# Von Neumann analysis of linearized discrete Tzitzeica PDE

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**Abstract.** This paper applies the von Neumann analysis to a discrete Tzitzeica PDE. Section 1 recalls some data from the fascinating history of Tzitzeica PDE, emphasizing on its geometrical and physical roots. Section 2 gives the Tzitzeica Lagrangian and its associated Tzitzeica Hamiltonian. Section 3 shows that the Tzitzeica PDE can be obtained via geometric dynamics. Section 4 refreshes the theory of integrators for two-parameter Lagrangian dynamics. Section 5 finds the discrete Tzitzeica equation. Section 6 motivates the von Neumann stability analysis. Section 7 presents the von Neumann analysis of dual variational integrator equation. Section 8 performs the von Neumann analysis of linearized discrete Tzitzeica equation. Section 9 proposes the study of an extended Tzitzeica PDE via a Laurent polynomial. Section 10 underlines the importance of the von Neumann analysis.

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**Key words**: Tzitzeica PDE; geometric dynamics; discrete Lagrangian dynamics; discrete Tzitzeica PDE; von Neumann analysis.

## 1 History of Tzitzeica PDE

The *Tzitzeica hyperbolic nonlinear PDE* is

$$(\ln h)_{uv} = h - \frac{1}{h^2}.$$

With a change of function $\ln h = \omega$, this equation rewrites as

$$\omega_{uv} = e^{\omega} - e^{-2\omega}. \tag{1.1}$$

This equation has a fascinating history underlined in the paper [10]. It first arose a century ago in the work of the greather Romanian geometer Tzitzeica [15], [16]. He arrived at it from the viewpoint of the geometry of surfaces, obtaining an associated

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linear representation and a Backlund transformation. For many decades after Tzitzeica's work, the equation (1.1) was not studied, two papers by Jonas [6], [7] being a notable exception. Thirty years ago, it was reintroduced within the area of soliton theory, see [2], [3], [5], [8], [10]-[14], [20]. In this setting, the PDE (1.1) is viewed as an integrable relativistic theory for a field $\omega(t; x)$ in two space-time dimensions, written in terms of light cone (characteristic) coordinates,

$$t = u - v; x = u + v. \tag{1.2}$$

Now, the Tzitzeica PDE (1.1) is known under various names, and has been studied from several perspectives, including geometry [see also, [4], [19]], classical soliton theory, quantum soliton theory, gas dynamics, link between the Tzitzeica equation and the $2D$ Toda equation, and integrable particle dynamics of relativistic Calogero-Moser type [10].

The Tzitzeica PDE (1.1) admits the homogeneous linearization

$$\omega_{uv} = 3\omega. \tag{1.3}$$

Also, it has an extension via a *Laurent polynomial*

$$\omega_{uv} = \sum_{n\in Z} a_n e^{n\omega}, \ \sum_{n\in Z} a_n = 0.$$

A Laurent polynomial is a Laurent series in which only finitely many coefficients $a_n$ are non-zero.

## 2 Tzitzeica Lagrangian, Tzitzeica Hamiltonian

The Tzitzeica hyperbolic PDE (1.1) is the Euler-Lagrange PDE provided by the first order *Tzitzeica Lagrangian*

$$L_1 = \frac{1}{2}\omega_u\omega_v + e^{\omega} + \frac{1}{2}e^{-2\omega}.$$

A Lagrangian of the form $L(\omega, \omega_u, \omega_v)$, whose Euler-Lagrange PDE coincides to Tzitzeica hyperbolic PDE (1.1), reduces to $L_1$.

Now we introduce the moments

$$p = \frac{\partial L_1}{\partial \omega_u} = \frac{1}{2}\omega_v, \ q = \frac{\partial L_1}{\partial \omega_v} = \frac{1}{2}\omega_u.$$

It follows the *Tzitzeica Hamiltonian*

$$H = p\omega_u + q\omega_v - L_1 = 2pq - e^{\omega} - \frac{1}{2}e^{-2\omega}$$

and the Hamiltonian PDEs

$$\frac{\partial \omega}{\partial u} = 2q, \ \frac{\partial \omega}{\partial v} = 2p, \ \frac{\partial p}{\partial u} + \frac{\partial q}{\partial v} = e^{\omega} - e^{-2\omega}.$$

The linearization of the last two exponential terms of $L_1$ produces

$$L_2 = \frac{1}{2}\omega_u\omega_v + \frac{3}{2}.$$

The Euler-Lagrange PDE associated to $L_2$ is $\omega_{uv} = 0$, different from (1.3).

# 3 Geometric dynamics and Tzitzeica PDE

To show that Tzitzeica hyperbolic PDE is connected to the *geometric dynamics theory* [17], let start with the semi-Riemannian (hyperbolic) manifold $(R^2; h^{11} = 0, h^{12} = h^{21} = \frac{1}{2}, h^{22} = 0)$ and the Riemannian manifold $(R, g = 1)$. We introduce the function $\omega : R^2 \to R, \omega = \omega(u, v)$ and the vector fields

$$X_1(u, v, \omega) = \alpha(u, v)e^{\omega} + \beta(u, v)e^{-\omega}$$

$$X_2(u, v, \omega) = \gamma(u, v)e^{\omega} + \delta(u, v)e^{-\omega}.$$

We obtain the evolution PDEs

$$\omega_u = \alpha(u, v)e^{\omega} + \beta(u, v)e^{-\omega}$$

$$\omega_v = \gamma(u, v)e^{\omega} + \delta(u, v)e^{-\omega}$$

whose complete integrability conditions are

$$\alpha_v = \gamma_u, \beta_v = \delta_u, \alpha\delta - \beta\gamma = 0.$$

The Lagrangian used in the geometric dynamics

$$L_2 = \frac{1}{2}h^{KL}g(\omega_K - X_K)(\omega_L - X_L)$$

$$= \frac{1}{2}(\omega_u - I(u, v)e^{\omega} - J(u, v)e^{-\omega})$$

$$\times(\omega_v - \gamma(u, v)e^{\omega} - \delta(u, v)e^{-\omega})$$

coincides to the Tzitzeica Lagrangian if and only if

$$\gamma\omega_u + \alpha\omega_v = -2, \delta\omega_u + \beta\omega_v = -1$$

$$\alpha\gamma = 0,\ \alpha\delta + \beta\gamma = 0,\ \beta\delta = 0.$$

# 4 Discrete two-parameter Lagrangian dynamics

Though continuous models are usually more convenient and yield results which are more transparent, the discrete models are also of interest being in fact *discrete dynamical systems.*

The theory of integrators for multi-parameter Lagrangian dynamics shows that instead of discretization of Euler-Lagrange PDEs we must use a discrete Lagrangian, a discrete action, and then discrete Euler-Lagrange equations. Of course, the discrete Euler-Lagrange equations associated to multi-time discrete Lagrangian can be solved successfully by the Newton method if it is convergent for a convenient step.

The discretization of a two-parameter Lagrangian $L(u, v, \omega(u, v))$ can be performed by using the *centroid rule* (see [17]-[18]) which consists in the substitution of the point $(u, v)$ with the fixed step $(k_1, k_2)$, of the point $\omega(u, v)$ with the fraction

$$\frac{\omega_{kl} + \omega_{k+1l} + \omega_{kl+1}}{3}$$

and of the partial velocities $\omega_\alpha,\ \alpha = 1, 2$, by the fractions

$$\frac{\omega_{k+1l} - \omega_{kl}}{k_1}, \frac{\omega_{kl+1} - \omega_{kl}}{k_2}.$$

One obtains a *discrete Lagrangian*

$$L_d : R^2 \times R^3 \to R,\ L_d(u_1, u_2, u_3) = L(k_1, k_2, \frac{u_1 + u_2 + u_3}{3}, \frac{u_2 - u_1}{k_1}, \frac{u_3 - u_1}{k_2}).$$

This determines the 2-dimensional *discrete action*

$$S : R^2 \times R^{(N_1+1)(N_2+1)} \to R,$$

$$S(k_1, k_2, A) = \sum_{k=0}^{N_1-1} \sum_{l=0}^{N_2-1} L(k_1, k_2, \omega_{kl}, \omega_{k+1l}, \omega_{kl+1}),$$

where

$$A = (\omega_{kl}), k = 0, ..., N_1,\ l = 0, ..., N_2.$$

The *discrete variational principle* consists in the characterization of the matrix $A$ for which the action $S$ is stationary, for any family

$$\omega_{kl}(\epsilon) \in R$$

with

$$k = 0, ..., N_1 - 1,\ l = 0, ..., N_2 - 1,$$

$$\epsilon \in I \subset R, 0 \in I,\ \omega_{kl}(0) = \omega_{kl}$$

and fixed elements

$$\omega_{0l},\ \omega_{N_1 l},\ \omega_{k0},\ \omega_{kN_2}.$$

The discrete variational principle is obtained using the first order variation of $S$. In other words the matrix (point) $A = (\omega_{kl})$ is stationary for the action $S$ if and only if (*discrete Euler-Lagrange equation*)

$$\sum_{\xi} \frac{\partial L}{\partial \omega_{kl}}(\xi) = 0, \tag{4.1}$$

where $\xi$ runs over three points,

$$(\omega_{kl}, \omega_{k+1l}, \omega_{kl+1})$$

$$(\omega_{k-1l}, \omega_{kl}, \omega_{k-1l+1}),\ \textit{left shift map}$$

$$(\omega_{kl-1}, \omega_{k+1l-1}, \omega_{kl}),\ \textit{right shift map}$$

and

$$k = 1, ..., N_1 - 1,\ l = 1, ..., N_2 - 1.$$

The *variational integrator* described by a discrete Euler-Lagrange equation works in three steps:

- Step 1: we give the lines

$$(\omega_{00}, \omega_{01}, ..., \omega_{0N}),\ (\omega_{10}, \omega_{11}, ..., \omega_{1N});$$

- Step 2: we denote

$$u = \omega_{kl+1}$$

$$A(kl) = \frac{\partial L}{\partial \omega_{kl}}(\omega_{k-1l}, \omega_{kl}, \omega_{k-1l+1})$$

$$B(kl) = \frac{\partial L}{\partial \omega_{kl}}(\omega_{kl-1}, \omega_{k+1l-1}, \omega_{kl})$$

$$f(u) = \frac{\partial L}{\partial \omega_{kl}}(\omega_{kl}, \omega_{k+1l}, u) + A(kl) + B(kl);$$

- Step 3: we solve the nonlinear equation

$$f(u) = 0$$

at each step $(k_1, k_2)$ using six points of starting as shown a part of the grid

$$\begin{array}{lll} \clubsuit & \clubsuit\omega_{k-1l} & \clubsuit\omega_{k-1l+1} \\ \clubsuit\omega_{kl-1} & \clubsuit\omega_{kl} & * \ u = \omega_{kl+1} \\ \clubsuit\omega_{k+1l-1} & \clubsuit\omega_{k+1l} & \circ \end{array}$$

Giving the boundary elements

$$\omega_{0l},\ \omega_{N_1 l},\ \omega_{k0},\ \omega_{kN_2},$$

the discrete Euler-Lagrange equation is solved by the Newton method if it is contractive for a small step $(k_1, k_2)$ (see [17]-[18]).

We introduce the *discrete momenta* via a *discrete Legendre transformation*

$$p^{kl} = \frac{\partial L}{\partial \omega_{kl}}(\omega_{kl}, \omega_{k+1l}, \omega_{kl+1}). \tag{4.2}$$

Then (4.1) becomes a linear initial value problem with constant coefficients

$$p^{kl} + p^{k-1l} + p^{kl-1} = 0, \tag{4.3}$$

called *dual variational integrator* equation. If (4.2) defines a bijection between $p^{kl}$ and $\omega_{kl+1}$ for given $\omega_{kl}$, $\omega_{k+1l}$, then we obtain a right one-step method $\phi^1 : (p^{kl}, \omega_{kl}) \to (p^{kl+1}, \omega_{kl+1})$ by composing the inverse discrete Legendre transform, a step with the discrete Euler-Lagrange equations, and the discrete Legendre transformation as shown in the diagram:

$$\begin{array}{ccc} (\omega_{kl}, \omega_{k+1l}, \omega_{kl+1}) & \overset{(4.1)}{\to} & (\omega_{kl+1}, \omega_{k+1l+1}, \omega_{kl+2}) \\ (4.2)\uparrow & & \downarrow(4.2) \\ (p^{kl}, \omega_{kl}) & \to & (p^{kl+1}, \omega_{kl+1}) \end{array}$$

If (4.2) defines a bijection between $p^{kl}$ and $\omega_{k+1l}$ for given $\omega_{kl}$, $\omega_{kl+1}$, then we obtain a left one-step method $\phi^2 : (p^{kl}, \omega_{kl}) \to (p^{k+1l}, \omega_{k+1l})$ by composing the inverse discrete Legendre transform, a step with the discrete Euler-Lagrange equations, and the discrete Legendre transformation as shown in the diagram:

$$\begin{array}{ccc} (\omega_{kl}, \omega_{k+1l}, \omega_{kl+1}) & \overset{(4.1)}{\to} & (\omega_{k+1l}, \omega_{k+2l}, \omega_{k+1l+1}) \\ (4.2) \uparrow & & \downarrow (4.2) \\ (p^{kl}, \omega_{kl}) & \to & (p^{k+1l}, \omega_{k+1l}) \end{array}$$

We summarize these considerations in the

**Theorem**. *The discrete variational principle gives the discrete Euler-Lagrange equations* (4.1) *and the momenta method*

$$p^{kl} = \frac{\partial L}{\partial \omega_{kl}}(\omega_{kl}, \omega_{k+1l}, \omega_{kl+1}),\ p^{k-1l} = \frac{\partial L}{\partial \omega_{kl}}(\omega_{k-1l}, \omega_{kl}, \omega_{k-1l+1})$$

$$p^{kl-1} = \frac{\partial L}{\partial \omega_{kl}}(\omega_{kl-1}, \omega_{k+1l-1}, \omega_{kl}) = -p^{kl} - p^{k-1l}.$$

# 5 Discrete Tzitzeica equation

We start from the discrete Tzitzeica Lagrangian

$$L = \frac{1}{2}\frac{\omega_{k+1l} - \omega_{kl}}{k_1}\frac{\omega_{kl+1} - \omega_{kl}}{k_2} + e^{(\omega_{kl}+\omega_{k+1l}+\omega_{kl+1})/3} + \frac{1}{2}e^{-2\cdot(\omega_{kl}+\omega_{k+1l}+\omega_{kl+1})/3}.$$

The *discrete Tzitzeica equation* (discrete Euler-Lagrange equation) associated to discrete Tzitzeica Lagrangian is

$$\begin{aligned} &\frac{1}{k_1k_2}(\omega_{k-1l} + \omega_{kl-1}) - \frac{1}{2k_1k_2}(\omega_{kl+1} + \omega_{k+1l} + \omega_{k-1l+1} + \omega_{k+1l-1}) \\ &+\frac{1}{3}e^{(\omega_{kl} + \omega_{k+1l} + \omega_{kl+1})/3} + \frac{1}{3}e^{(\omega_{k-1l} + \omega_{kl} + \omega_{k-1l+1})/3} \\ &+\frac{1}{3}e^{(\omega_{kl-1} + \omega_{k+1l-1} + \omega_{kl})/3} - \frac{1}{3}e^{-2(\omega_{kl} + \omega_{k+1l} + \omega_{kl+1})/3} \\ &-\frac{1}{3}e^{-2(\omega_{k-1l} + \omega_{kl} + \omega_{k-1l+1})/3} - \frac{1}{3}e^{-2(\omega_{kl-1} + \omega_{k+1l-1} + \omega_{kl})/3} = 0. \end{aligned}$$

This is a second order *nonlinear implicit finite difference equation.* The *singularity set* with respect to $u = \omega_{kl+1}$ is defined by the equation (see Figure 1)

$$e^{(\omega_{kl} + \omega_{k+1l} + \omega_{kl+1})/3} + 2e^{-2(\omega_{kl} + \omega_{k+1l} + \omega_{kl+1})/3} = \frac{9}{2k_1k_2}.$$

If we denote $\mathcal{Y} = e^{(\omega_{kl} + \omega_{k+1l} + u)/3}$, the singularity set with respect to u is given by positive solutions of third grade algebraic equation,

$$\mathcal{Y}^3 - \frac{9}{2k_1k_2}\mathcal{Y}^2 + 2 = 0.$$

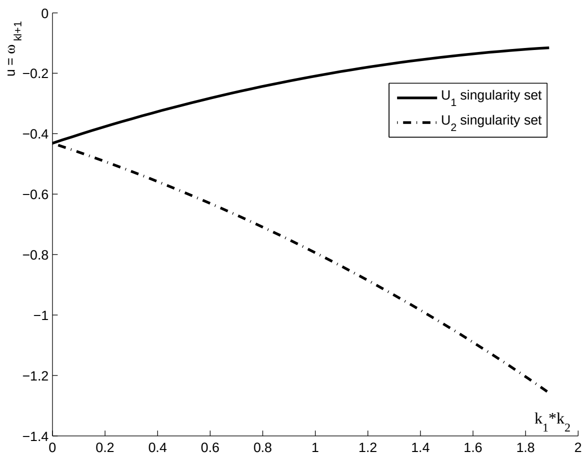
For $k_1k_2 > \dfrac{3}{\sqrt[3]{4}}$ the singularity set is empty.

For $k_1k_2 < \dfrac{3}{\sqrt[3]{4}}$ the previous implicit equation gives positive solutions:

$$u = U_1 - (\omega_{kl} + \omega_{k+1l}); u = U_2 - (\omega_{kl} + \omega_{k+1l})$$

and

$$U_{1,2} = 3\log((3/2 - 3cos(\pi/3 \pm \delta/3))/(k_1k_2)),$$
$$\delta = \arccos(-(6k_1k_2 + 8(k_1k_2)^3)/27).$$




**Singularity set with respect to $u$.**

## 6 Von Neumann stability analysis

The von Neumann analysis [1] is the most commonly used method of determining stability criteria. Unfortunately, it can only be used to establish necessary and suficient conditions for stability of linear initial value problems with constant coefficients. In case of non-linerities, this method can only be applied locally. If a non-linear difference equation is linearized in a small part of the solution domain, then the conditions for the applicability of this method are satisfied locally even though they are not satisfied over the whole solution domain.

The stability of numerical schemes is closely associated with numerical error. A finite difference scheme is stable if the errors made at one time step of the calculation do not cause the errors to increase (remain bounded) as the computations are continued. A *neutrally stable scheme* is one in which errors remain constant as the computations are carried forward. If the errors decay and eventually damp out, the numerical scheme is said to be *stable.* If, on the contrary, the errors grow with time the solution diverges and thus the numerical is said to be *unstable.* The stability of numerical schemes can be investigated by performing von Neumann stability analysis. For time-dependent problems, stability guarantees that the numerical method produces a bounded solution whenever the solution of the exact differential equation is bounded. Stability, in general, can be difficult to investigate, especially when equation under consideration is nonlinear. Unfortunately, von Neumann stability is necessary and sufficient for stability in the sense of Lax-Richtmyer (as used in the Lax equivalence theorem) only in certain cases: the PDE the finite difference scheme must be linear; the PDE must be constant-coefficient with periodic boundary conditions and have at least two independent variables; and the scheme must use no more than two time levels. It is necessary in a much wider variety of cases, however, and due to its relative simplicity it is often used in place of a more detailed stability analysis as a good guess at the restrictions (if any) on the step sizes used in the scheme.

# 7 Von Neumann analysis of dual variational integrator equation

To verify the stability of the dual variational equation

$$p^{kl} + p^{k-1l} + p^{kl-1} = 0,$$

we pass to the frequency domain, accepting that $u$ is a *spatial coordinate* and $v$ is a *temporal coordinate.* We need a 1D *discrete spatial Fourier transform* which can be obtained via the substitutions

$$p^{kl} \to P^l(\alpha)e^{j\alpha h},$$

where $\alpha$ denotes the *radian wave scalar.* We find a second order linear difference equation (*digital filter*)

$$P^l + P^l e^{-j\alpha h} + P^{l-1} = 0$$

that need its stability checked. For this purpose we introduce the *z-transform* $E(z,\alpha)$ and we must impose that the poles of the recursion do not lie outside the unit circle in the z-plane. To simplify, we accept the initial conditions $P^0 = 0$. One obtains the homogeneous linear equation

$$(1 + e^{-j\alpha h} + z^{-1})E = 0.$$

The pole $z = \dfrac{1}{1 + e^{-j\alpha h}}$ satisfies

$$|z| = \frac{1}{2|cos\dfrac{\alpha h}{2}|} \leq 1$$

if and only if $|cos\dfrac{\alpha h}{2}| \geq \dfrac{1}{2}$. This condition ensures that our scheme is marginally stable, and over the stability region, we have a relation in terms of the grid spacing $h$ and the wave number $\alpha$.

# 8 Von Neumann analysis of linearized discrete Tzitzeica equation

The linearization of discrete Tzitzeica equation is

$$\frac{1}{k_1k_2}(\omega_{k-1l}-\omega_{kl-1})+\omega_{kl}-\frac{1}{2k_1k_2}(\omega_{kl+1}+\omega_{k+1l}+\omega_{k-1l+1}+\omega_{k+1l-1})$$
$$+\frac{1}{3}(\omega_{k+1l}+\omega_{kl+1}+\omega_{k-1l}+\omega_{kl-1})+\frac{1}{3}(\omega_{k-1l+1}+\omega_{k+1l-1})=0.$$

To verify the stability of this finite difference scheme, we pass to the frequency domain, through what is called von Neumann analysis. For that

(1) we accept that $u$ is a *spatial coordinate* and $v$ is a *temporal coordinate*;

(2) we accept a uniform grid spacing in $u$, i.e., $h=k_1$ is constant, and an unbounded domain $R$;

(3) we denote by $\tau=k_2$ the "time" step regarding $v$ and we define the constant level sets $\frac{3}{2k_1k_2}=\rho$.

We need a 1D *discrete spatial Fourier transform* which can be obtained via the substitutions

$$\omega_{kl}\to\Omega_l(\alpha)e^{j\alpha h},$$

where $\alpha$ denotes the *radian wave scalar*. We find a second order linear difference equation (*digital filter*)

$$\rho(2\Omega_l e^{-j\alpha h}+2\Omega_{l-1}-\Omega_{l+1})-\rho(\Omega_l e^{j\alpha h}+\Omega_{l+1}e^{-j\alpha h}+\Omega_{l-1}e^{j\alpha h})$$
$$+3\Omega_l+\Omega_{l+1}+\Omega_{l-1}+\Omega_l e^{j\alpha h}+\Omega_l e^{-j\alpha h}+\Omega_{l+1}e^{-j\alpha h}+\Omega_{l-1}e^{j\alpha h}=0$$

that need its stability checked. For this purpose we introduce the *z-transform* $F(z,\alpha)$ and we must impose that the poles of the recursion do not lie outside the unit circle in the z-plane. To simplify, we accept the initial conditions $\Omega_0=0$. One obtains the homogeneous linear equation

$$(1-\rho)z(1+e^{-j\alpha h})F+z^{-1}(1+2\rho+(1-\rho)e^{j\alpha h})F$$
$$+((1+2\rho)e^{-j\alpha h}+(1-\rho)e^{j\alpha h}+3)F=0$$

The poles are the roots of the characteristic equation (see also [9])

$$(1-\rho)(1+e^{-j\alpha h})z^2+(1+2\rho+(1-\rho)e^{j\alpha h})$$
$$+((1+2\rho)e^{-j\alpha h}+(1-\rho)e^{j\alpha h}+3)z=0$$

with the unknown $z$. Explicitly, we have

$$a_2z^2+a_1z+a_0=0,$$

where

$$a_2=(1-\rho)(1+e^{-j\alpha h}),$$
$$a_1=(1+2\rho)e^{-j\alpha h}+(1-\rho)e^{j\alpha h}+3,$$
$$a_0=1+2\rho+(1-\rho)e^{j\alpha h}.$$

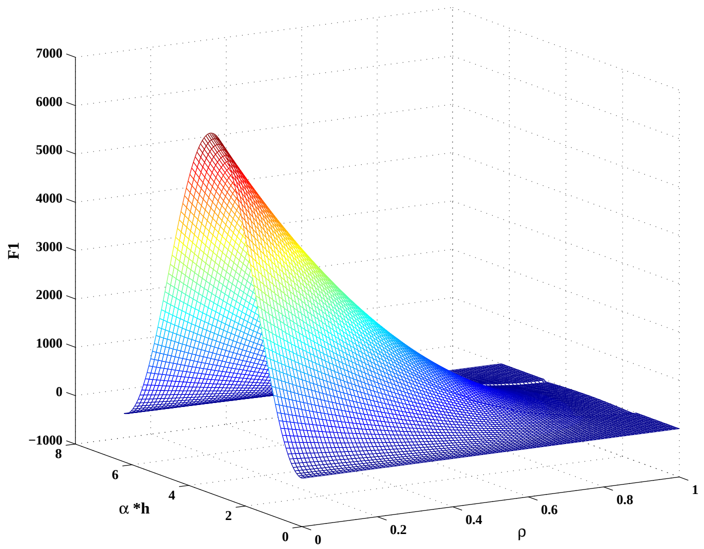
Since the solutions are determined by $z_{1,2} = \dfrac{-a_1 \pm \sqrt{\Delta}}{2a_2}$, $\Delta = a_1^2 - 4a_0a_2$, the condition

$$|z_i| \leq 1 \tag{8.1}$$

is equivalent to $F_{1,2} \geq 0$ with

$$F_{1,2} = 4(1-\rho)\sqrt{2 - 2\cos(\alpha h)} - |-a_1 \pm \sqrt{\Delta}|$$

(see Figure 2 and Figure 3).




**Graph of function** $F_1(\rho, \alpha h)$**.**

**Case 1**. For $\mathrm{Im}(a_2) = 0$ all the coefficients of the equation are real numbers. We find

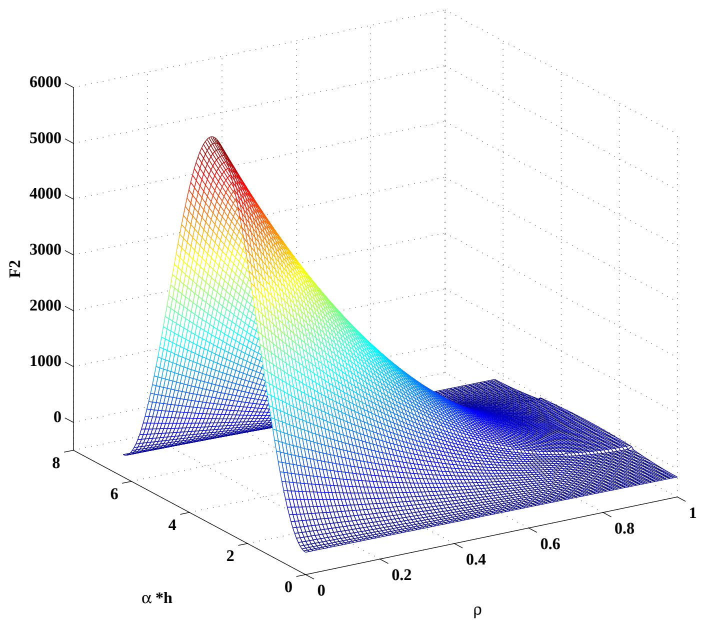
- $h = \dfrac{\pi}{\alpha}$, with $\cos(\alpha h) = -1$. We have $a_2 = 2(1-\rho)$, $a_1 = 1-\rho$, $a_0 = 3\rho$ and we find that $\dfrac{1}{13} < \rho < \dfrac{5}{18}$ ensure the marginally stablity to our scheme.
- $h = \dfrac{2\pi}{\alpha}$, when $\cos(\alpha h) = 1$.

  We have $a_2 = 0$, $a_1 = 5 + \rho$, $a_0 = 2 + \rho$. Condition (8.1) being satisfied for any $0 < \rho < 1$.

Over the stability region, the previous constraints give us a time step $\tau$, in terms of the grid spacing $h$, and implicitly in terms of the wave number $\alpha$.

**Case 2.** For $\mathrm{Im}(a_2) \neq 0$, we find that our condition (8.1) is satisfied also if $\cos(\alpha h) \neq 0$.

Positive values of functions $F_1$ and $F_2$ satisfy the constraints (8.1) on the time step $\tau$, the grid spacing $h$, and spatial wave number $\alpha$, describing the stability regions where our scheme is marginally stable.




**Graph of function** $F_2(\rho, \alpha h)$**.**

# 9 Open problem

Analyse the extended Tzitzeica PDE via a Laurent polynomial

$$\omega_{uv} = \sum_{n\in Z} a_n e^{n\omega}, \ \sum_{n\in Z} a_n = 0$$

and find applications in differential geometry and soliton theory.

# 10 Conclusion

The *von Neumann stability analysis* [1] is a procedure used to check the stability of finite difference schemes as applied to linear PDEs. The analysis is based on the Fourier decomposition of numerical error and was developed at Los Alamos National Laboratory after having been described in a 1947 article by British researchers Crank and Nicolson. Later, it also published in an article co-authored by von Neumann.

This paper is the first which applies the von Neumann analysis to proves the stability of the finite difference scheme for the linearized discrete Tzitzeica equation.

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