Canonical nonlinear connections in the multi-time Hamilton Geometry

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Abstract. In this paper we study some geometrical objects (d-tensors, multi-time semisprays of polymomenta and nonlinear connections) on the dual 1-jet vector bundle $J^{1*}(\mathcal{T}, M) \to \mathcal{T} \times M$. Several geometric formulas, which connect the last two geometrical objects, are also derived. Finally, a canonical nonlinear connection produced by a Kronecker h-regular multi-time Hamiltonian function is given.

M.S.C. 2000: 53B40, 53C60, 53C07.
Key words: dual 1-jet spaces, d-tensors, multi-time semisprays of polymomenta, Kronecker h-regular multi-time Hamiltonians, canonical nonlinear connections.

1 Introduction

From a geometrical point of view, we point out that the 1-jet spaces are fundamental ambient mathematical spaces used in the study of classical and quantum field theories (in their contravariant Lagrangian approach). For this reason, the differential geometry of these spaces was intensively studied by many authors (please see, for example, Saunders [18] or Asanov [1] and references therein). In this direction, it is important to note that, following the geometrical ideas initially stated by Asanov in [1], a *multi-time Lagrange contravariant geometry on 1-jet spaces* (in the sense of distinguished connection, torsions and curvatures) was recently constructed by Neagu and Udrişte [14], [16], [17] and published by Neagu in the book [15]. This geometrical theory is a natural multi-parameter extension on 1-jet spaces of the already classical *Lagrange geometrical theory on the tangent bundle* elaborated by Miron and Anastasiei [12]. Note that recent new geometrical developments, which relies on the multi-time Lagrange contravariant geometrical ideas from [15], are given by Udrişte and his co-workers in the paper [19].

From the point of view of physicists, the differential geometry of the dual 1-jet spaces was also studied because the dual 1-jet spaces represent the *polymomentum phase spaces* for the *covariant Hamiltonian formulation of the field theory* (this is a natural *multi-parameter*, or *multi-time*, extension of the classical Hamiltonian formalism from Mechanics). Thus, in order to quantize the covariant Hamiltonian field

Balkan Journal of Geometry and Its Applications, Vol.14, No.2, 2009, pp. 1-12.

theory (this is the final purpose in the framework of quantum field theory), the *covariant Hamiltonian differential geometry* was developed in three distinct ways:

- the *multisymplectic covariant geometry* elaborated by Gotay, Isenberg, Marsden, Montgomery and their co-workers [7], [8];

- the *polysymplectic covariant geometry* investigated by Giachetta, Mangiarotti and Sardanashvily [6];

- the *De Donder-Weyl covariant Hamiltonian geometry* intensively studied by Kanatchikov (please see [9], [10] and references therein).

It is important to note that these three distinct geometrical-physics variants differ by the multi-time phase space and the geometrical techniques used in this study. Also, we point out that there are different point of views for the study of the multi-time Hamilton equations, which appear in first order field theory. Please see, for example, Duca and Udrişte's paper [5].

Inspired by the Cartan covariant Hamiltonian approach of classical Mechanics, the studies of Miron [11], Atanasiu [2], [3] and their co-workers led to the development of the *Hamilton geometry on the cotangent bundle* exposed in the book [13]. Thus, in such a physical and geometrical context, suggested by the multi-time framework of the De Donder-Weyl covariant Hamiltonian formulation of Physical Fields, the aim of this paper is to present some basic geometrical concepts on dual 1-jet spaces (we refer to distinguished *(*written briefly, *d-)* tensors, multi-time semisprays of polymomenta and nonlinear connections), necessary to the development of a subsequent *multi-time covariant Hamilton geometry* (in the sense of *d-linear connections*, *d-torsions* and *d-curvatures* [4]), which to be a natural *multi-parameter*, or *poly-momentum*, generalization of the *Hamilton geometry on the cotangent bundle* [13].

2 The dual 1-jet vector bundle $J^{1*}(\mathcal{T}, M)$

We start our geometrical study considering two smooth real manifolds $\mathcal{T}^m$ and M^n having the dimensions m, respectively n, and which are coordinated by $(t^a)_{a=\overline{1,m}}$, respectively $(x^i)_{i=\overline{1,n}}$. We point out that, throughout this paper, the indices a, b, c, d, f, g run over the set $\{1, 2, \ldots, m\}$ and the indices i, j, k, l, r, s run over the set $\{1, 2, \ldots, n\}$.

Let us consider the 1-jet space $E \overset{not}{=} J^1(\mathcal{T} \times M) \to \mathcal{T} \times M$, coordinated by (t^a, x^i, x^i_a), where x^i_a behave as *partial derivatives.*

Remark 2.1. From a physical point of view, the manifold $\mathcal{T}$ can be regarded as a *temporal* manifold or, better, a *multi-time* manifold, while the manifold M can be regarded as a *spatial* one. In this way, the coordinates x^i_a are regarded as *partial velocities.* In other words, the 1-jet vector bundle $J^1(\mathcal{T}, M) \to \mathcal{T} \times M$ can be regarded as a *bundle of configurations* for "multi-time" physical events.

It is well known that the transformations of coordinates on the 1-jet vector bundle $J^1(\mathcal{T}, M)$ are given by

$$\begin{cases} \tilde{t}^a = \tilde{t}^a(t^b) \\ \tilde{x}^i = \tilde{x}^i(x^j) \\ \tilde{x}^i_a = \dfrac{\partial \tilde{x}^i}{\partial x^j}\dfrac{\partial t^b}{\partial \tilde{t}^a}x^j_b, \end{cases} \tag{2.1}$$

where $\det(\partial\tilde{t}^a/\partial t^b) \neq 0$ and $\det(\partial\tilde{x}^i/\partial x^j) \neq 0$.

Now, using the general theory of vector bundles (please see [12], for example), let us consider the *dual 1-jet vector bundle* $E^* \overset{not}{=} J^{1*}(\mathcal{T}, M) \to \mathcal{T} \times M$, whose local coordinates are denoted by (t^a, x^i, p^a_i).

Remark 2.2. According to the Kanatchikov's physical terminology [9], which generalizes the Hamiltonian terminology from Analytical Mechanics, the coordinates p^a_i are called *polymomenta* and the dual 1-jet space E^* is called the *polymomentum phase space*.

It is easy to see that the transformations of coordinates on the dual 1-jet space E^* have the expressions

$$\begin{cases} \tilde{t}^a = \tilde{t}^a(t^b) \\ \tilde{x}^i = \tilde{x}^i(x^j) \\ \tilde{p}^a_i = \dfrac{\partial x^j}{\partial \tilde{x}^i}\dfrac{\partial \tilde{t}^a}{\partial t^b}p^b_j, \end{cases} \tag{2.2}$$

where $\det(\partial\tilde{t}^a/\partial t^b) \neq 0$ and $\det(\partial\tilde{x}^i/\partial x^j) \neq 0$. In the sequel, doing a transformation of coordinates (2.2) on E^*, we obtain

Proposition 2.3. *The elements of the local natural basis* $\left\{\dfrac{\partial}{\partial t^a}, \dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial p^a_i}\right\}$ *of the Lie algebra of vector fields* $\mathcal{X}(E^*)$ *transform by the rules*

$$\begin{aligned} \frac{\partial}{\partial t^a} &= \frac{\partial \tilde{t}^b}{\partial t^a}\frac{\partial}{\partial \tilde{t}^b} + \frac{\partial \tilde{p}^b_j}{\partial t^a}\frac{\partial}{\partial \tilde{p}^b_j}, \\ \frac{\partial}{\partial x^i} &= \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial}{\partial \tilde{x}^j} + \frac{\partial \tilde{p}^b_j}{\partial x^i}\frac{\partial}{\partial \tilde{p}^b_j}, \\ \frac{\partial}{\partial p^a_i} &= \frac{\partial x^i}{\partial \tilde{x}^j}\frac{\partial \tilde{t}^b}{\partial t^a}\frac{\partial}{\partial \tilde{p}^b_j}. \end{aligned} \tag{2.3}$$

Proposition 2.4. *The elements of the local natural cobasis* $\{dt^a, dx^i, dp^a_i\}$ *of the Lie algebra of covector fields* $\mathcal{X}^*(E^*)$ *transform by the rules*

$$\begin{aligned} dt^a &= \frac{\partial t^a}{\partial \tilde{t}^b}d\tilde{t}^b, \\ dx^i &= \frac{\partial x^i}{\partial \tilde{x}^j}d\tilde{x}^j, \\ dp^a_i &= \frac{\partial p^a_i}{\partial \tilde{t}^b}d\tilde{t}^b + \frac{\partial p^a_i}{\partial \tilde{x}^j}d\tilde{x}^j + \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial t^a}{\partial \tilde{t}^b}d\tilde{p}^b_j. \end{aligned} \tag{2.4}$$

3 d-Tensors, multi-time semisprays of polymomenta and nonlinear connections

It is well known the importance of tensors in the development of a fertile geometry on a vector bundle. Following the geometrical ideas developed in the books [12] and [13], in our study upon the geometry of the dual 1-jet bundle E^* a central role is played by the *distinguished tensors* or, briefly, *d-tensors*.

Definition 3.1. A geometrical object $T = \left(T^{ai(k)(d)\ldots}_{bj(c)(l)\ldots}\right)$ on the dual 1-jet vector bundle E^*, whose local components, with respect to a transformation of coordinates (2.2) on E^*, transform by the rules

$$T^{ai(k)(d)\ldots}_{bj(c)(l)\ldots} = \tilde{T}^{ep(r)(h)\ldots}_{fq(g)(s)\ldots} \frac{\partial t^a}{\partial \tilde{t}^e} \frac{\partial x^i}{\partial \tilde{x}^p} \left(\frac{\partial x^k}{\partial \tilde{x}^r} \frac{\partial \tilde{t}^g}{\partial t^c}\right) \frac{\partial \tilde{t}^f}{\partial t^b} \frac{\partial \tilde{x}^q}{\partial x^j} \left(\frac{\partial \tilde{x}^s}{\partial x^l} \frac{\partial t^d}{\partial \tilde{t}^h}\right) \ldots,$$

is called a d-tensor or a distinguished tensor field on the dual 1-jet space E^*.

Example 3.2. If $H : E^* \to \mathbb{R}$ is a Hamiltonian function depending on the polymomenta p^a_i, then the local components

$$G^{(i)(j)}_{(a)(b)} = \frac{1}{2} \frac{\partial^2 H}{\partial p^a_i \partial p^b_j}$$

represent a d-tensor field $\mathbb{G} = \left(G^{(i)(j)}_{(a)(b)}\right)$ on the dual 1-jet space E^*, which is called the *fundamental vertical metrical d-tensor associated to the Hamiltonian function of polymomenta H*.

Example 3.3. Let us consider the d-tensor $\mathbb{C}^* = \left(\mathbb{C}^{(a)}_{(i)}\right)$, where $\mathbb{C}^{(a)}_{(i)} = p^a_i$. The distinguished tensor $\mathbb{C}^*$ is called the *Liouville-Hamilton d-tensor field of polymomenta* on the dual 1-jet space E^*.

Example 3.4. Let $h_{ab}(t)$ be a semi-Riemannian metric on the temporal manifold $\mathcal{T}$. The geometrical object $\mathbb{L} = \left(L^{(c)}_{(j)ab}\right)$, where $L^{(c)}_{(j)ab} = h_{ab} p^c_j$, is a d-tensor field on E^*, which is called the *polymomentum Liouville-Hamilton d-tensor field associated to the metric $h_{ab}(t)$*.

Example 3.5. Using the preceding metric $h_{ab}(t)$, we can construct the d-tensor field $\mathbb{J} = \left(J^{(i)}_{(a)bj}\right)$, where $J^{(i)}_{(a)bj} = h_{ab}\delta^i_j$. The distinguished tensor $\mathbb{J}$ is called the *d-tensor of h-normalization on the dual 1-jet vector bundle E^**.

Definition 3.6. A set of local functions $\underset{1}{G} = \left(\underset{1}{G}{}^{(b)}_{(j)i}\right)$, which transform by the rules

$$2\underset{1}{\widetilde{G}}{}^{(c)}_{(k)r} = 2\underset{1}{G}{}^{(b)}_{(j)i} \frac{\partial \tilde{t}^c}{\partial t^b} \frac{\partial x^i}{\partial \tilde{x}^r} \frac{\partial x^j}{\partial \tilde{x}^k} - \frac{\partial x^i}{\partial \tilde{x}^r} \frac{\partial \tilde{p}^c_k}{\partial t^a} p^a_i, \tag{3.1}$$

is called a temporal semispray on the dual 1-jet vector bundle E^*.

Example 3.7. If $\varkappa^a_{bc}(t)$ are the Christoffel symbols of a semi-Riemannian metric $h_{ab}(t)$ of the temporal manifold $\mathcal{T}$, then the local components

(3.2)
$$\underset{1}{\overset{0}{G}}{}^{(a)}_{(j)k} = \frac{1}{2}\varkappa^a_{bc}p^b_j p^c_k$$

represent a temporal semispray $\underset{1}{\overset{0}{G}}$ on the dual 1-jet vector bundle E^*.

Definition 3.8. The temporal semispray $\underset{1}{\overset{0}{G}}$ given by (3.2) is called the canonical temporal semispray associated to the temporal metric $h_{ab}(t)$.

Definition 3.9. A set of local functions $\underset{2}{G} = \left(\underset{2}{G}{}^{(b)}_{(j)i}\right)$, which transform by the rules

(3.3)
$$2\underset{2}{\widetilde{G}}{}^{(d)}_{(s)k} = 2\underset{2}{G}{}^{(b)}_{(j)i}\frac{\partial\tilde{t}^d}{\partial t^b}\frac{\partial x^i}{\partial\tilde{x}^k}\frac{\partial x^j}{\partial\tilde{x}^s} - \frac{\partial x^i}{\partial\tilde{x}^k}\frac{\partial\tilde{p}^d_s}{\partial x^i},$$

is called a spatial semispray on the dual 1-jet vector bundle E^*.

Example 3.10. If $\gamma^i_{jk}(x)$ are the Christoffel symbols of a semi-Riemannian metric $\varphi_{ij}(x)$ of the spatial manifold M, then the local components

(3.4)
$$\underset{2}{\overset{0}{G}}{}^{(b)}_{(j)k} = -\frac{1}{2}\gamma^i_{jk}p^b_i$$

define a spatial semispray $\underset{2}{\overset{0}{G}}$ on the dual 1-jet space E^*.

Definition 3.11. The spatial semispray $\underset{2}{\overset{0}{G}}$ given by (3.4) is called the canonical spatial semispray associated to the spatial metric $\varphi_{ij}(x)$.

Definition 3.12. A pair $G = \left(\underset{1}{G}, \underset{2}{G}\right)$, consisting of a temporal semispray $\underset{1}{G}$ and a spatial semispray $\underset{2}{G}$, is called a multi-time semispray of polymomenta on the dual 1-jet space E^*.

Definition 3.13. The pair $\overset{0}{G} = \left(\underset{1}{\overset{0}{G}}, \underset{2}{\overset{0}{G}}\right)$, given by the local functions (3.2) and (3.4), is called the canonical semispray of polymomenta associated to the pair of semi-Riemannian metrics $h_{ab}(t)$ and $\varphi_{ij}(x)$.

Definition 3.14. A pair of local functions $N = \left(\underset{1}{N}{}^{(c)}_{(k)a}, \underset{2}{N}{}^{(c)}_{(k)i}\right)$ on E^*, which transform by the rules

(3.5)
$$\begin{aligned}\underset{1}{\widetilde{N}}{}^{(b)}_{(j)d} &= \underset{1}{N}{}^{(c)}_{(k)a}\frac{\partial\tilde{t}^b}{\partial t^c}\frac{\partial x^k}{\partial\tilde{x}^j}\frac{\partial t^a}{\partial\tilde{t}^d} - \frac{\partial t^a}{\partial\tilde{t}^d}\frac{\partial\tilde{p}^b_j}{\partial t^a},\\ \underset{2}{\widetilde{N}}{}^{(b)}_{(j)r} &= \underset{2}{N}{}^{(c)}_{(k)i}\frac{\partial\tilde{t}^b}{\partial t^c}\frac{\partial x^k}{\partial\tilde{x}^j}\frac{\partial x^i}{\partial\tilde{x}^r} - \frac{\partial x^i}{\partial\tilde{x}^r}\frac{\partial\tilde{p}^b_j}{\partial x^i},\end{aligned}$$

is called a nonlinear connection on the dual 1-jet bundle E^*.

Remark 3.15. The nonlinear connections are very important in the study of the differential geometry of the dual 1-jet space E^* because they produce the *adapted distinguished 1-forms*

$$\delta p_i^a = dp_i^a + \underset{1}{N}{}_{(i)b}^{(a)} dt^b + \underset{2}{N}{}_{(i)j}^{(a)} dx^j,$$

which are necessary for the adapted local description of the *d-linear connections*, *d-torsions* or *d-curvatures*. For more details, please see the paper [4].

Now, let us expose the connection between the notions of multi-time semispray of polymomenta and nonlinear connection on the dual 1-jet space E^*. Thus, in our context, using the transformation rules (3.1), (3.3) and (3.5) of the geometrical objects taken in study, we can easily prove the following statements:

Proposition 3.16. *i) If* $\underset{1}{G}{}_{(j)k}^{(a)}$ *are the components of a temporal semispray* $\underset{1}{G}$ *on* E^* *and* $\varphi_{ij}(x)$ *is a semi-Riemannian metric on the spatial manifold* M*, then the local components*

$$\underset{1}{N}{}_{(r)b}^{(a)} = \varphi^{jk} \frac{\partial \underset{1}{G}{}_{(j)k}^{(a)}}{\partial p_i^b} \varphi_{ir}$$

represent the temporal components of a nonlinear connection N_G *on* E^*.

ii) Conversely, if $\underset{1}{N}{}_{(i)b}^{(a)}$ *are the temporal components of a nonlinear connection* N *on* E^**, then the local components*

$$\underset{1}{G}{}_{(i)j}^{(a)} = \frac{1}{2} \underset{1}{N}{}_{(i)b}^{(a)} p_j^b$$

represent a temporal semispray $\underset{1}{G}_N$ *on* E^*.

Proposition 3.17. *i) If* $\underset{2}{G}{}_{(j)i}^{(b)}$ *are the components of a spatial semispray* $\underset{2}{G}$ *on* E^**, then the local components*

$$\underset{2}{N}{}_{(j)i}^{(b)} = 2\underset{2}{G}{}_{(j)i}^{(b)}$$

represent the spatial components of a nonlinear connection N_G *on* E^*.

ii) Conversely, if $\underset{2}{N}{}_{(j)i}^{(b)}$ *are the spatial components of a nonlinear connection* N *on* E^**, then the local functions*

$$\underset{2}{G}{}_{(j)i}^{(b)} = \frac{1}{2} \underset{2}{N}{}_{(j)i}^{(b)}$$

represent a spatial semispray $\underset{2}{G}_N$ *on* E^*.

Remark 3.18. The Propositions 3.16 and 3.17 emphasize that a multi-time semispray of polymomenta $G = \left(\underset{1}{G}, \underset{2}{G}\right)$ on the dual 1-jet space E^* naturally induces a nonlinear connection N_G on E^* and vice-versa, N induces G_N.

Definition 3.19. The nonlinear connection N_G on the dual 1-jet space E^* is called the canonical nonlinear connection associated to the multi-time semispray of polymomenta $G = \left(\underset{1}{G}, \underset{2}{G}\right)$ and vice-versa.

Corollary 3.20. *The canonical nonlinear connection* $\overset{0}{N} = \left(\underset{1}{\overset{0}{N}}{}^{(a)}_{(i)b}, \underset{2}{\overset{0}{N}}{}^{(a)}_{(i)j}\right)$ *produced by the canonical multi-time semispray of polymomenta* $\overset{0}{G} = \left(\underset{1}{\overset{0}{G}}, \underset{2}{\overset{0}{G}}\right)$ *associated to the pair of semi-Riemannian metrics* $(h_{ab}(t), \varphi_{ij}(x))$ *has the local components*

$$\underset{1}{\overset{0}{N}}{}^{(a)}_{(i)b} = \varkappa^a_{cb} p^c_i \quad \textit{and} \quad \underset{2}{\overset{0}{N}}{}^{(a)}_{(i)j} = -\gamma^k_{ij} p^a_k.$$

4 Kronecker h-regularity. Canonical nonlinear connections

Let us consider a smooth multi-time Hamiltonian function $H : E^* \to \mathbb{R}$, locally expressed by

$$E^* \ni (t^a, x^i, p^a_i) \to H(t^a, x^i, p^a_i) \in \mathbb{R},$$

whose *fundamental vertical metrical d-tensor* is defined by

$$G^{(i)(j)}_{(a)(b)} = \frac{1}{2} \frac{\partial^2 H}{\partial p^a_i \partial p^b_j}.$$

In the sequel, let us fix $h = (h_{ab}(t^c))$, a semi-Riemannian metric on the temporal manifold $\mathcal{T}$, together with a d-tensor $g^{ij}(t^c, x^k, p^c_k)$ on the dual 1-jet space E^*, which is symmetric, has the rank $n = \dim M$ and a constant signature.

Definition 4.1. A multi-time Hamiltonian function $H : E^* \to \mathbb{R}$, having the fundamental vertical metrical d-tensor of the form

$$G^{(i)(j)}_{(a)(b)}(t^c, x^k, p^c_k) = \frac{1}{2} \frac{\partial^2 H}{\partial p^a_i \partial p^b_j} = h_{ab}(t^c) g^{ij}(t^c, x^k, p^c_k),$$

is called a Kronecker h-regular multi-time Hamiltonian function.

Definition 4.2. A pair $MH^n_m = (E^* = J^{1*}(\mathcal{T}, M), H)$, where $m = \dim \mathcal{T}$ and $n = \dim M$, consisting of the dual 1-jet space and a Kronecker h-regular multi-time Hamiltonian function $H : E^* \to \mathbb{R}$, is called a multi-time Hamilton space.

Remark 4.3. In the particular case $(\mathcal{T}, h) = (\mathbb{R}, \delta)$, a multi-time Hamilton space will be called a *relativistic rheonomic Hamilton space. In this case, we use the notation* $RRH^n = (J^{1*}(\mathbb{R}, M), H)$.

Example 4.4. Let us consider the following Kronecker h-regular multi-time Hamiltonian function $H_1 : E^* \to \mathbb{R}$, defined by

$$H_1 = \frac{1}{mc} h_{ab}(t) \varphi^{ij}(x) p^a_i p^b_j, \tag{4.1}$$

where $h_{ab}(t)$ ($\varphi_{ij}(x)$, respectively) is a semi-Riemannian metric on the temporal (spatial, respectively) manifold $\mathcal{T}$ (M, respectively) having the physical meaning of *gravitational potentials*, and m and c are the known constants from Physics representing the *mass of the test body* and the *speed of light*. Then, the multi-time Hamilton space

$$\mathcal{G}MH_m^n = (E^*, H_1)$$

defined by the multi-time Hamiltonian function (4.1) is called (please see [13]) the *multi-time Hamilton space of gravitational field.*

Example 4.5. Using preceding notations, let us consider the Kronecker h-regular multi-time Hamiltonian function $H_2 : E^* \to \mathbb{R}$, defined by

$$H_2 = \frac{1}{mc} h_{ab}(t)\varphi^{ij}(x)p_i^a p_j^b - \frac{2e}{mc^2} A_{(a)}^{(i)}(x)p_i^a + \frac{e^2}{mc^3} F(t,x), \tag{4.2}$$

where $A_{(a)}^{(i)}(x)$ is a d-tensor on E^* having the physical meaning of *potential d-tensor of an electromagnetic field*, e is the *charge of the test body* and the function $F(t,x)$ is given by

$$F(t,x) = h^{ab}(t)\varphi_{ij}(x)A_{(a)}^{(i)}(x)A_{(b)}^{(j)}(x).$$

Then, the multi-time Hamilton space

$$\mathcal{ED}MH_m^n = (E^*, H_2)$$

defined by the multi-time Hamiltonian function (4.2) is called (please see [13]) the *autonomous multi-time Hamilton space of electrodynamics.* The non-dynamical character (the independence of the temporal coordinates t^c) of the spatial gravitational potentials $\varphi_{ij}(x)$ motivated us to use the term *"autonomous"*.

Example 4.6. More general, if we take on E^* a symmetric d-tensor field $g_{ij}(t,x)$ having the rank n and a constant signature, we can define the Kronecker h-regular multi-time Hamiltonian function $H_3 : E^* \to \mathbb{R}$, setting

$$H_3 = h_{ab}(t)g^{ij}(t,x)p_i^a p_j^b + U_{(a)}^{(i)}(t,x)p_i^a + \mathcal{F}(t,x), \tag{4.3}$$

where $U_{(a)}^{(i)}(t,x)$ is a d-tensor field on E^* and $\mathcal{F}(t,x)$ is a function on E^*. Then, the multi-time Hamilton space

$$\mathcal{NED}MH_m^n = (E^*, H_3)$$

defined by the multi-time Hamiltonian function (4.3) is called the *non-autonomous multi-time Hamilton space of electrodynamics.* The dynamical character (the dependence of the temporal coordinates t^c) of the spatial gravitational potentials $g_{ij}(t,x)$ motivated us to use the word *"non-autonomous"*.

An important role and, at the same time, an obstruction for the subsequent development of a geometrical theory for the multi-time Hamilton spaces, is represented by the following result:

Theorem 4.7 (of characterization of the multi-time Hamilton spaces). *If we have $m = \dim \mathcal{T} \geq 2$, then the following statements are equivalent:*

(i) H is a Kronecker h-regular multi-time Hamiltonian function on E^*.

(ii) The multi-time Hamiltonian function H reduces to a multi-time Hamiltonian function of non-autonomous electrodynamic kind, that is we have

$$H = h_{ab}(t)g^{ij}(t,x)p_i^a p_j^b + U_{(a)}^{(i)}(t,x)p_i^a + \mathcal{F}(t,x). \tag{4.4}$$

Proof. *(ii)* $\Longrightarrow$ *(i)* It is obvious (even if we have $m = 1$).

(i) $\Longrightarrow$ *(ii)* Let us suppose that $m = \dim \mathcal{T} \geq 2$ and let us consider that H is a Kronecker h-regular multi-time Hamiltonian function, that is we have

$$\frac{1}{2}\frac{\partial^2 H}{\partial p_i^a \partial p_j^b} = h_{ab}(t^c)g^{ij}(t^c, x^k, p_k^c).$$

*(**1**°)* Firstly, let us suppose that there exist two distinct indices a and b, from the set $\{1, \ldots, m\}$, such that $h_{ab} \neq 0$. Let k (c, respectively) be an arbitrary element of the set $\{1, \ldots, n\}$ ($\{1, \ldots, m\}$, respectively). Deriving the above relation, with respect to the variable p_k^c, and using the Schwartz theorem, we obtain the equalities

$$\frac{\partial g^{ij}}{\partial p_k^c} h_{ab} = \frac{\partial g^{jk}}{\partial p_i^a} h_{bc} = \frac{\partial g^{ik}}{\partial p_j^b} h_{ac}, \quad \forall\, a, b, c \in \{1, \ldots, m\}, \quad \forall\, i, j, k \in \{1, \ldots, n\}.$$

Contracting now with h^{cd}, we deduce that

$$\frac{\partial g^{ij}}{\partial p_k^c} h_{ab} h^{cd} = 0, \quad \forall\, d \in \{1, \ldots, m\}.$$

In this context, the supposing $h_{ab} \neq 0$, together with the fact that the metric h is non-degenerate, imply that $\dfrac{\partial g_{ij}}{\partial p_k^c} = 0$, for any two arbitrary indices k and c. Consequently, we have $g^{ij} = g^{ij}(t^d, x^r)$.

*(**2**°)* Let us suppose now that $h_{ab} = 0$, $\forall\, a \neq b \in \{1, \ldots, m\}$. It follows that

$$h_{ab} = h_a(t)\delta_b^a, \quad \forall\, a, b \in \{1, \ldots, m\},$$

where $h_a(t) \neq 0$, $\forall\, a \in \{1, \ldots, m\}$. In these conditions, the relations

$$\frac{\partial^2 L}{\partial p_i^a \partial p_j^b} = 0, \quad \forall\, a \neq b \in \{1, \ldots, m\}, \quad \forall\, i, j \in \{1, \ldots, n\},$$

$$\frac{1}{2h_a(t)} \frac{\partial^2 L}{\partial p_i^a \partial p_j^a} = g_{ij}(t^c, x^k, p_k^c), \quad \forall\, a \in \{1, \ldots, m\}, \quad \forall\, i, j \in \{1, \ldots, n\},$$

are true. If we fix now an index a in the set $\{1, \ldots, m\}$, we deduce from the first relations that the local functions $\dfrac{\partial L}{\partial p_i^a}$ depend only by the coordinates (t^c, x^k, p_k^a). Considering $b \neq a$ another index from the set $\{1, \ldots, m\}$, the second relations imply

$$\frac{1}{2h_a(t)} \frac{\partial^2 L}{\partial p_i^a \partial p_j^a} = \frac{1}{2h_b(t)} \frac{\partial^2 L}{\partial p_i^b \partial p_j^b} = g_{ij}(t^c, x^k, p_k^c), \quad \forall\, i, j \in \{1, \ldots, n\}.$$

Because the first term of the above equality depends only by the coordinates (t^c, x^k, p_k^a), while the second term depends only by the coordinates (t^c, x^k, p_k^b), and because we have $a \neq b$, we conclude that $g^{ij} = g^{ij}(t^d, x^r)$.

Finally, the equalities

$$\frac{1}{2}\frac{\partial^2 H}{\partial p_i^a \partial p_j^b} = h_{ab}(t^c)g^{ij}(t^c, x^k), \quad \forall\, a, b \in \{1, \ldots, m\}, \quad \forall\, i, j \in \{1, \ldots, n\},$$

imply that the multi-time Hamilton function H is one of kind (4.4). □

Corollary 4.8. *The* fundamental vertical metrical d-tensor of a *Kronecker h-regular multi-time Hamiltonian function H has the form*

$$G^{(i)(j)}_{(a)(b)} = \frac{1}{2}\frac{\partial^2 H}{\partial p^a_i \partial p^b_j} = \begin{cases} h_{11}(t)g^{ij}(t, x^k, p_k), & m = \dim \mathcal{T} = 1 \\ h_{ab}(t^c)g^{ij}(t^c, x^k), & m = \dim \mathcal{T} \geq 2. \end{cases} \tag{4.5}$$

Remark 4.9. i) It is obvious that the Theorem 4.7 is an obstruction in the development of a fertile geometrical theory for the multi-time Hamilton spaces. This obstruction will be surpassed in other future paper by the introduction of the more general geometrical concept of *generalized multi-time Hamilton space*. The generalized multi-time Hamilton geometry on the dual 1-jet space E^* will be constructed using only a Kronecker h-regular *fundamental vertical metrical d-tensor (not necessarily provided by a Hamiltonian function)* $G^{(i)(j)}_{(a)(b)} = h_{ab}(t^c)g^{ij}(t^c, x^k, p^c_k)$, together with an "*a priori*" given nonlinear connection N on E^*.

ii) In the case $m = \dim \mathcal{T} \geq 2$, the Theorem 4.7 obliges us to continue our geometrical study of the multi-time Hamilton spaces channeling our attention upon the *non-autonomous multi-time Hamilton spaces of electrodynamics.*

In the sequel, following the geometrical ideas of Miron from [11], we will show that any Kronecker h-regular multi-time Hamiltonian function H produces a natural nonlinear connection on the dual 1-jet bundle E^*, which depends only by H. In order to do that, let us take a Kronecker h-regular multi-time Hamiltonian function H, whose fundamental vertical metrical d–tensor is given by (4.5). Also, let us consider the *generalized spatial Christoffel symbols* of the d-tensor g_{ij}, given by

$$\Gamma^k_{ij} = \frac{g^{kl}}{2}\left(\frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right).$$

In this context, using preceding notations, we can give the following result:

Theorem 4.10. *The pair of local functions* $N = \left(\underset{1}{N}{}^{(a)}_{(i)b}, \underset{2}{N}{}^{(a)}_{(i)j}\right)$ *on* E^**, where*

$$\begin{aligned} &\underset{1}{N}{}^{(a)}_{(i)b} = \varkappa^a_{cb} p^c_i, \\ &\underset{2}{N}{}^{(a)}_{(i)j} = \frac{h^{ab}}{4}\left[\frac{\partial g_{ij}}{\partial x^k}\frac{\partial H}{\partial p^b_k} - \frac{\partial g_{ij}}{\partial p^b_k}\frac{\partial H}{\partial x^k} + g_{ik}\frac{\partial^2 H}{\partial x^j \partial p^b_k} + g_{jk}\frac{\partial^2 H}{\partial x^i \partial p^b_k}\right], \end{aligned} \tag{4.6}$$

represents a nonlinear connection on E^**, which is called the* canonical nonlinear connection of the multi-time Hamilton space $MH^n_m = (E^*, H)$.

Proof. Taking into account the classical transformation rules of the Christoffel symbols $\varkappa^a_{bc}$ of the temporal semi-Riemannian metric h_{ab}, by direct local computations, we deduce that the temporal components $\underset{1}{N}{}^{(a)}_{(i)b}$ from (4.6) verify the first transformation rules from (3.5) (please see also the Corollary 3.20).

In the particular case when $m = \dim \mathcal{T} = 1$, the spatial components

$$\underset{2}{N}{}^{(1)}_{(i)j} = \frac{h^{11}}{4}\left[\frac{\partial g_{ij}}{\partial x^k}\frac{\partial H}{\partial p_k} - \frac{\partial g_{ij}}{\partial p_k}\frac{\partial H}{\partial x^k} + g_{ik}\frac{\partial^2 H}{\partial x^j \partial p_k} + g_{jk}\frac{\partial^2 H}{\partial x^i \partial p_k}\right]$$

become (except the multiplication factor h^{11}) exactly the canonical nonlinear connection from the classical Hamilton geometry (please see [11] or [13, pp. 127]).

For $m = \dim \mathcal{T} \geq 2$, the Theorem 4.7 (more exactly, the formula (4.4)) leads us to the following expression for the spatial components $\underset{2}{N}{}^{(a)}_{(i)j}$ from (4.6):

$$\underset{2}{N}{}^{(a)}_{(i)j} = -\Gamma^k_{ij} p^a_k + T^{(a)}_{(i)j}, \tag{4.7}$$

where

$$T^{(a)}_{(i)j} = \frac{h^{ab}}{4} \left[\frac{\partial g_{ij}}{\partial x^k} U^{(k)}_{(b)} + g_{ik} \frac{\partial U^{(k)}_{(b)}}{\partial x^j} + g_{jk} \frac{\partial U^{(k)}_{(b)}}{\partial x^i} \right].$$

Because $T^{(a)}_{(i)j}$ is a d-tensor on E^* (we prove this by local computations, studying the local transformation laws of $T^{(a)}_{(i)j}$), it follows that the spatial components $\underset{2}{N}{}^{(a)}_{(i)j}$ given by (4.7) transform as in the second laws of (3.5). □

Corollary 4.11. *For $m = \dim \mathcal{T} \geq 2$, the canonical nonlinear connection N of a multi-time Hamilton space $MH^n_m = (E^*, H)$ (given by (4.4)) has the components*

$$\underset{1}{N}{}^{(a)}_{(i)b} = \varkappa^a_{cb} p^c_i, \qquad \underset{2}{N}{}^{(a)}_{(i)j} = -\Gamma^k_{ij} p^a_k + \frac{h^{ab}}{4} \left(U_{ib\bullet j} + U_{jb\bullet i} \right),$$

where $U_{ib} = g_{ik} U^{(k)}_{(b)}$ and

$$U_{kb\bullet r} = \frac{\partial U_{kb}}{\partial x^r} - U_{sb} \Gamma^s_{kr}.$$

Proof. Using the expression (4.7), by computations, we find the required result. □

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