Hyper-Kähler structures on the tangent bundle of a Kähler manifold

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Dedicated to the 70-th anniversary
of Professor Constantin Udriste

Abstract. We construct a family of almost hyper-complex structures on the tangent bundle of a Kählerian manifold by using two anti-commuting almost complex structures obtained from the natural lifts of the Riemannian metric (see [11], [12], [13], [18]) and the integrable almost complex structure on the base manifold. Next we obtain an almost hyper-Hermitian metric obtained from the same natural lifts, related to the considered almost complex structures. We study the integrability conditions for the almost complex structures, obtaining that the base manifold must have constant holomorphic sectional curvature, and the conditions under which the considered almost hyper-Hermitian metric leads to a hyper-Kählerian structure on the tangent bundle.

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1 Introduction

Consider an $m(=2n)$-dimensional Riemannian manifold (M,g) and denote by $\tau : TM \longrightarrow M$ its tangent bundle. Several Riemannian and semi-Riemannian metrics can be used in order to obtain geometric properties of the tangent bundle TM of (M,g). They are induced from the Riemannian metric g on M by using some lifts of g. Among these metrics, we may quote the Sasaki metric and the complete lift of the metric g. On the other hand, the natural lifts of g to TM, induce some other Riemannian and pseudo-Riemannian geometric structures with many nice geometric properties (see [8], [7]). By similar methods one can get from g some natural almost complex structures on TM. If (M,g) has a structure of Kählerian manifold we can find some other Riemannian metrics and almost complex structures on its tangent bundle and from them we can get some almost hyper-Hermitian structures (see also [19], [20]). Similar results are obtained in the case of the cotangent bundle (see e.g. [3]).

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In the present paper we study a class of natural almost hyper-Hermitian structures (G, J_1, J_2), on the tangent bundle TM of a Kählerian manifold (M, g, J), induced from the Riemannian metric g and the integrable almost complex structure J. The metric G and the anti-commuting almost complex structures J_1, J_2 are obtained as natural lifts of diagonal type from g and J.

The manifolds, tensor fields and other geometric objects we consider in this paper are assumed to be differentiable of class C^∞ (i.e. smooth). We use the computations in local coordinates in a fixed local chart but many results may be expressed in an invariant form by using the vertical and horizontal lifts. Some quite complicate computations have been made by using the Ricci package under Mathematica for doing tensor computations. The well known summation convention is used throughout this paper, the range of the indices h, i, j, k, l being always $\{1, \dots, m = 2n\}$.

1. Hyper-complex structures on TM.

Let (M, g) be a smooth $m = (2n)$-dimensional Riemannian manifold and denote its tangent bundle by $\tau : TM \longrightarrow M$. Recall that there is a structure of a smooth $2m$-dimensional manifold on TM, induced from the structure of smooth m-dimensional manifold of M. From every local chart $(U, \varphi) = (U, x^1, \dots, x^m)$ on M, it is induced a local chart $(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \dots, x^m, y^1, \dots, y^m)$, on TM, as follows. For a tangent vector $y \in \tau^{-1}(U) \subset TM$, the first m local coordinates $x^1, \dots, x^m$ are the local coordinates $x^1, \dots, x^m$ of its base point $x = \tau(y)$ in the local chart (U, φ) (in fact we made an abuse of notation, identifying x^i with $\tau^* x^i = x^i \circ \tau,\ i = 1, \dots, m$). The last m local coordinates $y^1, \dots, y^m$ of $y \in \tau^{-1}(U)$ are the vector space coordinates of y with respect to the natural basis $((\frac{\partial}{\partial x^1})_{\tau(y)}, \dots, (\frac{\partial}{\partial x^m})_{\tau(y)})$, defined by the local chart (U, φ). Due to this special structure of differentiable manifold for TM, it is possible to introduce the concept of M-tensor field on it. An M-tensor field of type (p, q) on TM is defined by sets of n^{p+q} components (functions depending on x^i and y^i), with p upper indices and q lower indices, assigned to induced local charts $(\tau^{-1}(U), \Phi)$ on TM, such that the local coordinate change rule is that of the local coordinate components of a tensor field of type (p, q) on the base manifold M, when a change of local charts on M (and hence on TM) is performed (see [10] for further details); e.g., the components $y^i,\ i = 1, \dots, m$, corresponding to the last m local coordinates of a tangent vector y, assigned to the induced local chart $(\tau^{-1}(U), \Phi)$ define an M-tensor field of type $(1, 0)$ on TM. A usual tensor field of type (p, q) on M may be thought of as an M-tensor field of type (p, q) on TM. If the considered tensor field on M is covariant only, the corresponding M-tensor field on TM may be identified with the induced (pullback by τ) tensor field on TM. Some useful M-tensor fields on TM may be obtained as follows. Let $u : [0, \infty) \longrightarrow \mathbf{R}$ be a smooth function and let $\|y\|^2 = g_{\tau(y)}(y, y)$ be the square of the norm of the tangent vector $y \in \tau^{-1}(U)$. If δ^i_j are the Kronecker symbols (in fact, they are the local coordinate components of the identity tensor field I on M), then the components $u(\|y\|^2)\delta^i_j$ define an M-tensor field of type $(1, 1)$ on TM. Similarly, if $g_{ij}(x)$ are the local coordinate components of the metric tensor field g on M in the local chart (U, φ), then the components $u(\|y\|^2)g_{ij}$ define a symmetric M-tensor field of type $(0, 2)$ on TM. The components $g_{0i} = y^k g_{ki}$, as well as $u(\|y\|^2)g_{0i}$ define M-tensor fields of type $(0, 1)$ on TM. Of course, all the components considered above are in the induced local chart $(\tau^{-1}(U), \Phi)$.

We shall use the horizontal distribution HTM, defined by the Levi Civita connection $\dot{\nabla}$ of g, in order to define some first order natural lifts to TM of the Riemannian metric g on M. Denote by $VTM = \mathrm{Ker}\ \tau_* \subset TTM$ the vertical distribution on TM. Then we have the direct sum decomposition

$$TTM = VTM \oplus HTM. \tag{1.1}$$

If $(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \ldots, x^m, y^1, \ldots y^m)$ is a local chart on TM, induced from the local chart $(U, \varphi) = (U, x^1, \ldots, x^m)$, the local vector fields $\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^m}$ define a local frame for VTM over $\tau^{-1}(U)$ and the local vector fields $\frac{\delta}{\delta x^1}, \ldots, \frac{\delta}{\delta x^m}$ define a local frame for HTM over $\tau^{-1}(U)$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{0i} \frac{\partial}{\partial y^h}, \quad \Gamma^h_{0i} = y^k \Gamma^h_{ki}$$

and $\Gamma^h_{ki}(x)$ are the Christoffel symbols of g.

The set of vector fields $(\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^m}, \frac{\delta}{\delta x^1}, \ldots, \frac{\delta}{\delta x^m})$ defines a local frame on TM, adapted to the direct sum decomposition (1.1). Remark that

$$\frac{\partial}{\partial y^i} = (\frac{\partial}{\partial x^i})^V, \quad \frac{\delta}{\delta x^i} = (\frac{\partial}{\partial x^i})^H,$$

where X^V and X^H denote the vertical and horizontal lifts of the vector field X on M.

Now assume that (M, g, J) is a Kählerian manifold. The Riemannian metric g and the integrable almost complex structure J are related by

$$g(JX, JY) = g(X, Y), \quad \dot{\nabla} J = 0,$$

where $\dot{\nabla}$ is the Levi Civita connection of g. Recall that we have too the following relations

$$N = 0, \quad d\phi = 0,$$

where N is the Nijehuis tensor field of J and ϕ is the associated 2-form, defined by

$$\phi(X, Y) = g(X, JY).$$

Denote by g_{ij}, J^i_j the components of g, J in the local chart $(U, \varphi) = (U, x^1, \ldots, x^m)$. Introduce the components $J_{ij} = g_{ih} J^h_j$, obtained from the components of J by lowering the contravariance index on the first place (in fact, J_{ij} are the components of the fundamental 2-form ϕ defined by the Kählerian structure (g, J). Consider the following M-tensor fields on $\tau^{-1}(U)$, defined by the components

$$g_{i0} = g_{ih} y^h, \quad J_{i0} = J_{ih} y^h = -J_{0i}.$$

Lemma 1. *If $m > 1$ and $u_1, u_2, u_3, u_4, u_5, u_6$ are smooth functions on TM such that*

$$u_1 g_{ij} + u_2 g_{i0} g_{j0} + u_3 J_{i0} J_{j0} + u_4 g_{i0} J_{j0} + u_5 J_{i0} g_{j0} + u_6 J_{ij} = 0, \ y \in \tau^{-1}(U)$$

on the domain of any induced local chart on TM*, then* $u_1 = u_2 = u_3 = u_4 = u_5 = u_6 = 0$.

The proof is obtained easily by transvecting the given relation with $g^{ij}, J^{ij} = J^i_h g^{hj}, J^j_0 = J^j_h y^h$ and y^j (Recall that the functions $g^{ij}(x)$ are the components of the inverse of the matrix $(g_{ij}(x))$, associated to g in the local chart (U, φ) on M; moreover, the components $g^{ij}(x)$ define a tensor field of type $(2, 0)$ on M).

Remark. From a relation of the type

$$u_1\delta^i_j + u_2 y^i g_{j0} + u_3 J^i_0 J_{j0} + u_4 y^i J_{j0} + u_5 J^i_0 g_{j0} + u_6 J^i_j = 0,\ y \in \tau^{-1}(U)$$

it is obtained, in a similar way, $u_1 = u_2 = u_3 = u_4 = u_5 = u_6 = 0$.

Since we work in a fixed local chart (U, φ) on M and in the corresponding induced local chart $(\tau^{-1}(U), \Phi)$ on TM, we shall use the following simpler notations

$$\frac{\partial}{\partial y^i} = \partial_i, \quad \frac{\delta}{\delta x^i} = \delta_i.$$

Denote by

$$t = \frac{1}{2}\|y\|^2 = \frac{1}{2} g_{\tau(y)}(y, y) = \frac{1}{2} g_{ik}(x) y^i y^k, \quad y \in \tau^{-1}(U) \tag{1.2}$$

the energy density defined by g in the tangent vector y. We have $t \in [0, \infty)$ for all $y \in TM$. Let $C = y^i \frac{\partial}{\partial y^i} = y^V$ be the Liouville vector field on TM and consider the corresponding horizontal vector field $\widetilde{C} = y^i \frac{\delta}{\delta x^i} = y^H$ on TM, obtained in a similar way. Consider the real valued smooth functions $a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6, c_1, c_2, c_3, c_4, c_5, c_6, d_1, d_2, d_3, d_4, d_5, d_6$ defined on $[0, \infty) \subset \mathbf{R}$ and define two diagonal natural almost complex structures J_1, J_2 on TM, by using these coefficients, the Riemannian metric g and the integrable almost complex structure J

$$\tag{1.3} \begin{cases} J_1 X^H_y = a_1(t) X^V_y + a_2(t) g_{\tau(y)}(y, X) C_y + a_3(t) g_{\tau(y)}(Jy, X)(Jy)^V_y + \\ +a_4(t)(JX)^V_y + a_5(t) g_{\tau(y)}(X, y)(Jy)^V_y + a_6(t) g_{\tau(y)}(Jy, X) C_y, \\ \\ J_1 X^V_y = -(b_1(t) X^H_y + b_2(t) g_{\tau(y)}(y, X) \widetilde{C}_y + b_3(t) g_{\tau(y)}(Jy, X)(Jy)^H_y + \\ +b_4(t)(JX)^H_y + b_5(t) g_{\tau(y)}(X, y)(Jy)^H_y + b_6(t) g_{\tau(y)}(Jy, X) \widetilde{C}_y), \end{cases}$$

$$\tag{1.4} \begin{cases} J_2 X^H_y = c_1(t) X^V_y + c_2(t) g_{\tau(y)}(y, X) C_y + c_3(t) g_{\tau(y)}(Jy, X)(Jy)^V_y + \\ +c_4(t)(JX)^V_y + c_5(t) g_{\tau(y)}(X, y)(Jy)^V_y + c_6(t) g_{\tau(y)}(Jy, X) C_y, \\ \\ J_2 X^V_y = -(d_1(t) X^H_y + d_2(t) g_{\tau(y)}(y, X) \widetilde{C}_y + d_3(t) g_{\tau(y)}(Jy, X)(Jy)^H_y + \\ +d_4(t)(JX)^H_y + d_5(t) g_{\tau(y)}(X, y)(Jy)^H_y + d_6(t) g_{\tau(y)}(Jy, X) \widetilde{C}_y). \end{cases}$$

The expressions of J_1, J_2 in adapted local frames are

$$J_1\delta_i = J_1 H^h_i \partial_h, \quad J_1\partial_i = J_1 V^h_i \delta_h,$$
$$J_2\delta_i = J_2 H^h_i \partial_h, \quad J_2\partial_i = J_2 V^h_i \delta_h,$$

where the M-tensor fields $J_1H_i^h, J_1V_i^h, J_2H_i^h, J_2V_i^h$ are given by

$$\begin{aligned}
J_1H_i^h &= a_1\delta_i^h + a_2 g_{i0}y^h + a_3 J_{i0}J_0^h + a_4 J_i^h + a_5 g_{i0}J_0^h + a_6 J_{i0}y^h,\\
J_1V_i^h &= -(b_1\delta_i^h + b_2 g_{i0}y^h + b_3 J_{i0}J_0^h + b_4 J_i^h + b_5 g_{i0}J_0^h + b_6 J_{i0}y^h),\\
J_2H_i^h &= c_1\delta_i^h + c_2 g_{i0}y^h + c_3 J_{i0}J_0^h + c_4 J_i^h + c_5 g_{i0}J_0^h + c_6 J_{i0}y^h,\\
J_2V_i^h &= -(d_1\delta_i^h + d_2 g_{i0}y^h + d_3 J_{i0}J_0^h + d_4 J_i^h + d_5 g_{i0}J_0^h + d_6 J_{i0}y^h).
\end{aligned}$$

The matrices associated to J_1, J_2 have a diagonal form

$$J_1 = \begin{pmatrix} 0 & J_1H_i^h \\ J_1V_i^h & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & J_2H_i^h \\ J_2V_i^h & 0 \end{pmatrix}.$$

Remark that, one can consider the case of the general natural tensor fields J_1, J_2 on TM, when $J_1\delta_i, J_1\partial_i, J_2\delta_i, J_2\partial_i$ are expressed as combinations of ∂_h, δ_h. In this case we should have 48 coefficients and the computations would become really complicate. However, the results obtained in the general case do not differ too much from that obtained in the diagonal case.

We use the following notation:

$$\alpha = (a_1 + 2a_2t)(a_1 + 2a_3t) + (a_4 + 2a_5t)(a_4 - 2a_6t).$$

Proposition 2. *The operator J_1 defines an almost complex structure on TM if and only if the coefficients $b_1, b_2, b_3, b_4, b_5, b_6$ are expressed as*

$$(1.5)\qquad \begin{cases}
b_1 = \frac{a_1}{a_1^2+a_4^2}, \quad b_4 = \frac{-a_4}{a_1^2+a_4^2},\\
b_2 = \frac{1}{\alpha}[b_1(-a_1a_2 - 2a_2a_3t + 2a_5a_6t) + b_4(a_1a_5 - a_1a_6 - a_3a_4)],\\
b_3 = \frac{1}{\alpha}[b_1(-a_1a_3 - 2a_2a_3t + 2a_5a_6t) + b_4(a_1a_5 - a_1a_6 - a_2a_4)],\\
b_5 = \frac{1}{\alpha}[b_1(-a_1a_5 + a_2a_4 + a_3a_4) + b_4(a_4a_6 - 2a_2a_3t + 2a_5a_6t)],\\
b_6 = \frac{1}{\alpha}[b_1(-a_1a_6 - a_2a_4 - a_3a_4) + b_4(a_4a_5 + 2a_2a_3t - 2a_5a_6t)].
\end{cases}$$

Proof. The relations are obtained by some quite straightforward but long computations, from the property $J_1^2 = -I$ of J_1 and Lemma 1.

Remark. Using the first two relations (1.5) we may find the expressions of b_2, b_3, b_5, b_6 as functions of $a_1, a_2, a_3, a_4, a_5, a_6$ only. Remark that the parameters a_1, a_4 cannot vanish simultaneously and that $\alpha \neq 0$. A similar result is obtained from the condition for J_2 to be an almost complex structure on TM. In this case we can express the coefficients $d_1, d_2, d_3, d_4, d_5, d_6$ as functions of $c_1, c_2, c_3, c_4, c_5, c_6$. We shall use the following notation:

$$\beta = (c_1 + 2c_2t)(c_1 + 2c_3t) + (c_4 + 2c_5t)(c_4 - 2c_6t).$$

Then we get

$$(1.6)\qquad \begin{cases}
d_1 = \frac{c_1}{c_1^2+c_4^2}, \quad d_4 = \frac{-c_4}{c_1^2+c_4^2},\\
d_2 = \frac{1}{\beta}[d_1(-c_1c_2 - 2c_2c_3t + 2c_5c_6t) + d_4(c_1c_5 - c_1c_6 - c_3c_4)],\\
d_3 = \frac{1}{\beta}[d_1(-c_1c_3 - 2c_2c_3t + 2c_5c_6t) + d_4(c_1c_5 - c_1c_6 - c_2c_4)],\\
d_5 = \frac{1}{\beta}[d_1(-c_1c_5 + c_2c_4 + c_3c_4) + d_4(c_4c_6 - 2c_2c_3t + 2c_5c_6t)],\\
d_6 = \frac{1}{\beta}[d_1(-c_1c_6 - c_2c_4 - c_3c_4) + d_4(c_4c_5 + 2c_2c_3t - 2c_5c_6t)].
\end{cases}$$

Now we shall study the conditions under which the almost complex structures J_1, J_2 satisfy the relation $J_1J_2 + J_2J_1 = 0$, leading to the almost hyper-complex structure on TM.

Theorem 3. *The almost complex structures J_1, J_2 define an almost hyper-complex structure on TM if*

$$c_1 = a_4, \quad c_4 = -a_1, \tag{1.7}$$

$$\begin{aligned} c_3 = \ & (a_1^2a_5 + a_4^2a_5 - a_1^2a_6 - a_4^2a_6 - a_1^2c_2 - a_4^2c_2 + 2a_2a_3a_4t - 2a_1a_2a_6t- \\ & -2a_1a_3a_6t - 4a_4a_5a_6t + 2a_4a_6^2t - 2a_1a_3c_2t + 2a_4a_6c_2t - 2a_2a_4c_6t - 2a_3a_4c_6t+ \\ & +4a_1a_5c_6t - 2a_1c_2c_6t + 2a_4c_6^2t - 4a_2a_3a_6t^2 + 4a_5a_6^2t^2 - 4a_3c_2c_6t^2 + 4a_5c_6^2t^2)/ \\ & ((a_1 + 2a_2t)(a_1 + 2c_6t) + (a_4 + 2c_2t)(a_4 - 2a_6t)), \\ c_5 = \ & \tfrac{-1}{a_1+2c_6t}(a_1a_2 + a_1a_3 + a_4a_5 - a_4a_6 - a_4c_2 - a_4c_3 - a_1c_6+ \\ & +2a_2a_3t - 2a_5a_6t - 2c_2c_3t). \end{aligned}$$

Proof. From the relation $J_1V_h^kJ_2H_i^h + J_2V_h^kJ_1H_i^h = 0$ we get

$$\begin{aligned} & (-b_1c_1 + b_4c_4 - a_1d_1 + a_4d_4)\delta_i^k - (b_4c_1 + b_1c_4 + a_4d_1 + a_1d_4)J_i^k- \\ & -(b_5c_1 + b_4c_2 + b_3c_4 + b_1c_5 + a_5d_1 + a_4d_3 + a_2d_4 + a_1d_5+ \\ & +2b_5c_2t + 2b_3c_5t + 2a_5d_3t + 2a_2d_5t)g_{i0}J_0^k+ \\ & (-b_3c_1 - b_1c_3 + b_5c_4 - b_4c_6 - a_3d_1 - a_1d_3 + a_6d_4 + a_4d_5- \\ & -2b_3c_3t - 2b_5c_6t - 2a_3d_3t - 2a_6d_5t)J_{i0}J_0^k+ \\ & -b_2c_1 - b_1c_2 - b_6c_4 + b_4c_5 - a_2d_1 - a_1d_2 + a_5d_4 - a_4d_6- \\ & -2b_2c_2t - 2b_6c_5t - 2a_2d_2t - 2a_5d_6t)g_{i0}y^k- \\ & (b_6c_1 - b_4c_3 - b_2c_4 + b_1c_6 + a_6d_1 - a_4d_2 - a_3d_4 + a_1d_6+ \\ & +2b_6c_3t + 2b_2c_6t + 2a_6d_2t + 2a_3d_6t)J_{i0}y^k. \end{aligned}$$

Replacing $b_\alpha, d_\alpha;\ \alpha = 1, \dots, 6$ and using Lemma 1 we get the following relations (from the vanishing of the first two coefficients)

$$\begin{aligned} (a_1c_1 + a_4c_4)(a_1^2 + a_4^2 + c_1^2 + c_4^2) = 0, \\ (a_4c_1 - a_1c_4)(a_1^2 + a_4^2 - c_1^2 - c_4^2) = 0. \end{aligned}$$

Since $a_1^2 + a_4^2 \neq 0,\ c_1^2 + c_4^2 \neq 0$, we obtain the relations

$$c_1 = \pm a_4, \quad c_4 = \mp a_1.$$

From now on we shall consider only the case $c_1 = a_4, c_4 = -a_1$. The expressions of c_3, c_5 are obtained from the vanishing of the next 4 coefficients. Then the other relations obtained from $J_1J_2 + J_2J_1 = 0$ are identically fulfilled.

Remark that the final expression of c_5 is obtained after replacing the obtained expression of c_3.

Hence an almost hyper-complex structure on TM, of the considered type depends on 8 essential parameters $a_1, a_2, a_3, a_4, a_5, a_6, c_2, c_6$ (real valued smooth functions depending on the density energy $t \in [0, \infty)$. Remark that the functions $a_\alpha;\ \alpha = 1, \dots, 6$, must fulfill some supplementary conditions which assure the existence of the expressions obtained above.

Now we shall study the integrability problem for the obtained almost hyper-complex structure. The integrability conditions for such a structure are expressed with the help of various Nijenhuis tensor fields obtained from the tensor fields $J_1, J_2, J_3 = J_1 J_2$. For a tensor field K of type $(1,1)$ on a given manifold, we can consider its Nijenhuis tensor field N_K defined by

$$N_K(X,Y) = [KX, KY] - K[X, KY] - K[KX, Y] + K^2[X,Y],$$

where X, Y are vector fields on the given manifold. For two tensor fields K, L of type $(1,1)$ on the given manifold, we can consider the corresponding Nijenhuis tensor field $N_{K,L}$ defined by

$$N_{K,L}(X,Y) = [KX, LY] + [LX, KY] - K([X, LY] + [LX, Y]) -$$

$$-L([KX, Y] + [X, KY]) + (KL + LK)[X,Y].$$

The almost hyper-complex structure defined by J_1, J_2 is integrable iff $N_1 = 0,\ N_2 = 0$, where N_1, N_2 are the Nijenhuis tensor fields of J_1, J_2. Equivalently, the structure is integrable iff $N_1 + N_2 + N_3 = 0$, or iff $N_{12} = 0$, where N_3 is the Nijenhuis tensor field of $J_3 = J_1 J_2$ and $N_{12} = N_{J_1, J_2}$ is the Nijenhuis tensor field of J_1, J_2.

In the case of the almost hyper-complex structure defined on TM by the tensor fields J_1, J_2 the most convenient way to study its integrability is the using of the Nijenhuis tensor fields N_1, N_2.

Proposition 4. *If the almost hyper-complex structure defined by (J_1, J_2) on TM is integrable then the Kählerian manifold (M, g, J) has constant holomorphic sectional curvature.*

Proof. Recall the following formulas, useful in computing the expressions of N_1, N_2

$$[\partial_i, \partial_j] = 0, \quad [\partial_i, \delta_j] = -\Gamma^k_{ij}\partial_k, \quad [\delta_i, \delta_j] = -R^k_{0ij}\partial_k,$$

$$\delta_i y^h = -\Gamma^h_{i0}, \ \delta_i g_{jk} = \Gamma^h_{ij} g_{hk} + \Gamma^h_{ik} g_{jh}, \ \delta_i g_{j0} = g_{0h}\Gamma^h_{ij},$$

$$\delta_i J^k_l = -\Gamma^k_{ih} J^h_l + \Gamma^h_{il} J^k_h, \ \delta_i J^k_0 = -\Gamma^k_{ih} J^h_0, \ \delta_i J_{j0} = \Gamma^h_{ij} J_{h0}.$$

We have used the notations

$$R^k_{0ij} = y^h R^k_{hij}, \Gamma^k_{i0} = y^h \Gamma^k_{ih}, g_{j0} = g_{jh} y^h, J^k_0 = J^k_h y^h, J_{j0} = J_{jh} y^h.$$

Then we get

$$N_1(\delta_i, \delta_j) = (J_1 H^k_i \partial_k J_1 H^h_j - J_1 H^k_j \partial_k J_1 H^h_i + R^h_{0ij})\partial_h.$$

Remark that all the terms containing the Christoffel symbols cancel. Doing the necessary replacements, we get a relation of the following type

$$\alpha_1(J^h_i g_{j0} - J^h_j g_{i0}) + \alpha_2(g_{0i}\delta^h_j - g_{0j}\delta^h_i) + 2\alpha_3 J_{ij} y^h + \alpha_4(J_{i0} J^h_j - J_{j0} J^h_i) + 2\alpha_5 J_{ij} J^h_0 +$$

$$+\alpha_6(\delta_i^h J_{j0} - \delta_j^h J_{i0}) + R_{0ij}^h + \alpha_7(g_{i0}J_{j0} - g_{j0}J_{i0})y^h + \alpha_8(g_{i0}J_{j0} - g_{j0}J_{i0})J_0^h = 0,$$

where the coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8$, are functions of t, expressed with the help of the coefficients $a_1, a_2, a_3, a_4, a_5, a_6$ and their derivatives.

Differentiating this relation with respect to y^k, then taking $y = 0$, one gets

$$\alpha_1(0)(J_i^h g_{jk} - J_j^h g_{ik}) + \alpha_2(0)(g_{ki}\delta_j^h - g_{kj}\delta_i^h) + 2\alpha_3(0)J_{ij}\delta_k^h + \alpha_4(0)(J_{ik}J_j^h - J_{jk}J_i^h) +$$

$$+2\alpha_5(0)J_{ij}J_k^h + \alpha_6(0)(\delta_i^h J_{jk} - \delta_j^h J_{ik}) + R_{kij}^h = 0.$$

Then, using the well known (skew) symmetries of the components of R, as well as the (first) Bianchi identity and the invariance properties of R with respect to J, one finds

$$R_{kij}^h = c(g_{jk}\delta_i^h - g_{ik}\delta_j^h + J_i^h g_{kl}J_j^l - J_j^h g_{kl}J_i^l + 2J_k^h g_{il}J_j^l), \tag{1.8}$$

i.e. the Kählerian manifold (M, g, J) has constant holomorphic sectional curvature $4c$.

Replacing the obtained expression of R_{kij}^h in the relation $N_1(\delta_i, \delta_j) = 0$, and using Lemma 1, one obtains some further relations

$$a_3 = \frac{c - a_4 a_5}{a_1}, \quad a_6 = -\frac{a_2 a_4}{a_1}. \tag{1.9}$$

Then, replacing these expressions of a_3, a_6 in the remaining terms one gets

$$a_2 = \frac{a_1(a_1 a_1' + a_4 a_4' - c)}{a_1^2 + a_4^2 - 2a_1 a_1' t - 2a_4 a_4' t}, \quad a_5 = \frac{-a_2 a_4 + a_1 a_4' + 2a_2 a_4' t}{a_1}. \tag{1.10}$$

Next, computing the expressions $N_1(\partial_i, \partial_j), N_1(\delta_i, \partial_j)$, we get that they are identically zero.

Similar results are obtained from the integrability conditions for J_2, but we should prefer to present some other expressions (we shall assume that $c_4 \neq 0$)

$$\begin{cases} c_2 = -\frac{c_1 c_6}{c_4}, \quad c_5 = \frac{c - c_1 c_3}{c_4}, \\ c_3 = \frac{c_1 c_6 + c_1' c_4 - 2c_1' c_6 t}{c_4}, \quad c_6 = \frac{c_4(c - c_1 c_1' - c_4 c_4')}{c_1^2 + c_4^2 - 2c_1 c_1' t - 2c_4 c_4' t}. \end{cases} \tag{1.11}$$

Finally, by using the relations obtained in Theorem 3, one gets the expressions of c_2, c_3, c_5, c_6 as functions a_1, a_4 an their derivatives

$$\begin{cases} c_2 = \frac{a_4(a_1 a_1' + a_4 a_4' - c)}{a_1^2 + a_4^2 - 2a_1 a_1' t - 2a_4 a_4' t}, \quad c_3 = \frac{a_4(c - a_1 a_1') + a_4'(a_1^2 - 2ct)}{a_1^2 + a_4^2 - 2a_1 a_1' t - 2a_4 a_4' t}, \\ c_5 = \frac{a_1(a_4 a_4' - c) - a_1'(a_4^2 - 2ct)}{a_1^2 + a_4^2 - 2a_1 a_1' t - 2a_4 a_4' t}, \quad c_6 = \frac{a_1(a_1 a_1' + a_4 a_4' - c)}{a_1^2 + a_4^2 - 2a_1 a_1' t - 2a_4 a_4' t} \end{cases} \tag{1.12}$$

Remark that the values of c_3, c_5 obtained in (1.12) do coincide with the values of c_3, c_5 obtained in Theorem 3 after replacing c_2, c_6 obtained in (1.12) and a_2, a_3, a_5, a_6 obtained in (1.9) and (1.10).

2 Hyper-Kähler structures on TM

Consider a natural Riemannian metric G on TM of diagonal type induced from g and J and given by

$$(2.1)\quad \begin{cases} G_y(X^H,Y^H)=p_1(t)g_{\tau(y)}(X,Y)+p_2(t)g_{\tau(y)}(y,X)g_{\tau(y)}(y,Y)+ \\ +p_3(t)g_{\tau(y)}(JX,y)g_{\tau(y)}(JY,y)+p_4(t)(g_{\tau(y)}(JX,y)g_{\tau(y)}(Y,y)+ \\ +g_{\tau(y)}(JY,y)g_{\tau(y)}(X,y)), \\ \\ G_y(X^V,Y^V)=q_1(t)g_{\tau(y)}(X,Y)+q_2(t)g_{\tau(y)}(y,X)g_{\tau(y)}(y,Y)+ \\ +q_3(t)g_{\tau(y)}(JX,y)g_{\tau(y)}(JY,y)+q_4(t)(g_{\tau(y)}(JX,y)g_{\tau(y)}(Y,y)+ \\ +g_{\tau(y)}(JY,y)g_{\tau(y)}(X,y)), \\ \\ G_y(X^H,Y^V)=G_y(Y^V,X^H)=G_y(X^V,Y^H)=G_y(Y^H,X^V)=0, \end{cases}$$

where $p_1,p_2,p_3,p_4,q_1,q_2,q_3,q_4$ are smooth real valued functions defined on $[0,\infty)$. Remark that we have to find the conditions under which G is real Riemannian metric.

The expression of G in local adapted frames is defined by the following M-tensor fields

$$G_{ij}=G(\delta_i,\delta_j)=p_1g_{ij}+p_2g_{0i}g_{0j}+p_3J_{i0}J_{j0}+p_4(g_{i0}J_{j0}+g_{j0}J_{i0}),$$

$$H_{ij}=G(\partial_i,\partial_j)=q_1g_{ij}+q_2g_{0i}g_{0j}+q_3J_{i0}J_{j0}+q_4(g_{i0}J_{j0}+g_{j0}J_{i0})$$

and the associated $2m\times 2m$-matrix with respect to the adapted local frame

$$\left(\frac{\delta}{\delta x^1},\dots,\frac{\delta}{\delta x^m},\frac{\partial}{\partial y^1},\dots,\frac{\partial}{\partial y^m}\right)$$

has two $m\times m$-blocks on the first diagonal

$$G=\begin{pmatrix} G_{ij} & 0 \\ 0 & H_{ij}\end{pmatrix}.$$

We shall be interested in the conditions under which the metric G is almost Hermitian with respect to the almost complex structures J_1,J_2, considered in the previous section, i.e.

$$G(J_1X,J_1Y)=G(X,Y),\quad G(J_2X,J_2Y)=G(X,Y),$$

for all vector fields X,Y on TM.

From the relation

$$G(J_1\delta_i,J_1\delta_j)=G(\delta_i,\delta_j),$$

we get

$$(2.2)\qquad H_{kl}J_1H_i^kJ_1H_j^l=G_{ij},$$

from which we obtain the following expressions for p_1, p_2, p_3, p_4

$$\left\{\begin{array}{l} p_1 = (a_1^2 + a_4^2)q_1, \\ p_2 = (2a_1a_2 + 2a_4a_5 + 2a_2^2t + 2a_5^2t)q_1 + \\ (a_1 + 2a_2t)^2q_2 + (a_4 + 2a_5t)^2q_3 + 2(a_1 + 2a_2t)(a_4 + 2a_5t)q_4, \\ p_3 = (2a_1a_3 - 2a_4a_6 + 2a_3^2t + 2a_6^2t)q_1 + \\ (a_4 - 2a_6t)^2q_2 + (a_1 + 2a_3t)^2q_3 - 2(a_1 + 2a_3t)(a_4 - 2a_6t)q_4, \\ p_4 = (-a_2a_4 + a_3a_4 + a_1a_5 + a_1a_6 + 2a_3a_5t + 2a_2a_6t)q_1 + \\ +(-a_1a_4 - 2a_2a_4t + 2a_1a_6t + 2a_1a_6t)q_2 + (a_1 + 2a_3t)(a_4 + 2a_5t)q_3 + \\ +(a_1^2 - a_4^2 + 2a_1a_2t + 2a_1a_3t - 2a_4a_5t + 2a_4a_6t + 4a_2a_3t^2 + 4a_5a_6t^2)q_4. \end{array}\right. \tag{2.3}$$

Remark that from the conditions $G(J_1\partial_i, J_1\partial_j) = G(\partial_i, \partial_j)$, $G(J_1\delta_i, J_1\partial_j) = G(\delta_i, \partial_j)$, we do not obtain new essential relations fulfilled by $p's$ and $q's$.

Now we deal with the condition

$$G(J_2\delta_i, J_2\delta_j) = G(\delta_i, \delta_j),$$

from which we get

$$H_{kl}J_2H_i^kJ_2H_j^l = G_{ij}. \tag{2.4}$$

We find the coefficients p_1, p_2, p_3, p_4 expressed in function of the coefficients q_1, q_2, q_3, q_4 by formulas similar to (2.3), where the parameters a_1, a_2, a_4, a_5, a_6 are replaced by $c_1, c_2, c_3, c_4, c_5, c_6$ respectively. Next, we may write the system fulfilled by q_1, q_2, q_3, q_4, obtained by equalizing the obtained values for p_1, p_2, p_3, p_4. Remark that, due to the formula (1.7), the first equation, corresponding to p_1, is trivial. So, we get a homogeneous system consisting of 3 equations

$$\begin{aligned} & q_1(-2a_1a_2 - 2a_4a_5 + 2a_4c_2 - 2a_1c_5 - 2a_2^2t - 2a_5^2t + 2c_2^2t + 2c_5^2t) + \\ & \quad +q_2(-a_1^2 + a_4^2 - 4a_1a_2t + 4a_4c_2t - 4a_2^2t^2 + 4c_2^2t^2) + \\ & \quad +q_3(a_1^2 - a_4^2 - 4a_4a_5t - 4a_1c_5t - 4a_5^2t^2 + 4c_5^2t^2) + \\ & \quad +q_4(-4a_1a_4 - 4a_2a_4t - 4a_1a_5t - 4a_1c_2t + 4a_4c_5t - 8a_2a_5t^2 + 8c_2c_5t^2) = 0, \\ & q_1(-a_2a_4 + a_3a_4 + a_1a_5 + a_1a_6 - a_1c_2 + a_1c_3 - a_4c_5 - a_4c_6 + 2a_3a_5t + 2a_2a_6t - 2c_3c_5t - \\ & \quad -2c_2c_6t) + q_2(-2a_1a_4 - 2a_2a_4t + 2a_1a_6t - 2a_1c_2t - 2a_4c_6t + 4a_2a_6t^2 - 4c_2c_6t^2) + \\ & \quad +q_3(2a_1a_4 + 2a_3a_4t + 2a_1a_5t + 2a_1c_3t - 2a_4c_5t + 4a_3a_5t^2 - 4c_3c_5t^2) + q_4(2a_1^2 - 2a_4^2 + \\ & \quad +2a_1a_2t + 2a_1a_3t - 2a_4a_5t + 2a_4a_6t - 2a_4c_2t - 2a_4c_3t - 2a_1c_5t + 2a_1c_6t + 4a_2a_3t^2 + \\ & \quad +4a_5a_6t^2 - 4c_2c_3t^2 - 4c_5c_6t^2) = 0, \\ & q_1(2a_1a_3 - 2a_4a_6 - 2a_4c_3 - 2a_1c_6 + 2a_3^2t + 2a_6^2t - 2c_3^2t - 2c_6^2t) + \\ & \quad +q_2(-a_1^2 + a_4^2 - 4a_4a_6t - 4a_1c_6t + 4a_6^2t^2 - 4c_6^2t^2) + \\ & \quad +q3(a_1^2 - a_4^2 + 4a_1a_3t - 4a_4c_3t + 4a_3^2t^2 - 4c_3^2t^2) + \\ & \quad +q_4(-4a_1a_4 - 4a_3a_4t + 4a_1a_6t - 4a_1c_3t - 4a_4c_6t + 8a_3a_6t^2 - 8c_3c_6t^2) = 0. \end{aligned}$$

The matrix of this system has the rank 2 and we may obtain its general solution depending on two parameters

$$q_1 = \lambda, \quad q_3 = \mu, \tag{2.5}$$

$$\begin{aligned}
&q_4 = ((a_3a_4 - a_1a_5 + a_1c_2 - a_4c_6 + 2a_3c_2t - 2a_5c_6t)\lambda +\\
&\quad +(2a_3a_4t - 2a_1a_5t + 2a_1c_2t - 2a_4c_6t + 4a_3c_2t^2 - 4a_5c_6t^2)\mu)/\\
&\quad (a_1^2 + a_4^2 + 2a_1a_2t - 2a_4a_6t + 2a_4c_2t + 2a_1c_6t - 4a_6c_2t^2 + 4a_2c_6t^2),\\
&q_1 + 2tq_2 =\\
&\quad = ((a_1^4 + 2a_1^2a_4^2 + a_4^4 + 4a_1^3a_2t + 4a1^3a_3t + 4a_1a_2a_4^2t + 4a_1a_3a_4^2t + 4a_1^2a_4a_5t +\\
&\quad +4a_4^3a_5t - 4a_1^2a_4a_6t - 4a_4^3a_6t + 4a_1^2a_2^2t^2 + 16a_1^2a_2a_3t^2 + 4a_1^2a_3^2t^2 + 8a_2a_3a_4^2t^2 +\\
&\quad +4a_3^2a_4^2t^2 + 8a_1a_2a_4a_5t^2 + 4a_1^2a_5^2t^2 + 4a_4^2a_5^2t^2 - 8a_1a_2a_4a_6t^2 - 8a_1a_3a_4a_6t^2 -\\
&\quad -8a_1^2a_5a_6t^2 - 16a_4^2a_5a_6t^2 + 4a_4^2a_6^2t^2 + 8a_1a_3a_4c_2t^2 - 8a_1^2a_5c_2t^2 + 4a_1^2c_2^2t^2 -\\
&\quad -8a_3a_4^2c_6t^2 + 8a_1a_4a_5c_6t^2 - 8a_1a_4c_2c_6t^2 + 4a_4^2c_6^2t^2 + 16a_1a_2^2a_3t^3 + 16a_1a_2a_3^2t^3 +\\
&\quad +16a_2a_3a_4a_5t^3 - 16a_2a_3a_4a_6t^3 - 16a_1a_2a_5a_6t^3 - 16a_1a_3a_5a_6t^3 - 16a_4a_5^2a_6t^3 +\\
&\quad +16a_4a_5a_6^2t^3 + 16a_3^2a_4c_2t^3 - 16a_1a_3a_5c_2t^3 + 16a_1a_3c_2^2t^3 - 16a_3a_4a_5c_6t^3 +\\
&\quad +16a_1a_5^2c_6t^3 - 16a_3a_4c_2c_6t^3 - 16a_1a_5c_2c_6t^3 + 16a_4a_5c_6^2t^3 + 16a_2^2a_3^2t^4 -\\
&\quad -32a_2a_3a_5a_6t^4 + 16a_5^2a_6^2t^4 + 16a_3^2c_2^2t^4 - 32a_3a_5c_2c_6t^4 + 16a_5^2c_6^2t^4)(\lambda + 2t\mu))/\\
&\quad (a_1^2 + a_4^2 + 2a_1a_2t - 2a_4a_6t + 2a_4c_2t + 2a_1c_6t - 4a_6c_2t^2 + 4a_2c_6t^2)^2.
\end{aligned}$$

The explicit expression of q_2 is obtained from the expression of $q_1 + 2tq_2$ and is more complicate. Next, the expressions of p_1, p_2, p_3, p_4 are obtained from (2.3).

$$\begin{aligned}
&p_1 = (a_1^2 + a_4^2)\lambda,\\
&p_1 + 2tp_2 = (a_1^2 + a_4^2 + 2a_1a_2t + 2a_1a_3t + 2a_4a_5t - 2a_4a_6t + 4a_2a_3t^2 - 4a_5a_6t^2)^2\\
&\quad (a_1^2 + a_4^2 + 4a_1a_2t + 4a_4c_2t + 4a_2^2t^2 + 4c_2^2t^2)(\lambda + 2t\mu)/(a_1^2 + a_4^2 + 2a_1a_2t - 2a_4a_6t +\\
&\quad +2a_4c_2t + 2a_1c_6t - 4a_6c_2t^2 + 4a_2c_6t^2)^2,\\
&p_1 + 2tp_3 = (a_1^2 + a_4^2 + 2a_1a_2t + 2a_1a_3t + 2a_4a_5t - 2a_4a_6t + 4a_2a_3t^2 - 4a_5a_6t^2)^2\\
&\quad (a_1^2 + a_4^2 - 4a_4a_6t + 4a_1c_6t + 4a_6^2t^2 + 4c_6^2t^2)(\lambda + 2t\mu)/(a_1^2 + a_4^2 + 2a_1a_2t - 2a_4a_6t +\\
&\quad +2a_4c_2t + 2a_1c_6t - 4a_6c_2t^2 + 4a_2c_6t^2)^2,\\
&p_4 = (-a_2a_4 + a_1a_6 + a_1c_2 + a_4c_6 + 2a_2a_6t + 2c_2c_6t)(a_1^2 + a_4^2 + 2a_1a_2t + 2a_1a_3t +\\
&+2a_4a_5t - 2a_4a_6t + 4a_2a_3t^2 - 4a_5a_6t^2)^2(\lambda + 2t\mu))/(a_1^2 + a_4^2 + 2a_1a_2t - 2a_4a_6t +\\
&+2a_4c_2t + 2a_1c_6t - 4a_6c_2t^2 + 4a_2c_6t^2)^2.
\end{aligned}$$

If we assume that the almost hyper-complex structure defined by J_1, J_2 is integrable, the expressions of the coefficients in G are simpler.

For the almost hyper-Hermitian manifold (TM, G, J_1, J_2) the fundamental 2-forms ϕ_1, ϕ_2 are defined by

$$\phi_1(X, Y) = G(X, J_1Y), \quad \phi_2(X, Y) = G(X, J_2Y),$$

where X, Y are vector fields on TM.

Since we have a third almost complex structure $J_3 = J_1 J_2$ which is almost Hermitian with respect to G, we can consider a third 2-form ϕ_3 defined by $\phi_3(X,Y) = G(X, J_3 Y)$, next we have the fundamental 4 form Ω, defined by

$$\Omega = \phi_1 \wedge \phi_1 + \phi_2 \wedge \phi_2 + \phi_3 \wedge \phi_3.$$

The almost hyper-Hermitian manifold (TM, G, J_1, J_2) is hyper-Kählerian if the almost complex structures J_1, J_2 are parallel with respect to the Levi Civita connection ∇ defined by G, i.e. $\nabla J_1 = 0, \nabla J_2 = 0$. Equivalently, (TM, G, J_1, J_2) is hyper-Kählerian if and only if the almost hyper-complex structure (J_1, J_2) is integrable and the 4-form Ω is closed, i.e. $N_1 = 0, N_2 = 0, d\Omega = 0$. The condition for Ω to be closed is equivalent to the conditions for ϕ_1, ϕ_2 (and hence for ϕ_3 too) to be closed i.e. $d\phi_1 = 0,\ d\phi_2 = 0$. In our case, it is more convenient to study the conditions under which the 2-forms ϕ_1, ϕ_2 are closed.

The expressions of ϕ_1, ϕ_2 in adapted local frames are

$$\phi_1 = \phi_{1,jk} Dy^j \wedge dx^k, \quad \phi_2 = \phi_{2,jk} Dy^j \wedge dx^k,$$

where

$$\begin{aligned} &Dy^j = dy^j + \Gamma^j_{i0} dx^i, \\ &\phi_{1,jk} = G(\partial_j, J_1 \delta_k) = H_{jh} J_1 H^h_k = -J_1 V^h_j G_{hk}, \\ &\phi_{2,jk} = G(\partial_j, J_2 \delta_k) = H_{jh} J_2 H^h_k = -J_2 V^h_j G_{hk}. \end{aligned}$$

Replacing the expressions of H_{jh} and $J_1 H^h_k, J_2 H^h_k$, we find the following expressions

$$\begin{aligned} \phi_{1,jk} = {} & a_1 q_1 g_{jk} + a_4 q_1 J_{jk} + (a_2 q_1 + a_1 q_2 + a_4 q_4 + 2a_2 q_2 t + 2a_5 q_4 t) g_{j0} g_{k0} + \\ & +(a_6 q_1 - a_4 q_2 + a_1 q_4 + 2a_6 q_2 t + 2a_3 q_4 t) g_{j0} J_{k0} + \\ & +(a_5 q_1 + a_4 q_3 + a_1 q_4 + 2a_5 q_3 t + 2a_2 q_4 t) J_{j0} g_{k0} + \\ & +(-a_3 q_1 - a_1 q_3 + a_4 q_4 - 2a_3 q_3 t - 2a_6 q_4 t) J_{j0} J_{k0}, \\ \phi_{2,jk} = {} & a_4 q_1 g_{jk} - a_1 q_1 J_{jk} + (c_2 q_1 + c_4 q_2 - a_1 4 q_4 + 2c_2 q_2 t + 2c_5 q_4 t) g_{j0} g_{k0} + \\ & +(c_6 q_1 + a_1 q_2 + a_4 q_4 + 2c_6 q_2 t + 2c_3 q_4 t) g_{j0} J_{k0} + \\ & +(c_5 q_1 - a_1 q_3 + a_4 q_4 + 2c_5 q_3 t + 2c_2 q_4 t) J_{j0} g_{k0} + \\ & +(-c_3 q_1 - a_4 q_3 - a_1 q_4 - 2c_3 q_3 t - 2c_6 q_4 t) J_{j0} J_{k0}. \end{aligned}$$

The final expressions of ϕ_1, ϕ_2 are obtained by replacing the values of q_1, q_2, q_3, q_4 obtained from (2.5), then the values of c_3, c_5, obtained in Theorem 3. We get for $\phi_{1,jk}$ an expression of the type

$$\phi_{1,jk} = \alpha_1 g_{jk} + \alpha_2 J_{jk} + \alpha_3 J_{jk} + \alpha_4 J_{0j} g_{0k} + \alpha_5 g_{0j} J_{0k} + \alpha_6 J_{0j} J_{0k},$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ are functions of t, expressed with the help of the coefficients $a_1, a_2, a_3, a_4, a_5, a_6, c_2, c_6$. A similar expression is obtained for $\phi_{2,jk}$.

Now we shall compute the expression of $\mathrm{d}\phi_1$ by using the following formulas

$$\mathrm{d}\alpha_r = \alpha'_r g_{0i} Dy^i, \quad r = 1,2,3,4,5,6, \quad \mathrm{d}g_{jk} = (\Gamma^h_{ij} g_{hk} + \Gamma^h_{ik} g_{jh}) \mathrm{d}x^i,$$

$$\mathrm{d}g_{0j} = \mathrm{d}g_{j0} = g_{ji}Dy^i + g_{0h}\Gamma^h_{ji}\mathrm{d}x^i, \mathrm{d}J_{jk} = (\Gamma^h_{ij}J_{hk} + \Gamma^h_{ik}J_{jh})\mathrm{d}x^i,$$

$$\mathrm{d}J_{j0} = -\mathrm{d}J_{0j} = J_{h0}\Gamma^h_{ji}\mathrm{d}x^i + J_{ji}Dy^i, \quad \mathrm{d}Dy^h = \Gamma^h_{ij}Dy^i \wedge \mathrm{d}x^j + \frac{1}{2}R^h_{0ij}\mathrm{d}x^i \wedge \mathrm{d}x^j.$$

We get the cancellation of all terms containing $\mathrm{d}x^i \wedge Dy^j \wedge \mathrm{d}x^k$ in the expression of $\mathrm{d}\phi_1$. Next the terms containing $\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$ are

$$\frac{1}{6}(\phi_{1,hk}R^h_{0ij} + \phi_{1,hi}R^h_{0jk} + \phi_{1,hj}R^h_{0ki})\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k.$$

From the vanishing of this term and under the assumption that the base manifold (M, g, J) has constant holomorphic sectional curvature $4c$, we get that the factors λ, μ are related by

$$\lambda = -\left(\frac{a_1^2 + a_4^2}{c} + 2t\right)\mu. \tag{2.6}$$

Finally, assuming that this relation as, well as the integrability conditions for the almost hyper-complex structure defined by J_1, J_2 are fulfilled, we get that $\alpha_5 = \alpha_6 = 0$ and the expression of $d\phi_1$ becomes

$$\mathrm{d}\phi_1 = (\alpha_1' g_{0i}g_{jk} + \alpha_2' g_{0i}J_{jk} + \alpha_3 g_{0j}g_{ki} + \alpha_4 g_{0j}J_{ik})Dy^i \wedge Dy^j \wedge \mathrm{d}x^k,$$

where the coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are given by

$$\begin{aligned}
&\alpha_1 = a_1\lambda, \quad \alpha_2 = a_4\lambda,\\
&\alpha_3 = \frac{\lambda(-a_1^2a_1' + a_1'a_4^2 - 2a_1a_4a_4' + 2a_1c - 2a_1'ct)}{a_1^2 + a_4^2 - 2ct},\\
&\alpha_4 = \frac{\lambda(-2a_1a_1'a_4 + a_1^2a_4' - a_4'a_4^2 + 2a_4c - 2a_4'ct)}{a_1^2 + a_4^2 - 2ct}.
\end{aligned} \tag{2.7}$$

Doing the necessary alternation in the relation $\mathrm{d}\phi_1 = 0$, we get the equations

$$\alpha_1' = \alpha_3, \quad \alpha_2' = \alpha_4.$$

Then, after some simple computations, we obtain that the coefficients λ, μ are given by

$$\lambda = \frac{k}{a_1^2 + a_4^2 - 2ct}, \quad \mu = \frac{-ck}{(a_1^2 + a_4^2)^2 - 4c^2t^2},$$

where k is a constant.

Under the same assumptions that the relation (2.6) is true and that the integrability conditions for the almost hyper-complex structure defined by J_1, J_2 are fulfilled, we get the following expression expression for the 2-form ϕ_2

$$\phi_2 = (\alpha_2 g_{jk} - \alpha_1 J_{jk} + \alpha_4 g_{0j}g_{0k} - \alpha_3 g_{0j}J_{0k})Dy^j \wedge dx^k.$$

If $d\phi_1 = 0$ we have that $d\phi_2 = 0$ too, so that the structure (G, J_1, J_2) on TM becomes Kählerian. We write down the explicit expressions of the coefficients involved in the expressions of (G, J_1, J_2). First of all, from the integrability condition $N_1 = 0$, we get

$$a_2 = \frac{a_1(a_1a_1' + a_4a_4' - c)}{a_1^2 + a_4^2 - 2a_1a_1't - 2a_4a_4't}, \quad a_3 = \frac{a_1'a_4^2 - a_1a_4a_4' + a_1c - 2a_1'ct}{a_1^2 + a_4^2 - 2a_1a_1't - 2a_4a_4't}, \tag{2.8}$$

$$a_5 = \frac{-a_1a_1'a_4 + a_1^2a_4' + a_4c - 2a_4'ct}{a_1^2 + a_4^2 - 2a_1a_1't - 2a_4a_4't},\ a_6 = \frac{-a_4(a_1a_1' + a_4a_4' - c)}{a_1^2 + a_4^2 - 2a_1a_1't - 2a_4a_4't}.$$

The values of c_2, c_3, c_5, c_6 are given by (1.12). Next we have

$$p_1 = \frac{k(a_1^2 + a_4^2)}{a_1^2 + a_4^2 - 2ct},\ p_2 = p_3 = \frac{ck}{a_1^2 + a_4^2 - 2ct},\ p_4 = 0, \tag{2.9}$$

$$q_1 = \lambda = \frac{k}{a_1^2 + a_4^2 - 2ct}, q_3 = \mu = \frac{-ck}{(a_1^2 + a_4^2)^2 - 4c^2t^2}, q_4 = \frac{k(a_1'a_4 - a_1a_4')}{(a_1^2 + a_4^2)^2 - 4c^2t^2}.$$

The expression of q_2 is quite complicate and can be obtained from (2.5). Hence we may state

Theorem 5. *Consider the almost hyper-Hermitian structure* (G, J_1, J_2) *defined as above on the tangent bundle* TM *of the Kählerian manifold* M. *This structure is hyper-Kählerian if and only if the almost complex structures* J_1, J_2 *are integrable (hence the Proposition 4 and the relations* (1.9), (1.10), (1.11), (1.12) *are fulfilled) and and the relations* (2.6), (2.7), (2.8), (2.9) *are fulfilled by the hyper-Hermitian metric* G.

The case where $a_4 = 0$

We shall study a special case when $a_4 = 0$. In this case we shall obtain some much more simple formulas and results and many of them are related to those obtained in [19], [20]. However, our parametrization is quite different.

Since the integrability conditions for the almost complex structure J_1 we get that the condition $a_4 = 0$ implies the conditions $a_5 = 0,\ a_6 = 0$. We are interested in the integrable case, so that we shall assume from the beginning $a_4 = a_5 = a_6 = 0$. According to the result obtained in Proposition 2, the tensor field J_1 defines almost complex structure on TM if and only if

$$\begin{cases} b_1 = \frac{1}{a_1},\ b_2 = -\frac{a_2}{a_1(a_1+2a_2t)},\ b_3 = -\frac{a_3}{a_1(a_1+2a_3t)}, \\ b_4 = 0,\ \ b_5 = 0,\ \ b_6 = 0. \end{cases} \tag{2.10}$$

he integrability condition for J_1 gives

$$a_2 = \frac{a_1a_1' - c}{a_1 - 2a_1't},\ a_3 = \frac{c}{a_1}. \tag{2.11}$$

Next, from (1.5), (1.6), the Theorem 3 and the integrability conditions for J_2, we get

$$\begin{cases} c_1 = 0,\ \ c_2 = 0,\ \ c_3 = 0,\ \ c_4 = -a_1, c_5 = \frac{-c}{a_1},\ \ c_6 = \frac{a_1a_1'-c}{a_1-2a_1't}, \\ d_1 = 0,\ \ d_2 = 0,\ \ d_3 = 0,\ \ d_4 = \frac{1}{a_1}, d_5 = \frac{c-a_1a_1'}{a_1(a_1^2-2ct)},\ \ d_6 = \frac{c}{a_1(a_1^2+2ct).} \end{cases} \tag{2.12}$$

Finally, in the case where (TM, G, J_1, J_2) is hyper-Kähler, we have

$$\begin{cases} p_1 = \frac{ka_1^2}{a_1^2-2ct},\ p_2 = \frac{ck}{a_1^2-2ct},\ p_3 = \frac{ck}{a_1^2-2ct},\ p_4 = 0, \\ q_1 = \frac{k}{a_1^2-2ct},\ q_1 + 2q_2t = \frac{k(a_1-2a_1't)^2(a_1^2+2ct)}{(a_1^2-2ct)^3},\ q_3 = \frac{-ck}{a_1^4-4c^2t^2},\ q_4 = 0. \end{cases} \tag{2.13}$$

Hence we may state

Theorem 6. *In the case where* $a_4 = 0$*, the almost hyper-Hermitian structure* (G, J_1, J_2) *defined on the tangent bundle* TM *becomes hyper-Kählerian if the conditions* (2.10)-(2.13) *are fulfilled.*

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