Complete Finsler manifolds and adapted coordinates

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Abstract. Recently we studied a certain second order differential equation which leads to the definition of some coordinate systems on Finsler manifolds. Here we consider two kinds of these coordinate systems and study connection dependence effect. More precisely we prove that the existence of such a coordinate system on a Finsler manifold endowed with a Berwald connection, reduces the Finsler structure to a locally Minkowskian one. Next, in the case of Cartan connection it is proved that existence of an special case of this coordinate system implies that the underlying Finsler manifold be isometric into sphere. Meanwhile some results on totally geodesic and umbilical hypersurfaces are obtained.

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1 Introduction

Coordinate systems play an essential role in the study of global differential geometry, particularly the Riemannian adapted coordinate system appears often in the study of a Riemannian manifold (M, g), see for instance Tashiro work [13] . In Physics, this coordinate system is closely connected to the study of collineations in General Relativity [8]. This coordinate system is also appeared in the study of pseudo-Riemannian manifolds, in form of a certain conformal transformation, see for example [7] and [9]. In Finsler geometry this coordinate system plays somehow a parallel role to the normal coordinate system on Riemannian geometry and leads to a classification of complete Finsler manifolds [2]. For the study of conformal vector fields on tangent bundle of Finsler manifolds, one can refer to [4].

Hence, it might be interesting and useful to study in this paper more tidily this coordinate system and find effect of different Finsler connections. More precisely, we prove the following theorems.

Theorem 1: *Let (M, g) be a Finsler manifold endowed with a Berwald connection. If there is an adapted coordinate system on M, then (M, g) is a locally Minkowskian space.*

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Indeed when we use an adapted coordinate system the role of Cartan connection is essential and can not be replaced by that of Berwald connection.

In particular we define a restricted adapted coordinate system what is called here special adapted coordinate system. In the case of Cartan connection we have the following result for special adapted coordinate system.

Theorem 3: *Let (M, g) be a complete connected Finsler manifold of dimension $n \geq 2$ endowed with a Cartan connection. If there is a special adapted coordinate system on M, then (M, g) is isometric to an n-sphere with a certain Finsler metric form.*

Meanwhile in Theorem 2, some necessary conditions for the related level sets, to be totally geodesic or umbilical is obtained. For our purpose, we adopt very often the notations of [3] and [1].

2 Preliminaries

Let M be a real n-dimensional manifold of class C^∞. We denote by $TM \to M$ the bundle of tangent vectors and by $\pi : TM_0 \to M$ the fiber bundle of non-zero tangent vectors. A *Finsler structure* on M is a function $F : TM \to [0, \infty)$, with the following properties: (I) F is differentiable (C^∞) on TM_0; (II) F is positively homogeneous of degree one in y, i.e. $F(x, \lambda y) = \lambda F(x, y), \forall \lambda > 0$, where we denote an element of TM by (x, y). (III) The Hessian matrix of F^2 is positive definite on TM_0; $(g_{ij}) := \left(\frac{1}{2}\left[\frac{\partial^2}{\partial y^i \partial y^j} F^2\right]\right)$. A *Finsler manifold* is a pair of a differentiable manifold M and a Finsler structure F on M. The tensor field $g = (g_{ij})$ is called the *Fundamental Finsler tensor* or *Finsler metric tensor*. Here, we denote a Finsler manifold by (M, g).

Let $V_v TM = ker \pi_*^v$ be the set of the vectors tangent to the fiber through $v \in TM_0$. Then a *vertical vector bundle* on M is defined by $VTM := \bigcup_{v \in TM_0} V_v TM$. A *non-linear connection* or a *horizontal distribution* on TM_0 is a complementary distribution HTM for VTM on TTM_0. Therefore we have the decomposition $TTM_0 = VTM \oplus HTM$. HTM is a *vector bundle* completely determined by the non-linear differentiable functions $N_i^j(x, y)$ on TM, called coefficients of the non-linear connection. Let HTM be a non-linear connection on TM and ∇ a linear connection on VTM, then the pair (HTM, ∇) is called a *Finsler connection* on the manifold M. Using the local coordinates (x^i, y^i) on TM we have the local field of frames $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\}$ on TTM. It is well known that we can choose a local field of frames $\{\frac{\delta}{\delta x_i}, \frac{\partial}{\partial y_i}\}$ adapted to the above decomposition, that is, $\frac{\delta}{\delta x_i} \in \Gamma(HTM)$ and $\frac{\partial}{\partial y_i} \in \Gamma(VTM)$ set of vector fields on HTM and VTM respectively, where $\frac{\delta}{\delta x_i} := \frac{\partial}{\partial x_i} - N_i^j \frac{\partial}{\partial y_j}$, and we use the *Einstein summation convention.*

Let $(M, g(x, y))$ be a Finsler manifold then a Finsler connection is called a *metric Finsler connection* if g is parallel with respect to ∇. According to the Miron terminology [10] this is equivalent to say that g is both horizontally and vertically metric. The *Cartan connection* is a metric Finsler connection for which the Deflection , horizontal and vertical torsion tensor fields vanish.

Let (M, g) be an n-dimensional Finsler manifold and $TM \to M$ the bundle of its tangent vectors. A global approach to the general Finsler connection, Cartan covariant derivatives and its applications is given in [5] and [6].

Let $\rho : M \to [0, \infty)$ be a scalar function on M and consider the following second order differential equation

$$\nabla^H \nabla^H \rho = \phi g, \tag{2.1}$$

where ∇^H is the Cartan horizontal covariant derivative and ϕ is a function of x alone, then we say that Eq. (2.1) has a solution ρ. The connected component of a regular hypersurface defined by $\rho =$ *constant*, is called a *level set of* ρ. We denote by $\mathtt{grad}\rho$ the gradient vector field of ρ which is locally written in the form $\mathtt{grad}\rho = \rho^i \frac{\partial}{\partial x^i}$, where $\rho^i = g^{ij}\rho_j$, $\rho_j = \frac{\partial \rho}{\partial x^j}$ and $i, j, ...$ run over the range $1, ..., n$. The partial derivatives ρ_j are defined on the manifold M while ρ^i the components of $\mathtt{grad}\rho$ may be defined on its slit tangent bundle TM_0. Hence, $\mathtt{grad}\rho$ can be considered as a section of $\pi^*TM \to TM_0$, the pulled-back tangent bundle over TM_0, and its trajectories lie on TM_0.

One can easily verify that the canonical projection of the trajectories of the vector field $\mathtt{grad}\rho$ are geodesic arcs on M [2]. Therefore, once a non-trivial solution of (2.1) exist on M, we can choose a local coordinates $(u^1 = t, u^2, ..., u^n)$ such that t is the parameter of the geodesic containing the projection of a trajectory of the vector field $\mathtt{grad}\rho$ and also the level sets of ρ are given by $t =$constant. These geodesics are called *t-geodesics*. Since in this local coordinates, the level sets of ρ are given by $t =$constant, ρ may be considered as a function of t only. In the sequel we will refer to these level sets or hyper-surfaces as *t-levels* and this local coordinate system as *adapted coordinates*. The parameter t may be regarded as the arc-length parameter of t-geodesics.

Let (M, g) be a Finsler manifold and ρ a non-trivial solution of Eq. (2.1) on M. Then in an adapted coordinates, components of the Finsler metric tensor g are given by $g_{\alpha 1} = g_{1\alpha} = 0$, where here and every where in this paper, the Greek indices $\alpha, \beta, \gamma, ...$ run over the range $2, 3, ..., n$.

If $g(\mathtt{grad}\rho, \mathtt{grad}\rho) = 0$ in some points of M, then M possesses some interesting properties. A point o of (M, g) is called a *critical point* of ρ if the vector field $\mathtt{grad}\rho$ vanishes at o, or equivalently if $\rho'(o) = 0$. All other points are called ordinary points of ρ on M.

Let $\rho : M \to [0, \infty)$ be a scalar function on M. If there is a non-trivial solution of

$$\nabla^H \nabla^H \rho + C^2 \rho g = 0, \tag{2.2}$$

where ∇^H is the Cartan horizontal covariant derivative, then we say that there is an *special adapted coordinate system* on M. In fact in Eq. (2.1) we assume $\phi = -C^2\rho$, where C is a constant.

2.1 Finsler Submanifolds.

Let $\overline{M}$ and M be two differentiable manifolds of dimension m and $m+n$ and let (u^α) and (x^i) be the local coordinates on $\overline{M}$ and M, respectively. We denote by the pairs (u^α, v^α) and (x^i, y^i) consisting of position and direction, the line elements of $T\overline{M}$ and TM, where $\alpha, \beta, ...$ and $i, j, ...$ run over the range $1, ..., m$ and $1, ..., m+n$ respectively. Let $f : \overline{M} \to M$ be a smooth mapping, given by $(u^1, ..., u^m) \to x^i(u^1, ..., u^m)$, where $i = 1, ..., m+n$. The differential or tangent mapping of f is

$$f_* : T_u\overline{M} \to T_x M,$$
$$(u^\alpha, v^\alpha) \to (x^i(u), y^i(u,v)),$$

where $y^i(u,v) = B^i_\alpha v^\alpha$ and $B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$.
If f_* is injective at every point u of $\overline{M}$, that is, rank $[B^i_\alpha] = m$, then $\overline{M}$ is called an *immersed submanifold* or simply *submanifold* of M.
Next consider an $(m+n)$-dimensional Finsler manifold (M,g). The Finsler structure F induces on $T\overline{M}$ a Finsler structure $\overline{F}$ defined by $\overline{F}(u,v) := F(x(u), y(u,v))$. Putting $\overline{g}_{\alpha\beta} := \frac{1}{2}\frac{\partial^2 \overline{F}^2}{\partial v^\alpha \partial v^\beta}$, one obtain by direct calculation

$$\overline{g}_{\alpha\beta}(u,v) = g_{ij}(x(u), y(u,v))B^{ij}_{\alpha\beta}, \tag{2.3}$$

where $B^{ij}_{\alpha\beta} = B^i_\alpha B^j_\beta$. Therefore the pair $(\overline{M}, \overline{g})$ is a Finsler manifold, called *Finsler submanifold* of (M,g).

2.2 Induced Finsler connections.

Next we consider a Finsler connection (HTM, ∇) on the Finsler manifold (M,g) and look for the induced geometric objects on the Finsler submanifold $(\overline{M}, \overline{g})$. We denote by h and v the projection morphisms of $TT\overline{M}_0$ on $HT\overline{M}$ and $VT\overline{M}$ respectively. Then we have the *Gauss formula* as follows

$$\nabla_X Y^v = \overline{\nabla}_X Y^v + B(X, Y^v), \tag{2.4}$$

where $X, Y \in \Gamma(T\overline{M}_0)$, $Y^v \in \Gamma(VT\overline{M})$, $\overline{\nabla}_X Y^v \in \Gamma(VT\overline{M})$ and $B(X,Y^v) \in \Gamma(VT\overline{M}^\perp)$. B is a $\Gamma(VT\overline{M}^\perp)$-valued bilinear mapping on $\Gamma(T\overline{M}_0) \times \Gamma(VT\overline{M})$ called the *second fundamental form* of (M,g). It is easy to check that ∇ is a linear connection on the vertical vector bundle VTM of $(\overline{M}, \overline{g})$. Thus $(HT\overline{M}, \overline{\nabla})$ is a Finsler connection on $(\overline{M}, \overline{g})$, called the *induced Finsler connection.* By means of the second fundamental form B, one defines the *horizontal*(h)- and *vertical* (v)- *second fundamental forms* H and V of $(\overline{M}, \overline{g})$, by the bilinear mappings

$$\begin{gathered} H : \Gamma(HT\overline{M}) \times \Gamma(HT\overline{M}) \to \Gamma(VT\overline{M}^\perp), \\ (X^h, Y^h) \to B(X^h, QY^h), \end{gathered} \tag{2.5}$$

and

$$\begin{gathered} V : \Gamma(VT\overline{M}) \times \Gamma(VT\overline{M}) \to \Gamma(VT\overline{M}^\perp), \\ (X^v, Y^v) \to B(X^v, Y^v), \end{gathered} \tag{2.6}$$

respectively, for any $X, Y \in \Gamma(TT\overline{M})$, where Q is defined by

$$Q : \Gamma(TT\overline{M}) \to \Gamma(TT\overline{M}),$$
$$X \to QX,$$

that is for $X = X^i \frac{\delta}{\delta x^i} + \dot{X}^i \frac{\partial}{\partial y^i}$ we have $QX = \dot{X}^i \frac{\delta}{\delta x^i} + X^i \frac{\partial}{\partial y^i}$.

Let $(\overline{M}, \overline{g})$ be a Finsler submanifold of a Finsler manifold (M, g). Then $(\overline{M}, \overline{g})$ is called a *totally umbilical Finsler submanifold* if there exists a smooth function ρ on $\overline{M}$ such that on each coordinate neighborhood $U \subset \overline{M}$ we have

$$h_{\alpha\beta} = \rho g_{\alpha\beta}, \qquad \forall \alpha, \beta, \in \{1, ..., m\},$$

where $h_{\alpha\beta}$ is the components of the second fundamental form of $(\overline{M}, \overline{g})$. The *h-totally* and *v-totally umbilical Finsler submanifolds* are defined by replacing the second fundamental form in the above equation, with h-second and v-second fundamental forms, respectively.

3 Finsler connections and adapted coordinates

In this section, we study the properties of adapted coordinate system related to the Berwald and Cartan Finsler connections. Let ρ be a solution of Eq. 2.1, we consider an adapted coordinate system (u^i, U) on M, where U is the open set of all ordinary points of ρ. We denote by (u^i, v^i) the line element on TM_0 related to this coordinate system. We have the following proposition.

Lemma 1. *[2] Let (M, g) be a Finsler manifold and U the set of all ordinary points of a non-trivial solution ρ of Eq. 2.1 on M. Then there is a local coordinate system (u^i, U), such that the coefficients of the non-linear connection vanish.*

Case (a) Berwald connection and adapted coordinates
Let (M, g) be a Finsler manifold endowed with a Berwald connection, such that there exists a non-trivial solution ρ of Eq. 2.1 on M. Then we have the following theorem.

Theorem 1. *Let (M, g) be a Finsler manifold endowed with a Berwald connection. If there is an adapted coordinate system on M, then (M, g) is a locally Minkowskian space.*

Proof. Let's consider an adapted coordinate system (u^i, U) on M. Then from the L-metrical property of the Berwald connection, we have $\frac{\partial F}{\partial u^i} = v_r G^r_i$, where F is the Finsler structure of (M, g) and $v_r = g_{ir} v^i$. By means of Lemma 1, we get $\partial_i F = 0$, that is, F depends only on y and by definition, (M, g) is a locally Minkowskian space.
□

Case (b) Cartan connection and adapted coordinates
As another consequence of the Lemma 1, we have ${\Gamma^*}^i_{jk} = \gamma^i_{jk}$, where ${\Gamma^*}^i_{jk}$ are coefficients of the Cartan connection, and $\gamma^i_{jk}(u, v)$ are formal Christoffel symbols. Thus we have the following proposition.

Lemma 2. *Let (M, g) be a Finsler manifold endowed with the Cartan connection. In the adapted coordinate system the coefficients of Cartan connection reduce to the formal Christoffel symbols.*

Let $i = i^l \frac{\partial}{\partial x^l}$ be the unit vector field in the direction of Y and let $\tau = \sqrt{g(Y, Y)}$ denotes the length of Y. Then we have

$$\rho^l = \tau i^l, \qquad i^l = \frac{1}{\tau} \rho^l. \tag{3.1}$$

Differentiating covariantly[1] the first equation of (3.1), and using (2.1), we have

$$(\nabla^H_k \tau) i_l + \tau(\nabla^H_k i_l) = \phi g_{lk}. \tag{3.2}$$

Contracting this equation with i^l, we get

$$\nabla^H_k \tau = \phi i_k. \tag{3.3}$$

Replacing this equation in (3.2), gives

$$\nabla^H_k i_l = \frac{\phi}{\tau}(g_{lk} - i_k i_l). \tag{3.4}$$

We remark that, the set of ordinary points of ρ is an open set and in the sequel we shall confine our consideration to the open set U of ordinary points.
Let $\overline{M}(p)$ be the t-level through an ordinary point p in $U \subset M$. The unit vector field i is normal to $\overline{M}(p)$ at any point of $\overline{M}(p)$. Let's consider the local coordinates (u^α) in $\overline{M}(p)$ such that $\overline{M}(p)$ be expressed by the parametric equations $x^i = x^i(u^\alpha)$ in U, where, here and in the sequel the indices $i, j, k, ...$ run over the range $1, 2, ..., n$ and $\alpha, \beta, \gamma, ...$ run over the range $2, 3, ..., n$.

The induced metric tensor $\overline{g}_{\gamma\beta}$ of $\overline{M}(p)$ is given by $\overline{g}_{\gamma\beta} = g_{ij} B^i_\gamma B^j_\beta$ where $B^i_\gamma = \partial_\gamma x^i$ and $\partial_\gamma = \frac{\partial}{\partial u^\gamma}$. Then the h-second fundamental form of $\overline{M}$ defined by $h_{\gamma\beta} := (\nabla^H_\gamma B^k_\beta) i_k$ becomes

$$(\nabla^H_\gamma B^k_\beta) i_k = (\partial_\gamma B^k_\beta + \Gamma^{*\ k}_{ij} B^i_\gamma B^j_\beta - \overline{\Gamma}^{*\alpha}_{\gamma\beta} B^k_\alpha) i_k, \tag{3.5}$$

where $\overline{\Gamma}^{*\alpha}_{\gamma\beta}$ are Finsler connection's coefficients in $\overline{M}$ with induced metric $\overline{g}_{\gamma\beta}$ and $\Gamma^{*\ k}_{i\gamma} = \Gamma^{*\ k}_{ij} B^j_\gamma$. Thus we have

$$\partial_\beta B^k_\gamma + \Gamma^{*\ k}_{ij} B^i_\gamma B^j_\beta - \overline{\Gamma}^{*\alpha}_{\gamma\beta} B^k_\alpha = h_{\gamma\beta} i^k. \tag{3.6}$$

Since $B^k_\beta i_k = 0$, we have

$$h_{\gamma\beta} = -(\nabla^H_j i_k) B^j_\gamma B^k_\beta. \tag{3.7}$$

Substituting (3.4) into this equation, we get

$$h_{\gamma\beta} = h \overline{g}_{\gamma\beta}, \qquad \text{where} \quad h = \frac{-\phi}{\tau}. \tag{3.8}$$

When $\phi(p) = 0$, then $h = 0$ and hence the components of the h-second fundamental form $h_{\gamma\beta}$ vanish. Therefore, the t-level $\overline{M}(p)$ is h-totally geodesic. When $\phi(p) \neq 0$, by definition $\overline{M}(p)$ is h-totally umbilical. Thus we have the following result.

Theorem 2. *The level set $\overline{M}(p)$ is h-totally geodesic if $\phi(p) = 0$, and it is h-totally umbilical if $\phi(p) \neq 0$.*

[1] Everywhere in this section, by covariant derivative we mean the Cartan horizontal covariant derivative.

If we take the last $(n-1)$ coordinates (u^α) of (u^i) where α run over the range $2, ..., n$; as a local coordinate system in each t-level in U, then $B^i_\alpha = \partial_\alpha u^i = \delta^i_\alpha$, and for induced metric tensor $\overline{g}_{\alpha\beta}$ on each t-level $\overline{M}(p)$, we have

$$\overline{g}_{\alpha\beta} = g_{\alpha\beta}.$$

Since the components of the unit vector field i are $i^k = \delta^k_1$ in adapted coordinate system (u^i, U), the equation (3.6) for $k = 1$ reduces to

$$\Gamma^{*1}_{\gamma\beta} = h_{\gamma\beta} = h g_{\gamma\beta}. \tag{3.9}$$

Using $\rho^l = g^{il}\rho_i$ and the fact that in this coordinate ρ is a function of u^1, (3.1) reduce to $\tau = \rho_1 = \rho'$, where prime denotes the ordinary differentiation with respect to u^1. Thus from (3.3) and (3.8) we get

$$\tau = \rho', \quad \phi = \rho'', \quad h = \frac{-\phi}{\tau} = \frac{-\rho''}{\rho'}. \tag{3.10}$$

By replacing h in (3.9) and in according to the Theorem 2, we have $\frac{\partial g_{\gamma\beta}}{\partial u^1} = \frac{2\rho''}{\rho'} g_{\gamma\beta}$. Therefore by integrating, the components $g_{\gamma\beta}$ are written in the form

$$g_{\gamma\beta} = \rho'^2 f_{\gamma\beta}, \tag{3.11}$$

where $f_{\gamma\beta}$ are functions of the $(n-1)$ coordinates u^α. Since the metric tensor $g_{\gamma\beta}$ is positive definite, so is the matrix $(f_{\gamma\beta})$. Therefore by choosing an adapted coordinate system we have the following proposition.

Proposition 1. *Let (M, g) be a Finsler manifold endowed with a Cartan connection. If there is an adapted coordinate system on M, then the metric form of M is given by*

$$ds^2 = (du^1)^2 + \rho'^2 f_{\gamma\beta} du^\gamma du^\beta. \tag{3.12}$$

In what follows we shall sometimes write u for the first coordinate u^1 of an adapted coordinate system. The length of the gradient vector field is constant on each t-level through an ordinary point, and all points of a t-level are ordinary, that is, $\rho' \neq 0$ in any point of a t-level. Therefore from definition of a t-level, it is a closed submanifold. Let $\overline{M}$ be an $(n-1)$-dimensional manifold diffeomorphic to a t-level $(\overline{M}(p), \overline{g}_{\gamma\beta})$ and having $f_{\gamma\beta}$ as Finsler metric tensor. We can consider $f_{\gamma\beta}$ as a positive constant coefficient of Finsler metric $g_{\gamma\beta}$, so $f_{\gamma\beta}$ is a Finsler metric. Since the coefficients ρ'^2 are positive constant in every t-level, the Finsler manifold $(\overline{M}, f_{\gamma\beta})$ and t-levels neighboring $(\overline{M}(p), \overline{g}_{\gamma\beta})$ are locally homothetically diffeomorphic to each other. Therefore the connection coefficients constructed from $f_{\gamma\beta}$ on $\overline{M}$ have the same expressions $\overline{\Gamma}^{*\alpha}_{\beta\gamma}$ as those of the Finsler induced metric $\overline{g}_{\gamma\beta}$ in $\overline{M}(p)$, see for more details [2].

We notice that, along any geodesic with arc-length u, the equation (2.1) reduces to an ordinary differential equation

$$\frac{d^2\rho}{du^2} = \phi(\rho), \tag{3.13}$$

where ϕ is a function of ρ and it is differentiable in ρ at ordinary points of ρ.

In equation (2.1) for $k = l = 1$, we have $\nabla^H_1 \rho_1 = \frac{\delta \rho_1}{\delta u} - \Gamma^{*i}_{\ 11}\rho_i$ and this equation in adapted coordinate system is given by $\nabla^H_1 \rho_1 = \frac{d^2\rho}{du^2}$.

Case(c) Cartan connection and special adapted coordinates

Theorem 3. *Let (M, g) be a complete connected Finsler manifold of dimension $n \geq 2$ endowed with a Cartan connection. If there is a special adapted coordinate system on M, then (M, g) is isometric to an n-sphere of radius $1/C$ with a certain Finsler metric form.*

Proof. Let (M, g) be an n-dimensional Finsler manifold which admits a special adapted coordinate system. That is, there is a non-trivial solution of Eq. (2.2) where $K = C^2 > 0$ is constant. Along any geodesic with arc-length t, Eq. (2.2) reduces to the following ordinary differential equation, see for instance [1] or [5].

$$\frac{d^2\rho}{dt^2} + K\rho = 0. \tag{3.14}$$

By a suitable choice of the arc-length t, a solution of Eq. (3.14) is given by

$$\rho(t) = A\cos\sqrt{K}t, \tag{3.15}$$

and its first derivative is

$$\rho'(t) = -A\sqrt{K}\sin\sqrt{K}t. \tag{3.16}$$

We can see at a glance, that it might appear two critical points corresponding to $t = 0$ and $t = \frac{\pi}{\sqrt{K}}$ on M, where these points are periodically repeated. Thus ρ has exactly two critical points on M and by an extension of the Milnor theorem to Finsler manifolds [1], (M, g) is homeomorphic to an n-sphere. Hence, if ρ is a non-trivial solution of Eq. (3.14), then it can be written in the following form

$$\rho(t) = \frac{-1}{\sqrt{K}}\cos\sqrt{K}t\ , \quad (A = \frac{-1}{\sqrt{K}}). \tag{3.17}$$

Taking into account Eq. (3.12), the metric form of M becomes

$$ds^2 = dt^2 + (\sin\sqrt{K}t)^2\overline{ds}^2, \tag{3.18}$$

where $\overline{ds}^2$ is the metric form of a t-level of ρ given by $\overline{ds}^2 = f_{\gamma\beta}du^\gamma du^\beta$. This is the polar form of a Finsler metric on a standard sphere of radius $1/\sqrt{K}$ [12]. Therefore, (M, g) is isometric to an n-sphere of radius $1/C$ with the Finsler metric form (3.18), where $\overline{ds}^2$ is the metric form of a t-level of ρ given by $\overline{ds}^2 = f_{\gamma\beta}du^\gamma du^\beta$. This completes the proof of the theorem. □

Related results can be found in [11].

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