H-convex Riemannian submanifolds

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Abstract. Having in mind the well known model of Euclidean convex hypersurfaces [4], [5] and the ideas in [1], many authors defined and investigated the convex hypersurfaces of a Riemannian manifold. As it was proved by the first author in [7], there follows the interdependence between convexity and Gauss curvature of the hypersurface. This paper defines and studies the H-convexity of a Riemannian submanifold of arbitrary codimension, replacing the normal versor of a hypersurface with the mean curvature vector of the submanifold. The main results include: some properties of H-convex submanifolds, a characterization of the Chen definition of strictly H-convexity for submanifolds in real space forms [2], [3] and examples.

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1 Convex hypersurfaces in Riemannian manifolds

Let (N, g) be a complete finite-dimensional Riemannian manifold and M be an oriented hypersurface whose induced Riemannian metric is also denoted by g. We denote by ω the 1-form associated to the unit normal vector field ξ on the hypersurface M.

Let x be a point in $M \subset N$ and V a neighborhood of x in N such that $\exp_x : T_xN \to V$ is a diffeomorphism. The real-valued function defined on V by

$$F(y) = \omega_x(\exp_x^{-1}(y))$$

has the property that the set

$$TGH_x = \{y \in V \mid F(y) = 0\}$$

is a *totally geodesic hypersurface at x*, tangent to M at x. This hypersurface is the common boundary of the sets

$$TGH_x^- = \{y \in V \mid F(y) \leq 0\},\ TGH_x^+ = \{y \in V \mid F(y) \geq 0\}.$$

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Definition. The hypersurface M is called *convex* at $x \in M$ if there exists an open set $U \subset V \subset N$ containing x such that $M \cap U$ is contained either in TGH_x^- or in TGH_x^+.

A hypersurface M convex at x is said to be *strictly convex* at x if

$$M \cap U \cap TGH_x = \{x\}.$$

In [7] it was obtained a necessary condition for a hypersurface of a Riemannian manifold to be convex at a given point.

Theorem 1.1 *If M is an oriented hypersurface in N, convex at $x \in M$, then the bilinear form*

$$\Omega_x : T_xM \times T_xM \to R,\ \Omega_x(X, Y) = g(h(X, Y), \xi),$$

where ξ is the normal versor at x, and h is the second fundamental form of M, is semidefinite.

The converse of Theorem 1.1 is not true. To show this, we consider the surface $M : x^3 = (x^1)^2 + (x^2)^3$ in R^3. One observes that $0 \in M$, $\xi(0) = (0, 0, 1)$ and $TGH_0 : x^3 = 0$ is the plane tangent to M at the origin. On the other hand, if

$$c : I \to M,\ c(t) = (x^1(t), x^2(t), x^3(t))$$

is a C^2 curve such that $0 \in I$ and $c(0) = 0$, then $(x^3)''(0) = 2((x^1)'(0))^2$ and hence the function

$$f : I \to R,\ f(t) = \langle c(t) - 0, \xi(0)\rangle$$

satisfies the relations $f(t) = x^3(t)$ and $f''(0) = (x^3)''(0) = 2((x^1)'(0))^2$.

Since $\Omega_0(\dot{c}\ (0), \dot{c}\ (0)) = f''(0)$, and c is an C^2 arbitrary curve, one gets that Ω_0 is positive semidefinite. However M is not convex at the origin because the tangent plane $TGH_0 : x^3 = 0$ cuts the surface along the semicubic parabola

$$x^3 = 0, (x^1)^2 + (x^2)^3 = 0$$

and consequently in any neighborhood of the origin there exist points of the surface placed both below the tangent plane and above the tangent plane.

If the bilinear form Ω is definite at the point $x \in M$, then the hypersurface M is strictly convex at x.

The next results [7] establish a connection between the Riemannian manifolds admitting a function whose Hessian is positive definite and their convex hypersurfaces.

Theorem 1.2 *Suppose that the Riemannian manifold (N, g) supports a function $f : N \to R$ with positive definite Hessian. On each compact oriented hypersurface M in N there exists a point $x \in M$ such that the bilinear form $\Omega(x)$ is definite.*

Theorem 1.3 *If the Riemannian manifold (N, g) supports a function $f : N \to R$ with positive definite Hessian, then*

1) there is no compact minimal hypersurface in N;

2) if the hypersurface M is connected and compact and its Gauss curvature is nowhere zero, then M is strictly convex.

Theorem 1.4 *Let (N, g) be a connected and complete Riemannian manifold and $f : N \to R$ a function with positive definite Hessian. If x_0 is a critical point of f and $a_0 = f(x_0)$, then for any real number $a \in Im\, f \setminus \{a_0\}$, the hypersurface $M_a = f^{-1}\{a\}$ is strictly convex.*

2 H-convex Riemannian submanifolds

Having in mind the model of convex hypersurfaces in Riemannian manifolds, we define the H-convexity of a Riemannian submanifold of arbitrary codimension, replacing the normal versor of a hypersurface with the mean curvature vector of the submanifold.

Let (N, g) be a complete finite-dimensional Riemannian manifold and M be a submanifold in N of dimension n whose induced Riemannian metric is also denoted by g. Let x be a point in $M \subset N$, with $H_x \neq 0$ and V a neighborhood of x in N such that $\exp_x : T_xN \to V$ is a diffeomorphism. We denote by ω the 1-form associated to the *mean curvature vector* H of M.

The real-valued function defined on V by

$$F(y) = \omega_x(\exp_x^{-1}(y))$$

has the property that the set

$$TGH_x = \{y \in V \mid \ F(y) = 0\}$$

is a totally geodesic hypersurface at x, tangent to Mat x. This hypersurface is the common boundary of the sets

$$TGH_x^- = \{y \in V \mid \ F(y) \leq 0\}, \quad TGH_x^+ = \{y \in V \mid \ F(y) \geq 0\}.$$

Definition. The submanifold M is called *H-convex* at $x \in M$ if there exists an open set $U \subset V \subset N$ containing x such that $M \cap U$ is contained either in TGH_x^- or in TGH_x^+.

A submanifold M, which is H-convex at x, is called *strictly H-convex* at x if

$$M \cap U \cap TGH_x = \{x\}.$$

The next result is a necessary condition for a submanifold of a Riemannian manifold to be H-convex at a given point.

Theorem 2.1 *If M is a submanifold in N, H-convex at $x \in M$, then the bilinear form*

$$\Omega_x : T_xM \times T_xM \to R, \ \Omega_x(X, Y) = g(h(X, Y), H),$$

where h is the second fundamental form of M, is positive semidefinite.

Proof. We suppose that there is a open set $U \subset V \subset N$ which contains the point x such that $M \cap U \subset TGH_x^+$.

For an arbitrary vector $X \in T_xM$, let $c : I \to M \cap U$ be a C^2 curve, where I is a real interval such that $0 \in I$ and $c(0) = x$, $\dot{c}\,(0) = X$. As $c(I) \subset M \cap U \subset TGH_x^+$ the function $f = F \circ c : I \to R$ satisfies

$$f(t) \geq 0, \ \forall t \in I. \tag{2.1}$$

It follows that 0 is a global minimum point for f, and hence

$$0 = f^{'}(0) = \omega_x(d\exp_x^{-1}(c(0)))(\dot{c}\,(0)) = \omega_x(X), \tag{2.2}$$

$$0 \leq f^{''}(0) = \omega_x(d^2\exp_x^{-1}(c(0)))(\dot{c}\,(0), \dot{c}\,(0)) \tag{2.3}$$

$$+\omega_x(d\exp_x^{-1}(c(0)))(\ddot{c}\,(0)) = \omega_x(\ddot{c}\,(0)) = \Omega_x(X,X).$$

Since $X \in T_xM$ is an arbitrary vector, we obtain that Ω_x is positive semidefinite.

Remark. We consider $\{e_1, e_2, ..., e_n\}$ an orthonormal frame in T_xM. Since

$$\text{Trace}(\Omega_x) = g(\sum_{i=1}^{n} h(e_i, e_i), H_x) = ng(H_x, H_x) > 0,$$

the quadratic form Ω_x cannot be negative semidefinite, therefore $M \cap U$ cannot be contained in TGH_x^-. So, if the submanifold M is H-convex at the point x, then there exists an open set $U \subset V \subset N$ containing x such that $M \cap U$ is contained in TGH_x^+.

In the sequel, we prove that if the bilinear form Ω_x is positive definite, then the submanifold M is strictly H-convex at the point x. For this purpose we introduce a function similar to the height function used in the study of the hypersurfaces of an Euclidean space.

We fix $x \in M \subset N$ and a neighborhood V of x for which $\exp_x : T_xN \to V$ is a diffeomorphism. The function

$$F_{\omega_x} : V \to R,\ F_{\omega_x}(y) = \omega_x(\exp_x^{-1}(y)$$

has the property that it is affine on geodesics radiating from x.

We consider an arbitrary vector $X \in T_xM$ and a curve $c : I \to V$ such that $0 \in I$, $c(0) = x$, $\dot{c}\,(0) = X$. The function $f = F_{\omega_x} \circ c : I \to R$ satisfies

$$f'(0) = \omega_x(d\exp_x^{-1}(c(0))(\dot{c}\,(0)) = \omega_x(\dot{c}\,(0)) = \omega_x(X) = g(H, X) = 0$$

and hence $x \in M$ is a critical point of F_{ω_x}.

Theorem 2.2 *Let M be a submanifold in N. If the bilinear form Ω_x is positive definite, then M is strictly H-convex at the point x.*

Proof. The point $x \in M$ is a critical point of F_{ω_x} and $F_{\omega_x}(x) = 0$. On the other hand one observes that

$$\text{Hess}^N F_{\omega_x} = \text{Hess}^M F_{\omega_x} - \mathrm{d}F_{\omega_x}(\Omega H).$$

As F_{ω_x} is affine on each geodesic radiating from x, it follows $\text{Hess}^N F_{\omega_x} = 0$. It remains that

$$\text{Hess}^M F_{\omega_x}(x) = \Omega_x$$

and hence $\text{Hess}^M F_{\omega_x}$ is positive definite at the point x. In this way x is a strict local minimum point for F_{ω_x} in $M \cap V$, i.e., the submanifold M is strictly H-convex at x.

Remark. 1) The bilinear form Ω_x is positive (semi)definite if and only if the Weingarten operator A_H is positive (semi)definite.

2) If M is an hypersurface in N, x is a point in M with $H_x \neq 0$, then M is H-convex at x if and only if M is convex at x.

A class of strictly H-convex submanifolds into a Riemannian manifold is made from the curves which have the mean curvature nonzero.

Theorem 2.3 *Let (N, g) be a Riemannian manifold and $c : I \to N$ a regular curve which have the mean curvature nonzero, where I is an real interval. Then c is a strictly H-convex submanifold of N.*

Proof. We fix $t \in I$. As $T_{c(t)}c = \mathrm{Sp}\{\dot{c}\ (t)\}$, we obtain

$$H_{c(t)} = \frac{h(\dot{c}\ (t), \dot{c}\ (t))}{\left\|\dot{c}\ (t)\right\|^2}.$$

Since $\Omega(\dot{c}\ (t), \dot{c}\ (t)) = g(h(\dot{c}\ (t), \dot{c}\ (t)), H_{c(t)}) = \left\|\dot{c}\ (t)\right\|^2 \left\|H_{c(t)}\right\|^2 > 0$, the quadratic form Ω is positive definite. It follows that the curve c is a strictly H-convex submanifold of N.

3 H-convex Riemannian submanifolds in real space forms

Let us consider (M, g) a Riemannian manifold of dimension n. We fix $x \in M$ and $k \in \overline{2, n}$. Let L be a vector subspace of dimension k in T_xM. If $X \in L$ is a unit vector, and $\{e'_1, e'_2, ..., e'_k\}$ is an orthonormal frame in L, with $e'_1 = X$, we denote

$$\mathrm{Ric}_L(X) = \sum_{j=2}^{k} k(e'_1 \wedge e'_j),$$

where $k(e'_1 \wedge e'_j)$ is the sectional curvature given by $\mathrm{Sp}\{e'_1, e'_j\}$. We define *the Ricci curvature of k-order* at the point $x \in M$,

$$\theta_k(x) = \frac{1}{k-1} \min_{\substack{L, \dim L = k, \\ X \in L, \|X\| = 1}} Ric_L(X).$$

B. Y. Chen showed [2], [3] that the eigenvalues of the Weingarten operator of a submanifold in a real space form and the Ricci curvature of k-order satisfies the next inequality.

Theorem 3.1 *Let $(\widetilde{M}(c), \widetilde{g})$ be a real space form of dimension m and $M \subset \widetilde{M}(c)$ a submanifold of dimension n, and $k \in \overline{2, n}$. Then*
(i) $A_H \geq \frac{n-1}{n}(\theta_k(x) - c)I_n$.
(ii) If $\theta_k(x) \neq c$, then the previous inequality is strict.

Corollary 3.2 *If M is a submanifold of dimension n in the real space form $\widetilde{M}(c)$ of dimension m, $x \in M$ and there is a natural number $k \in \overline{2, n}$ such that $\theta_k(x) > c$, then M is strictly H-convex at the point x.*

The converse of previous corollary is also true in the case of hypersurfaces in a real space form.

Theorem 3.3 *If M is a hypersurface of dimension n of a real space form $\widetilde{M}(c)$ and M is strictly H-convex at a point x, then*

$$\theta_k(x) > c,\ \forall\ k \in \overline{2, n}.$$

Proof. Let x be a point in M, let H be the mean curvature of M and π a 2-plane in T_xM. We consider $\{X, Y\}$ an orthonormal frame in π and $\xi = \frac{H_x}{\|H_x\|}$. The second fundamental form of the submanifold M satisfies the relation

$$(3.1) \qquad h(U,V) = \frac{\Omega_x(U,V)}{\|H_x\|}\xi, \ \forall U,\, V \in T_xM.$$

On the other hand, the Gauss equation can be written

$$(3.2) \quad \widetilde{R}(X,Y,X,Y) = R(X,Y,X,Y) - \widetilde{g}(h(X,X),h(Y,Y)) + \widetilde{g}(h(X,Y),h(X,Y)).$$

Using the relation (3.1) and the fact that $\widetilde{M}(c)$ has the sectional curvature c, we obtain

$$(3.3) \qquad R(X,Y,X,Y) = c + \frac{1}{\|H_x\|^2}(\Omega_x(X,X)\Omega_x(Y,Y) - \Omega_x(X,Y)^2).$$

On the other hand, Ω_x is positive definite because M is strictly H-convex at the point x. From the Cauchy inequality, using the fact that X and Y are linear independent vectors, it follows

$$(3.4) \qquad \Omega_x(X,X)\Omega_x(Y,Y) - \Omega(X,Y)^2 > 0.$$

From (3.3) and (3.4) we find

$$(3.5) \qquad R(X,Y,X,Y) > c,$$

which means that the sectional curvature of M at the point x is strictly greater than c. Using the definition of Ricci curvatures, it follows that

$$\theta_k(x) > c, \ \forall \ k \in \overline{2,n}.$$

Let M be a submanifold of dimension n in the m dimensional sphere $S^m \subset R^{m+1}$. We denote with $\langle\ ,\ \rangle$ the metrics induced on S^m and M by the standard metric of R^{m+1}, with $\nabla,\ \nabla'$ and $\widetilde{\nabla}$ the Levi-Civita connections on M, S^m and R^{m+1} and with h the second fundamental form of M in R^{m+1}, with $h^{'}$ the second fundamental form of M in $S^m$and with $\widetilde{h}$ the second fundamental form of S^m in R^{m+1}.

Let X,Y be two vector fields tangents to M. The Gauss formula gives

$$(3.6) \qquad \nabla^{'}_X Y = \nabla_X Y + h^{'}(X,Y)$$

and

$$(3.7) \qquad \widetilde{\nabla}_X Y = \nabla^{'}_X Y + \widetilde{h}(X,Y) = \nabla_X Y + h^{'}(X,Y) + \widetilde{h}(X,Y).$$

Therefore

$$(3.8) \qquad h(X,Y) = h^{'}(X,Y) + \widetilde{h}(X,Y).$$

We fix a point $x \in M$ and an orthonormal frame $\{e_1, e_2, ..., e_n\}$ in T_xM. From the relation (3.8), one gets

$$(3.9) \qquad h(e_i,e_i) = h^{'}(e_i,e_i) + \widetilde{h}(e_i,e_i), \forall i \in \overline{1,n}$$

and hence

$$H = H' + \frac{1}{n}\sum_{i=1}^{n}\widetilde{h}(e_i, e_i), \tag{3.10}$$

where H is the mean curvature vector field of $M \subset R^{m+1}$ and H' is the mean curvature vector field of $S^m \subset R^{m+1}$. We introduce the quadratic forms

$$\Omega,\ \Omega' : T_xM \times T_xM \to R,$$

$$\Omega(X,Y) = \langle h(X,Y), H\rangle,\ \Omega'(X,Y) = \langle h'(X,Y), H'\rangle.$$

From (3.8) and (3.10), we obtain

$$\Omega(X,Y) = \langle h(X,Y), H\rangle = \langle h'(X,Y) + \widetilde{h}(X,Y), H' + \frac{1}{n}\sum_{i=1}^{n}\widetilde{h}(e_i,e_i)\rangle. \tag{3.11}$$

Using the fact that $h'(X,Y)$ and H' are tangent vectors at S^m, and $\widetilde{h}(X,Y)$ and $\sum_{i=1}^{n}\widetilde{h}(e_i,e_i)$ are normal vectors at S^m, one gets

$$\begin{aligned}\Omega(X,Y) &= \langle h'(X,Y), H'\rangle + \langle\widetilde{h}(X,Y), \frac{1}{n}\sum_{i=1}^{n}\widetilde{h}(e_i,e_i)\rangle \\ &= \Omega'(X,Y) + \langle\widetilde{h}(X,Y), \frac{1}{n}\sum_{i=1}^{n}\widetilde{h}(e_i,e_i)\rangle.\end{aligned} \tag{3.12}$$

Based on these considerations, we formulate the next

Theorem 3.4 *We consider a point $x \in M$.*

(i) If M is a submanifold in S^m, H-convex at x, then M is strictly H-convex at x, as submanifold in R^{m+1}.

(ii) If the Weingarten operator A_H of $M \subset R^{m+1}$ satisfies the inequality $A_H > I_n$, then M is a submanifold in S^m, strictly H-convex at x.

Proof. We denote with $\widetilde{X}$ the position vector field of S^m. The second fundamental form of $S^m \subset R^{m+1}$ is given by

$$\begin{aligned}\widetilde{h}(X,Y) &= \langle\widetilde{h}(X,Y), \widetilde{X}\rangle\widetilde{X} = \langle\widetilde{\nabla}_X Y, \widetilde{X}\rangle\widetilde{X} \\ &= -\langle Y, \widetilde{\nabla}_X\widetilde{X}\rangle\widetilde{X} = -\langle X,Y\rangle\widetilde{X},\ \forall X,Y \in \mathcal{X}(M).\end{aligned} \tag{3.13}$$

Using (3.13), we find

$$\frac{1}{n}\sum_{i=1}^{n}\widetilde{h}(e_i,e_i) = -\frac{1}{n}\sum_{i=1}^{n}\langle e_i,e_i\rangle\widetilde{X} = -\widetilde{X}. \tag{3.14}$$

From (3.12), (3.13), (3.14) and $\forall X,Y \in T_xM$, one gets

$$\Omega(X,Y) = \Omega'(X,Y) + \langle\langle X,Y\rangle\widetilde{X}, \widetilde{X}\rangle = \Omega'(X,Y) + \langle X,Y\rangle \tag{3.15}$$

We read (3.15) in two ways: (i) If M is a submanifold in S^m, H-convex at x, then $\Omega'(x)$ is positive semidefinite. Using the fact that $\langle,\ \rangle$ is positive definite, it follows that $\Omega(x)$ is positive definite, therefore M is a strictly H-convex submanifold in R^{m+1} at x. (ii) If $A_H > I_n$, then $\langle A_H X, X\rangle > \|X\|^2$, $\forall\ X \in T_x M$. Therefore

$$\Omega'(X,X) = \Omega(X,X) - \|X\|^2 = \langle h(X,X), H\rangle - \|X\|^2$$

$$= \langle A_H X, X\rangle - \|X\|^2 > 0, \forall X \in T_x M.$$

Consequently M is a submanifold in S^m, strictly H-convex at x.

Corollary 3.5 *If M is a minimal submanifold in S^m, then M is strictly H-convex at x as submanifold in R^{m+1}.*

Proof. Using the fact that M is minimal in S^m, one gets $\Omega' = 0$, therefore $\Omega(X,Y) = \langle X, Y\rangle$, $\forall\ X, Y \in \mathcal{X}(M)$. Consequently Ω is positive definite.

References

[1] R.L. Bishop, *Infinitesimal convexity implies local convexity*, Indiana Univ. Math. J. 24, 2 (1974), 169-172.

[2] B.Y. Chen, *Mean curvature shape operator of isometric immersions in real-space-forms*, Glasgow Math. J. 38 (1996), 87-97.

[3] B.Y. Chen, *Relations between Ricci curvature shape operator for submanifolds with arbitrary codimensions,* Glasgow Math. J. 41(1999), 33-41.

[4] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, vol 1,2, Interscience, New York, 1963, 1969.

[5] J.A. Thorpe, *Elementary topics in differential geometry*, Springer-Verlag 1979.

[6] C. Udrişte, *Convex hypersuprafaces*, Analele Şt. Univ. Al. I. Cuza, Iaşi, 32 (1986), 85-87.

[7] C. Udrişte, *Convex Functions and Optimization Methods on Riemannian Manifolds*, Mathematics and Its Applications, 297, Kluwer Academic Publishers Group, Dordrecht, 1994.

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