)$ we find

$$\overline{R}^{\,i}_{jrl} = R^{\,i}_{jrl} + \delta^i_j(p_{rl} - p_{lr}) + 2\Omega^{mi}_{jr}q_{ml} - 2\Omega^{mi}_{jl}q_{mr},$$

where $p_{rl} = p_{r/l} + p_r\sigma_l$, $q_{rl} = q_{r/l} - q_r q_l + \frac{1}{2}g_{rl}\rho + q_r\sigma_l$ and $\rho = g^{ij}q_iq_j$.

We get

$$\overline{A}^i_{jrl} = A^i_{jrl} + 2\Omega^{mi}_{jr}q_{ml} - 2\Omega^{mi}_{jl}q_{mr}\,.$$

The previous relation leads to

$$\overline{A}_{jr} = A_{jr} - (n-2)q_{jr} - g_{jr}\widetilde{q}\,,$$

where $\widetilde{q} = Trq$. Therefore $\widetilde{q} = -\dfrac{\overline{k} - k}{2(n-1)}$ and we get

$$q_{jr} = -\frac{1}{n-2}\left\{\overline{A}_{jr} - A_{jr} - g_{jr}\frac{\overline{r} - k}{2(n-1)}\right\}\,.$$

Hence $B^{\,i}_{jrl} = \overline{B}^{\,i}_{jrl}$.

Theorem 2.4. *Let* $(M, \hat{g}, W)$ *be a Weyl manifold,* $n > 3$ *and* ∇ *the conformal Weyl connection. Then there exist the 1-forms* p *and* q *such that the semi-symmetric connection*

$$(\star\star\star) \qquad \overline{\nabla}_X Y = \nabla_X Y + q(Y)X + p(X)Y - g(X,Y)q^{\#}$$

has vanishing curvature tensor if and only if the tensor B *is zero.*

Proof. " $\Rightarrow$ " is obvious.

" $\Leftarrow$ " If $B^i_{jkl} = 0$, one considers the following two systems of equations

$$\begin{cases} p_{r/l} = p_{rl} \\ p_{rl} - p_{lr} = -\dfrac{1}{n}R^s_{srl}\,, \end{cases}$$

$$\begin{cases} q_{r/l} = q_{rl} + q_r q_l - \dfrac{1}{2}g_{rl}\rho \\ q_{rl} = \dfrac{1}{n-2}\left[A_{rl} - \frac{k}{2(n-1)}g_{rl}\right]\,. \end{cases}$$

We prove that if $B^i_{jrl} = 0$, $n > 3$, then the previous systems have solutions. From $p_{r/l} - p_{l/r} = \Phi_{r/l} - \Phi_{l/r} = -\dfrac{1}{n}R^s_{srl}$, where $\Phi_r == -\dfrac{1}{2}\left(W(g)\right)_r$, one has

$$p_r = -\Phi_r + \frac{\partial h}{\partial x^r}\,,$$

where h is arbitrary smooth mapping.

Since $B^i_{jrl} = 0$, using $\sum_{r,l,h}^{C} A^i_{jrl/h} = 0$, we get

$$\Omega^{mi}_{jl} q_{mr/h} - \Omega^{mi}_{jr} q_{ml/h} + \Omega^{mi}_{jh} q_{ml/r} -$$

$$-\Omega^{mi}_{jl} q_{mh/r} + \Omega^{mi}_{jr} q_{mh/l} - \Omega^{mi}_{jh} q_{mr/l} = 0\,.$$

Hence $(n-3)(q_{jr/l} - q_{jl/r}) = 0$. Because $n > 3$ the integrability conditions

$$q_{jr/l} - q_{jl/r} = 0$$

are satisfied.

Remark 2.5. The previous result remains valid when replace ∇ by a semi-symmetric conection, compatible with the Weyl structure W.

Open problems. Let $(M, \hat{g}, W)$, $(M, \hat{g}, \overline{W})$ be Weyl manifolds and $\pi, \overline{\pi}$ be closed 1-forms.

Let ∇ and $\overline{\nabla}$ be conformal π (resp. $\overline{\pi}$) -semi-symmetric connections.

1) The characterisation of the invariance of sectionale curvature.

2) The study of properties of the deformation algebra $\mathcal{U}(M, \overline{\nabla} - \nabla)$.

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