On $N(k)$-quasi Einstein manifolds satisfying certain conditions

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Abstract. This paper deals with $N(k)$-quasi Einstein manifolds satisfying the conditions $R(\xi, X) \cdot C = 0$ and $R(\xi, X) \cdot \widetilde{C} = 0$, where C and $\widetilde{C}$ denote the Weyl conformal curvature tensor and the quasi-conformal curvature tensor, respectively.

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1 Introduction

The notion of a quasi-Einstein manifold was introduced by M. C. Chaki in [2]. A non-flat n-dimensional Riemannian manifold (M, g) is said to be a *quasi Einstein manifold* if its Ricci tensor S satisfies

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \qquad X,Y \in TM \tag{1.1}$$

for some smooth functions a and $b \neq 0$, where η is a nonzero 1-form such that

$$g(X,\xi) = \eta(X), \qquad g(\xi,\xi) = \eta(\xi) = 1 \tag{1.2}$$

for the associated vector field ξ. The 1-form η is called the associated 1-form and the unit vector field ξ is called the generator of the manifold. If $b = 0$ then the manifold is reduced to an Einstein manifold. If the generator ξ belongs to k-nullity distribution $N(k)$ then the quasi Einstein manifold is called as an $N(k)$*-quasi Einstein manifold* [6]. In [6], it was proved that a conformally flat quasi-Einstein manifold is $N(k)$-quasi Einstein. Consequently, it was shown that a 3-dimensional quasi-Einstein manifold is an $N(k)$-quasi-Einstein manifold. The derivation conditions $R(\xi, X) \cdot R = 0$ and $R(\xi, X) \cdot S = 0$ were also studied in [6], where R and S denote the curvature and Ricci tensor, respectively. In [4], the derivation conditions $\mathcal{Z}(\xi, X) \cdot \mathcal{Z} = 0$, $\mathcal{Z}(\xi, X) \cdot R = 0$ and $R(\xi, X) \cdot \mathcal{Z} = 0$ on $N(k)$-quasi Einstein manifolds were studied, where $\mathcal{Z}$ is the concircular curvature tensor. Moreover, in [4], for an $N(k)$-quasi Einstein manifold,

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it was proved that $k = \frac{a+b}{n-1}$. In this study, we consider $N(k)$-quasi Einstein manifolds satisfying the conditions $R(\xi, X) \cdot C = 0$ and $R(\xi, X) \cdot \widetilde{C} = 0$. The paper is organized as follows: In Section 2, we give the definitions of Weyl conformal curvature tensor and quasi-conformal curvature tensor. In Section 3, we give a brief introduction about $N(k)$-quasi Einstein manifolds. In Section 4, we prove that for an $n \geq 4$ dimensional $N(k)$-quasi Einstein manifold, the condition $R(\xi, X) \cdot C = 0$ or $R(\xi, X) \cdot \widetilde{C} = 0$ holds on the manifold if and only if either $a = -b$ or the manifold is conformally flat.

2 Preliminaries

Let (M^n, g) be a Riemannian manifold. We denote by C and $\widetilde{C}$ the *Weyl conformal curvature tensor* [7] and the *quasi-conformal curvature tensor* [8] of (M^n, g) which are defined by

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}\{S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)QX - g(X,Z)QY\} \\ &\quad + \frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\} \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \widetilde{C}(X,Y)Z &= \lambda R(X,Y)Z + \mu\{S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)QX - g(X,Z)QY\} \\ &\quad - \frac{r}{n}[\frac{\lambda}{n-1} + 2\mu]\{g(Y,Z)X - g(X,Z)Y\}, \end{aligned} \tag{2.2}$$

respectively, where λ and μ are arbitrary constants, which are not simultaneously zero. Here Q is the Ricci operator defined by

$$S(X,Y) = g(QX,Y).$$

If $\lambda = 1$ and $\mu = -\frac{1}{n-2}$ then the quasi-conformal curvature tensor is reduced to the Weyl conformal curvature tensor. For an $n \geq 4$ dimensional Riemannian manifold if $C = 0$ then the manifold is said to be *conformally flat* [7], if $\widetilde{C} = 0$ then it is called as *quasi-conformally flat* [8]. $R \cdot C$ and $R \cdot \widetilde{C}$ are defined by

$$\begin{aligned} (R(U,X) \cdot C)(Y,Z,W) = R(U,X)C(Y,Z)W - C(R(U,X)Y,Z)W \\ -C(Y,R(U,X)Z)W - C(Y,Z)R(U,X)W. \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} (R(U,X) \cdot \widetilde{C})(Y,Z,W) = R(U,X)\widetilde{C}(Y,Z)W - \widetilde{C}(R(U,X)Y,Z)W \\ -\widetilde{C}(Y,R(U,X)Z)W - \widetilde{C}(Y,Z)R(U,X)W, \end{aligned} \tag{2.4}$$

respectively, for all vector fields U, X, Y, Z, W where $R(U,X)$ acts on C and $\widetilde{C}$ as a derivation [3].

3 $N(k)$-quasi Einstein manifolds

From (1.1) and (1.2) it follows that

$$S(X,\xi) = (a+b)\,\eta(X)\,, \tag{3.1}$$

$$r = na + b, \tag{3.2}$$

where r is the scalar curvature of M^n.

The *k-nullity distribution* $N(k)$ [5] of a Riemannian manifold M^n is defined by

$$N(k) : p \to N_p(k) = \{U \in T_pM \mid R(X,Y)U = k(g(Y,U)\,X - g(X,U)\,Y)\}$$

for all $X, Y \in TM^n$, where k is some smooth function. In a quasi-Einstein manifold M^n if the generator ξ belongs to some k-nullity distribution $N(k)$, then we get

$$R(\xi,Y)U = k(g(Y,U)\,\xi - \eta(U)Y) \tag{3.3}$$

and M^n is said to be an $N(k)$-*quasi Einstein manifold* [6]. In fact, k is not arbitrary as we see in the following:

Lemma 3.1. *[4] In an n-dimensional $N(k)$-quasi Einstein manifold it follows that*

$$k = \frac{a+b}{n-1}. \tag{3.4}$$

4 Main Results

In this section, we give the main results of the paper. At first, we give the following theorem:

Theorem 4.1. *Let M^n be an n-dimensional, $n \geq 4$, $N(k)$-quasi Einstein manifold. Then M^n satisfies the condition $R(\xi,X)\cdot C = 0$ if and only if either $a = -b$ or M is conformally flat.*

Proof. Assume that M^n, $(n \geq 4)$, is an $N(k)$-quasi Einstein manifold and satisfies the condition $R(\xi,X)\cdot C = 0$. Then from (2.3) we can write

$$\begin{aligned} 0 &= R(\xi,X)C(Y,Z)W - C(R(\xi,X)Y,Z)W \\ &\quad -C(Y,R(\xi,X)Z)W - C(Y,Z)R(\xi,X)W. \end{aligned} \tag{4.1}$$

So using (3.3) and (3.4) in (4.1) we find

$$\begin{aligned} 0 &= \frac{a+b}{n-1}\{C(Y,Z,W,X)\xi - \eta(C(Y,Z)W)X \\ &\quad -g(X,Y)C(\xi,Z)W + \eta(Y)C(X,Z)W \\ &\quad -g(X,Z)C(Y,\xi)W + \eta(Z)C(Y,X)W \\ &\quad -g(X,W)C(Y,Z)\xi + \eta(W)C(Y,Z)X\}\,. \end{aligned}$$

Then either $a+b=0$ or

$$
\begin{aligned}
0 \;=\; & C(Y,Z,W,X)\xi - \eta(C(Y,Z)W)X \\
& -g(X,Y)C(\xi,Z)W + \eta(Y)C(X,Z)W \\
& -g(X,Z)C(Y,\xi)W + \eta(Z)C(Y,X)W \\
& -g(X,W)C(Y,Z)\xi + \eta(W)C(Y,Z)X.
\end{aligned} \tag{4.2}
$$

Taking the inner product of (4.2) with ξ we get

$$
\begin{aligned}
0 \;=\; & C(Y,Z,W,X) - \eta(X)\eta(C(Y,Z)W) \\
& -g(X,Y)\eta(C(\xi,Z)W) + \eta(Y)\eta(C(X,Z)W) \\
& -g(X,Z)\eta(C(Y,\xi)W) + \eta(Z)\eta(C(Y,X)W) \\
& -g(X,W)\eta(C(Y,Z)\xi) + \eta(W)\eta(C(Y,Z)X).
\end{aligned} \tag{4.3}
$$

In view of (2.1), (1.1) and (3.3) we have

$$\eta(C(Y,Z)W) = 0. \tag{4.4}$$

So using (4.4) into (4.3) we obtain

$$C(Y,Z,W,X) = 0, \tag{4.5}$$

i.e., M^n is conformally flat. The converse statement is trivial. This completes the proof of the theorem. □

It is known [1] that a quasi-conformally flat manifold is either conformally flat or Einstein.

So we have the following corollary:

Corollary 4.2. *If (M^n, g) is a quasi-conformally flat $N(k)$-quasi Einstein manifold then it is conformally flat.*

As a generalization of Theorem 4.1 we have the following theorem:

Theorem 4.3. *Let M^n be an $N(k)$-quasi Einstein manifold. Then the condition $R(\xi,X)\cdot\widetilde{C} = 0$ holds on M^n if and only if either $a=-b$ or M^n is conformally flat with $\lambda = \mu(2-n)$.*

Proof. Since the manifold satisfies the condition $R(\xi,X)\cdot\widetilde{C} = 0$, by the use of (2.4)

$$
\begin{aligned}
0 \;=\; & R(\xi,Y)\widetilde{C}(U,V)W - \widetilde{C}(R(\xi,Y)U,V)W \\
& -\widetilde{C}(U,R(\xi,Y)V)W - \widetilde{C}(U,V)R(\xi,Y)W.
\end{aligned} \tag{4.6}
$$

Since M^n is $N(k)$-quasi Einstein by making use of (3.3) and (3.4) in (4.6) we get

$$
\begin{aligned}
0 = \frac{a+b}{n-1}\Big\{ & \widetilde{C}(U,V,W,Y)\xi - \eta(\widetilde{C}(U,V)W)Y \\
& -g(Y,U)\widetilde{C}(\xi,V)W + \eta(U)\widetilde{C}(Y,V)W \\
& -g(Y,V)\widetilde{C}(U,\xi)W + \eta(V)\widetilde{C}(U,Y)W \\
& -g(Y,W)\widetilde{C}(U,V)\xi + \eta(W)\widetilde{C}(U,V)Y\Big\}.
\end{aligned}
$$

Then either $a = -b$ or

$$\begin{aligned} 0 &= \widetilde{C}(U,V,W,Y)\xi - \eta(\widetilde{C}(U,V)W)Y \\ &-g(Y,U)\widetilde{C}(\xi,V)W + \eta(U)\widetilde{C}(Y,V)W \\ &-g(Y,V)\widetilde{C}(U,\xi)W + \eta(V)\widetilde{C}(U,Y)W \\ &-g(Y,W)\widetilde{C}(U,V)\xi + \eta(W)\widetilde{C}(U,V)Y. \end{aligned} \tag{4.7}$$

Assume that $a \neq -b$. Taking the inner product of (4.7) with ξ we obtain

$$\begin{aligned} 0 &= \widetilde{C}(U,V,W,Y) - \eta(\widetilde{C}(U,V)W)\eta(Y) \\ &-g(Y,U)\eta(\widetilde{C}(\xi,V)W) + \eta(U)\eta(\widetilde{C}(Y,V)W) \\ &-g(Y,V)\eta(\widetilde{C}(U,\xi)W) + \eta(V)\eta(\widetilde{C}(U,Y)W) \\ &-g(Y,W)\eta(\widetilde{C}(U,V)\xi) + \eta(W)\eta(\widetilde{C}(U,V)Y). \end{aligned} \tag{4.8}$$

On the other hand, from (2.2), (3.3) and (3.1) we have

$$\eta(\widetilde{C}(U,V)W) = \frac{b}{n}\left[\mu(n-2)+\lambda\right]\{g(V,W)\eta(U) - g(U,W)\eta(V)\}, \tag{4.9}$$

for all vector fields U, V, W on M^n. So putting (4.9) into (4.8) we obtain

$$\widetilde{C}(U,V,W,Y) = \frac{b}{n}\left[\mu(n-2)+\lambda\right]\{g(V,W)g(Y,U) - g(Y,V)g(U,W)\}.$$

Then using (2.2) we can write

$$\begin{aligned} &\lambda R(U,V,W,Y) + \mu\{S(V,W)g(Y,U) \\ &-S(U,W)g(V,Y) + g(V,W)S(Y,U) - g(U,W)S(V,Y)\} \\ &-\frac{na+b}{n}\left[\frac{\lambda}{n-1}+2\mu\right]\{g(Y,U)g(V,W) - g(Y,V)g(U,W)\} \\ =\ &\frac{b}{n}\left[\mu(n-2)+\lambda\right]\{g(V,W)g(Y,U) - g(Y,V)g(U,W)\}. \end{aligned} \tag{4.10}$$

Contracting (4.10) over Y and U we get

$$[\lambda + \mu(n-2)]\{S(V,W) - (a+b)g(V,W)\} = 0.$$

Since M^n is not Einstein $S(V,W) \neq (a+b)g(V,W)$ so we obtain $\lambda = \mu(2-n)$. Hence from (4.9)

$$\eta(\widetilde{C}(U,V)W) = 0 \tag{4.11}$$

holds for every vector fields U, V, W. So using (4.11) in (4.8) we obtain $\widetilde{C}(U,V,W,Y) = 0$. Then by the use of Corollary 4.2, the quasi-conformally flatness gives us the conformally flatness of the manifold. Conversely, if $\widetilde{C} = 0$ then the condition $R(\xi,X)\cdot\widetilde{C} = 0$ holds trivially. If $a = -b$ then $R(\xi,X) = 0$ then $R(\xi,X)\cdot\widetilde{C} = 0$. Hence the proof of the theorem is completed. □

So using Theorem 4.1 and Theorem 4.3 we have the following corollary:

Corollary 4.4. *Let M^n be an $N(k)$-quasi Einstein manifold. Then the following conditions are equivalent:*

i) $R(\xi, X) \cdot C = 0$ *with* $\lambda = \mu(2-n)$,

ii) $R(\xi, X) \cdot \widetilde{C} = 0$,

iii) M is conformally flat with $\lambda = \mu(2-n)$.

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